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In Operations Research as in Life,
some rules are made to be broken.
M.Z.

A Salwa & Younes,

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Me voilà à la page des remerciements, classique, cela fait partie des normes d'ailleurs ! Certains essayent d'y être plus originaux que "les normes", mais je trouve le résultat presque pareil à la fin. Le plus original (i.e., le plus simple) serait-t-il, et je l'entends sincèrement, de ne rien écrire et passer dire Merci aux gens ? Mais vous cher lecteur, vous ne le saurez jamais : tant pis pour l'originalité alors.

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Extended Abstract

Nous considérons le problème de gestion de la maintenance des routes sujets à une dégradation stochastique. Nous développons tout d'abord un nouveau processus pour modéliser la fissuration longitudinale d'une section de route. Ce processus est unique dans le sens où il permet de représenter la dégradation observable aussi bien que non-observable et il est conditionnel. Ce processus permet aussi de modéliser l'incertitude endogène, i.e., qui dépend de la décision. Nous utilisons par la suite ce processus stochastique pour dériver les matrices de décision pour le processus de décision markovien (MDP) avec incertitude endogène que nous utilisons pour formuler le problème d'optimisation de la maintenance. Pour cette nouvelle classe de MDP, nous somme capable d'établir des propriétés structurelles intéressantes que nous exploitons pour accélérer l'algorithme classique d'itération de politiques. Nous étendons ensuite la formulation MDP en un MDP partiellement observable (POMDP) et utilisons les propriétés établies pour le modèle MDP à fin de développer une nouvelle heuristique. Finalement, nous développons un programme stochastique multi-étapes pour l'optimisation de l'inspection ainsi que de la maintenance de tout le réseau routier.

Nous nous intéressons particulièrement au problème de fissuration longitudinale qui résulte des charges de trafic croissantes et des conditions environnementales variables. Ce phénomène est un processus à deux phases commençant avec (1) une phase d'initiation durant laquelle les couches inférieures de la route se dégradent même si la surface reste intacte, et qui évolue en (2) une phase de propagation durant laquelle la fissuration devient observable et la dégradation des sous-couches continue à évoluer.

Les données de dégradation disponibles présentent le pourcentage de fissuration longitudinale (PFL) comme un indicateur du niveau de fissuration pour chaque section de route. Cependant, le PFL ne permet pas de représenter le niveau de dégradation sous-jacente des couches inférieures pendant les deux phases d'initiation et de propagation. Nous proposons d'introduire un nouveau paramètre que nous appelons taux de croissance de la dégradation (TCD) et qui représente un potentiel de fissuration pendant la phase d'initiation et une vitesse instantanée de dégradation pendant la phase de propagation. Nous développons un modèle stochastique conditionnel à deux phases pour ces deux paramètres de dégradation.

Ce modèle est dit conditionnel parce que l'évolution de la dégradation n'est pas fonction de l'âge de la route, comme classiquement supposé avec les modèles non-stationnaires, mais de son état courant, i.e., du niveau courant de dégradation donné par (PFL, TCD). Le modèle est constitué de ce que nous avons appelé un processus "gamma conditionnel" pour le PFL et un processus gamma bilatéral pour le TCD. Nous définissons le processus gamma conditionnel comme un processus gamma avec une loi d'évolution fonction de l'état de la section, i.e., ses niveaux courants de PFL et TCD. La dépendance du processus du PFL de son niveau courant conduit à la perte de la propriété d'infini-divisibilité du processus gamma. Cependant, nous prouvons que avec l'introduction du processus du TCD et la relation de dépendance que nous avons établie avec le PFL, le processus gamma conditionnel est infini-divisible. Nous utilisons le processus gamma bilatéral, emprunté au domaine de la finance, pour modéliser le TCD. Nous exploitons aussi sa propriété d'être défini comme la différence de deux processus gamma indépendants pour dériver directement ses paramètres sans efforts additionnels d'estimation.

Nous considérons dans un premier temps l'optimisation d'une unique section de route. Cette dernière est périodiquement inspectée à fin d'évaluer son niveau de dégradation. Après inspection, le décideur choisit une parmi plusieurs actions de maintenance allant de "ne rien faire" au renouvellement total de la route. Ces actions ont aussi différents impacts immédiats et futures sue la section de route ainsi que sur son comportement. L'objectif est de définir des politiques de maintenance optimales et efficaces pour minimiser le coût total espéré sur un horizon de décision infini. Nous proposons de changer le processus de dégradation après différentes maintenances en considérant le type de la dernière maintenance comme un paramètre de l'état de la section. De plus, dans le contexte de la modélisation des effets des maintenances imparfaites, l'introduction du TCD permet la différentiation de ces effets.

Nous assumons tout d'abord que le problème est complètement observable formulons le problème de l'optimisation de la maintenance comme un MDP avec incertitude endogène où les matrices de transitions dépendantes de la décision sont déterminées en utilisant notre modèle de dégradation conditionnel. Nous résolvons le modèle MDP avec l'algorithme d'itération de politiques. pour dériver des politiques de maintenance optimales.

Les propriétés structurelles sont classiquement démontrées pour des MDP avec incertitude exogène (i.e., une matrice de transition unique), cependant nous pensons être les premiers à les établir pour le cas endogène. Nous présentons les conditions que doivent vérifier les structures de transitions et de coûts pour que les politiques optimales avec incertitude dépendante de la décision soient du type limite de contrôme. Nous établissons aussi une nouvelle propriété que nous appelons "garantie d'utilisation des actions" et qui assure à toutes les actions d'être sélectionnées au moins pour un état dans la politique optimale. En pratique, cette propriété aide les décideurs à définir l'ensemble des actions significatives au problème. De plus, nous utilisons cette propriétés pour accélérer l'algorithme classique de résolution des MDP (jusqu'à 70%).

Formuler le problème d'optimisation comme un MDP suppose que l'état de la section est complètement observable. Cependant, ce n'est pas le cas du TCD qui modélise la dégradation sous-jacente et non-observable. Nous étendons alors la formulation à un POMDP où l'observabilité partielle est modélisée en remplaçant la connaissance déterministe de l'état du système par une croyance sur le vrai état. Par conséquence, une solution exacte du problème partiellement observable est presque impossible, sinon pour des problème de petites tailles.

Bien que propriétés structurelles démontrées pour le problème complètement observable peuvent ne pas tenir au cas partiellement observable, nous les utilisons pour définir une heuristique de résolution efficace du POMDP. En effet, nous proposons une heuristique du type "grid-based" où la construction de la grille de vecteurs de croyance est basée sur la définition d'ensembles de séparation obtenus par les efficacité immédiate des actions et de leurs intersections.

La comparaison des résultats numériques obtenus par notre heuristique à ceux obtenus par la procédure basée sur une base régulière (grid-based approximation (GBA)) montre les gains importants en terme de temps de calcul (jusqu'à 99%) ainsi que la bonne qualité des résultats en terme de fonction objectif. Nous sommes aussi capable de résoudre des problèmes de grande taille qui ne sont pas résolus par le GBA en un intervalle de temps raisonnable.

Nous étendons encore notre approche d'optimisation de la maintenance ainsi que ses propriétés structurelles au cas des actions pouvant changer l'épaisseur de la route. Dans un contexte plus général, de telles actions peuvent correspondre à la possibilité de remplacer le système opérant par un nouveau système avec des caractéristiques plus performantes (e.g., nouvelle technologie).

Finalement, nous considérons le réseau des sections de route pour lequel nous adressons le problème de l'optimisation jointe de l'inspection et de la maintenance. Nous définissons l'objectif des politiques optimales non seulement du point de vue du gestionnaire, mais aussi di point de vue des usagers. Nous considérons ainsi, en plus des coûts de maintenance, des coûts usagers et environnementaux sujets à des contraintes de budget de connectivité et de qualité de trafic.

L'optimisation des décisions d'inspections simultanément que celles de maintenance rend le problème beaucoup plus complexe à cause de la dépendance des deux décisions. Ceci est en effet une seconde source d'incertitude endogène (en plus d'évolution de la dégradation qui dépend de la décision). Selon notre connaissance, notre modèle est le seul qui contient les deux types de sources d'endogéneité. Nous formulons le problème d'optimisation de l'inspection et de la maintenance du réseau routier comme un programme stochastique multi-étape. Nous croyons qu'avec ce modèle, nous avons ouvert l'intéressante perspective de recherche de définir des procédures de résolutions efficaces pour cette classe de modèles.

Bien que le modèle de dégradation ainsi que l'approche d'optimisation de la maintenance que nous développons dans ce travail sont pour le problème de gestion routière, nous pensons qu'ils peuvent facilement être utilisés pour d'autres applications. De plus, notre heuristique de résolution du POMDP peut être utilisées pour plusieurs types de problèmes présentant une structure d'investissement/risk.

Extended Abstract

We consider the problem of managing road maintenance subject to stochastic deterioration. We begin by developing a new stochastic process to model longitudinal cracking on a road section. This process is unique in that it takes into account both observable and non-observable deterioration and is state-dependent. The process also allows handling endogenous, i.e., decision-dependent uncertainty. Next we utilize this stochastic process to derive the transition matrices for a Markov decision process (MDP) with endogenous uncertainty that we use to optimize the maintenance decisions. For this new class of MDP we are able to establish appealing structure and exploit it to accelerate the classical policy iteration algorithm. We then extend the MDP to a partially observable MDP (POMDP) and use the MDP properties to develop a novel heuristic. Finally, we develop a multi-state stochastic program that considers inspection and maintenance decisions on the entire road network.

We are interested specifically in the longitudinal cracking problem that arises from increasing traffic loads and harsh environmental conditions on roads. This is a two-phase process that begins with (1) an initiation phase during which the inferior road layers are deteriorating even if its surface is free of cracks, and evolves to (2) a propagation phase during which the deterioration becomes observable and while the degradation in the inferior layers continues.

Available deterioration data uses the longitudinal cracking percentage (LCP) as the indicator of each road section cracking level. However, the LCP does not represent the underlying deterioration level of inferior road layers in both the initiation and propagation phases. We propose to introduce a novel parameter that we call the deterioration growth rate (DGR) that represents a potential of cracking in the initiation phase and an instantaneous deterioration speed during the propagation phase. We develop a two-phase, state-dependent stochastic model for these two parameters. This model is said to be state-dependent since the deterioration evolution is not based on the system age as classically assumed in non-stationary models, but on its current state, i.e., current deterioration level given by (LCP, DGR).

The model is comprised of what we term a “state-dependent gamma process” (SDG) for the LCP and a bilateral gamma process (BG) for the DGR. We define the SDG as a gamma process with an evolution law that is a function of the section state, i.e., its current levels of LCP and DGR. The dependence of the LCP process on its current level leads to the loss of the infinite divisibility property of the gamma process. Nevertheless, we prove that with the introduction of the DGR process and the dependence relationship that we establish with the LCP, that the SDG process is infinitely divisible. We use the BG process, borrowed from a financial context, to model the DGR and exploit its property of being defined as a difference of two independent gamma processes to directly derive its parameters without any need for additional estimation.

We first consider the optimization of a single road section that is periodically inspected in order to evaluate its deterioration level. After inspection, the decision maker selects one of several maintenance actions ranging from “do nothing” to total road renewal. These actions also have different immediate and future effects on the road section and its behavior. The objective is to define efficient and optimal maintenance policies that minimize the total expected cost over the infinite decision horizon. We propose to change the deterioration process after different actions by considering the last maintenance type as a state parameter. Moreover, in the context of imperfect maintenance effect modeling, the introduction of the DGR allows the differentiation of these effects.

We first assume that the problem is completely observable and formulate the maintenance optimization problem as an MDP with endogenous uncertainty where the decision-dependent transition matrices are determined using our state-dependent deterioration model. We solve the MDP using the policy iteration algorithm (PIA) to derive optimal maintenance policies.

Structural properties are normally proven for MDP with exogenous uncertainty (i.e., unique transition matrix), however we believe we are the first to establish them for the endogenous case. We give conditions on transition and cost structure under which optimal policies with decision-dependent uncertainty are of the type control limit type. We also establish a novel property that we term an “action utilization guarantee” that ensures that all the actions are selected at least for one state in the optimal policy. In practice, this property helps decision-makers to properly define the set of significant maintenance actions. Moreover, we use these properties to accelerate the PIA (up to 70%).

Formulating the optimization problem as an MDP assumes that the section state is perfectly observable. However, this is not the case for the DGR that models the underlying and non-observable deterioration. Therefore, we extend the formulation to a POMDP where partial observability is taken into account by replacing the deterministic knowledge upon the system state by a belief on states. Consequently, an exact solution of the partially observed problem is almost impossible, except for very small size problems.

Although structural properties proved for the fully observed problem may not hold for the partially observed problem, we use them to define an efficient heuristic solution procedure for it. We propose a grid-based heuristic where the construction of the belief grid is based on the definition of immediate-efficiency action cuts and their intersections. A comparison of numerical results derived by our heuristic to those obtained by the regular grid-based approximation (GBA) heuristic shows substantial savings (up to 99%) in computation time as well as its strong performance in terms of objective value. We are also able to solve large size problems that are not solved by the GBA in a reasonable amount of time.

We extend further our maintenance optimization approach and its structural properties to the case of actions that may change the road thickness. In a more general context, such actions may correspond to the possibility of replacing the operating system by a newer one with better characteristics (e.g., newer technology).

Finally, we consider the road section network for which we address the problem of joint inspection and maintenance optimization. We define the objective of optimal policies not only from a managerial point of view, but also from user one. Moreover, we consider connectivity and traffic quality constraints as well as user and environmental costs subject to a budget constraint. Optimizing inspections simultaneously with maintenance decisions renders the optimization problem much more complex because of the dependence of the two decisions. This is a second source of endogenous uncertainty (in addition to decision-dependent deterioration evolution) and is the only model we are aware of that contains both types. We formulate the road network inspection and maintenance optimization problem as a multi-stage stochastic program. We believe that with this model we have opened the interesting research perspective of defining efficient solution procedures for this class of model.

Although both the deterioration model and the maintenance optimization approach are developed in this work for road maintenance, we feel they can easily be utilized for other applications. Moreover, the heuristic solution procedure can be used for many types of problems with an investment/risk structure.

INTRODUCTION GÉNÉRALE

L'intérêt scientifique porté à la recherche opérationnelle et à l'aide à la décision n'est pas récent mais date de la deuxième guerre mondiale quand les motivations et les applications des techniques d'aide à la décision étaient essentiellement militaires (e.g., déploiement de radars, détermination de la taille des convois, logistique). Après la guerre et suite au succès connu dans le domaine militaire, les techniques d'aide à la décision se sont étendues à différentes applications civiles comme l'énergie, l'ingénierie industrielle, la gestion des travaux dans le secteur public ou, plus récemment, à l'ingénierie financière.

La théorie de la décision englobe une très grande variété de techniques qui permettent de fournir des solutions optimales ou presque-optimales pour des problèmes de décision complexes. Ces techniques sont continuellement développées pour améliorer l'efficacité des décisions ainsi que la qualité des solutions. De plus, ce domaine de recherche ayant pour but de gérer au mieux l'utilisation d'une technologie donnée ou la prestation d'un service, il interfère assez souvent avec plusieurs domaines comme la finance, l'ingénierie civile, etc.

La résolution d'un problème de décision passe par deux grandes étapes :

- *La formulation d'un modèle* qui décrit la réalité du problème considéré. On peut distinguer les modèles déterministes où tous les paramètres du problème sont supposés connus avec certitude et les modèles stochastiques où ces paramètres sont représentés par des variables aléatoires ou des processus stochastiques.

- *La résolution numérique* du modèle qui fournit une solution optimale ou presque optimale. Cette étape est en liaison étroite avec d'une part la formulation du problème d'optimisation et le développement informatiques des algorithmes de résolution et, d'autre part, de la puissance de calcul des ordinateurs.

Dans une grande majorité des cas, l'hypothèse de paramètres connus et déterministes s'avère être très restrictive surtout lorsque le résultat d'une expérimentation est fortement aléatoire ou encore lorsque de nombreuses incertitudes existent sur le modèle. Par exemple, en finance, les fluctuations des prix d'options sont totalement stochastiques et une gestion efficace des portfolios repose sur des modèles de prévision élaborés. Dans un contexte de gestion de production, l'incertitude peut provenir de la demande qui ne peut pas être connue exactement avant l'instant de sa réalisation, mais aussi de la performance incertaine des moyens de production ou de service présentant un risque aléatoire de défaillance.

Pour de tels problèmes, les techniques dynamiques d'aide à la décision dans l'incertain cherchent à définir des stratégies optimales (décisions pour l'immédiat) qui prennent en compte toutes les évolutions possibles et réalisations futures sur un horizon de temps qui peut être fini ou infini. Ainsi, une étape de compréhension et de modélisation du processus stochastique et de son évolution est nécessaire pour assurer l'efficacité des décisions optimales. Cependant, la prise en compte de l'incertain dans le processus de décision rend la modélisation du problème et sa résolution beaucoup plus compliquées.

Dans les travaux présentés dans ce rapport, nous nous intéressons particulièrement à l'optimisation de la gestion de la maintenance des réseaux routiers sujets à des phénomènes de dégradation stochastiques. Ces travaux s'inscrivent dans le cadre d'un projet appelé "State-Based Decision For Road Maintenance" (SBaDFoRM) qui est une collaboration entre l'Ecole des Mines de Nantes et l'Institut français des sciences et technologies des transports, de l'aménagement et des réseaux (IFSTTAR), fruit du regroupement entre le Laboratoire Centrale des Ponts et Chaussées et l'Institut National de Recherche sur les Transports et leurs Sécurité.

Le projet SBaDFoRM est financé par le conseil général de la région Pays de La Loire qui assure depuis 2006 la gestion du réseau routier départemental et national de la région. Nous nous intéressons particulièrement au réseau de routes nationales qui sont des voies importantes traversant de grandes parties du territoire national français. A l'exception de quelques sections, les routes nationales sont à usage gratuit et sont sollicitées par tous les types de véhicules. Ainsi, bien que les réseaux de routes nationales puissent différer d'une région à une autre de façon significative selon le type et l'intensité de sollicitations ainsi que les budgets consacrés à leur gestion, ils sont considérés comme représentatifs des réseaux de routes principales en France [30]. Les routes nationales sont initialement dimensionnées pour une durée de vie (durée de service au bout de laquelle les routes doivent être renouvelées) généralement estimée à 20 ans. Toutefois, pendant leurs durées de vie, ces routes présentent de nombreuses pathologies qui peuvent nuire à la sécurité et au confort des usagers. Des inspections et des actions de maintenance doivent alors être programmées régulièrement durant la durée de vie des chaussées pour assurer des conditions acceptables de service.

Face à des coûts de maintenance et de renouvellement très importants et des budgets généralement très réduits par rapport à la taille des réseaux, la gestion du réseau routier représente un enjeux socio-économique important pour la politique départementale. Dans ce contexte, l'ensemble des acteurs est à la recherche poursuit de manière continue une recherche active pour améliorer les pratiques courantes, et notamment la construction de nouveaux modèles d'optimisation des politiques de gestion d'entretien de ces réseaux.

Dans le cas particulier de gestion des réseaux routiers, l'intérêt croissant porté à l'optimisation des politiques de maintenance durant les dernières décennies peut être expliqué par le fait que la construction et l'aménagement de ces réseaux représentent de très grands investissements qui doivent être conservés par des activités d'entretien régulières. De plus, la sécurité et le confort des passagers sont mis en jeu quand les routes atteignent des niveaux de dégradation inacceptables dépassant les normes de sécurité.

La définition de politiques de maintenance efficaces est une tâche complexe car elle doit prendre en compte des contraintes économiques (e.g., budget, coût de maintenance), politiques, sociales, stratégiques (e.g., fermetures d'itinéraires et coûts associés), réglementaires (sécurité et confort des usagers) ainsi que le niveau de dégradation des chaussées et leurs évolutions futures.

Différents systèmes de gestion de chaussées (pavement management systems (PMS)) ont été proposés et développés dans les pays industrialisés dans le but de répondre aux exigences citées ci-dessus et assurer une gestion efficace des réseaux routiers. Une revue détaillée des PMSs existants peut être trouvée dans la référence [24].

Un PMS est constitué essentiellement de quatre composantes : l'inventaire, l'évaluation de l'état (condition) de la chaussée, le modèle de dégradation de la chaussée et la méthode de planification des maintenances [51]. Afin de contribuer au développement des PMS et de proposer une approche conditionnelle d'optimisation pour la maintenance des routes dans le cadre du projet SBaDFoRM, j'ai cherché durant les trois années de ma thèse à contribuer simultanément au développement et à l'amélioration des trois dernières composantes d'un PMS.

Dans le cadre de ce projet, mes travaux de thèse ont pour objectifs essentiels le développement d'un nouveau modèle stochastique de dégradation des chaussées et son intégration dans une approche conditionnelle pour l'optimisation de la maintenance afin de proposer des politiques optimales de gestion de l'entretien routier.

Le détail des contributions est résumé dans la première partie de ce rapport, alors que la deuxième partie en présente les approfondissements sous la forme des principales publications issues de cette thèse.

Plus précisément, la première partie se compose de deux chapitres. Le Chapitre 1 permet de positionner les travaux de thèse en décrivant le problème de fissuration des chaussées, les pratiques de maintenance disponibles aux décideurs ainsi qu'une brève analyse bibliographique des approches existantes d'optimisation de la maintenance. Cette étude nous permet de mettre en parallèle les modèles existants développés dans un contexte général de maintenance ainsi que les problématiques propre à notre problème d'entretien routier. De cette analyse, nous dégagerons les limites des modèles actuels et définirons les axes de recherche de la thèse. Le Chapitre 2 présente une synthèse des différentes contributions qui seront détaillées dans la seconde partie de ce manuscrit.

L'organisation de la deuxième partie reprend les publications principales. Le Chapitre 3 peut être vu comme une introduction au problème de gestion de l'entretien routier pour lequel nous avons introduit un nouveau paramètre de décision. Ce nouveau paramètre vient ainsi compléter l'information de dégradation sous-jacente essentielle pour modéliser le processus de fissuration d'une part et expliquer les effets de l'entretien d'autre part. Nous avons proposé une modélisation discrète sous la forme d'une chaîne de Markov et mis en évidence l'intérêt de notre proposition sur un exemple numérique. Ce travail a été publié dans les actes de la conférence de Institute of Industrial Engineering (USA, Miami, Juin 2009) [71].

La modélisation de l'état de la chaussée par une chaîne de Markov, i.e., par des variables aléatoires, est critiquable d'une part part parce qu'elle ne prend pas en compte de l'effet de la longueur de l'intervalle inter-inspection sur l'évolution de l'état (les transitions des états de la chaîne de Markov), et d'autre part pour les difficultés liées à l'optimisation de cet intervalle. Nous avons proposé d'étendre nos premiers travaux en se reposant sur le processus Gamma. Cette extension a imposé la construction d'un nouveau processus stochastique bivarié qui permet notamment de pallier certaines limites de stationnarité des processus de Lévy. Le Chapitre 4 présente le détail de ce modèle stochastique de dégradation bivarié et conditionnel et fait l'objet d'une soumission [73] pour publication dans le journal Naval Research Logistics. Par ailleurs, nous avons mis en évidence les bénéfices de notre modèle dans le contexte de maintenance conditionnelle pour une section de chaussée dans les travaux présentés et publiés dans [72, 77, 76, 78, 74] et [75].

Bien que proposant un ensemble d'actions d'entretien variées, les hypothèses de maintenance proposées ne permettent pas de bien décrire les contextes opératoires de ces entretiens et ainsi de modéliser leurs effets notamment par les changements de constitution des épaisseurs de la chaussée. Nous avons proposé de décomposer l'action d'entretien en deux phases : le décaissement et le chargement. Les variables de décision sont ainsi étendues aux épaisseurs de ces deux actions. La construction du modèle de maintenance et la conduite d'analyses numériques de performance sont présentées dans le Chapitre 5. Ces analyses permettent de mettre en évidence l'intérêt de notre approche sur plusieurs points. Le premier est la proposition d'un outil intégré pour optimiser la balance entre investissement en maintenance et gestion du risque de dégradation. Le second point est relatif à la prise en compte, dans le modèle décisionnel, d'évolutions technologiques moins sensibles à la dégradation. Ce travail a fait l'objet d'un article publié dans *Journal of Risk and Reliability* [77].

Jusqu'à présent, nous avons considéré l'observabilité du deuxième indicateur de fissuration alors qu'il ne peut être qu'estimé au vu des données disponibles. Cette hypothèse d'observabilité nous a permis de modéliser le problème d'optimisation comme un processus de décision markovien. Dans le Chapitre 6, nous levons cette hypothèse d'observabilité et nous proposons une modélisation de type processus de décision markovien partiellement observable (POMDP). Ce travail a été présenté à la conférence Reliability And Maintenability Symposium (RAMS) (Orlando, USA, Janvier 2011) [76].

Avec la formulation POMDP, l'espace des paramètres de décision passe de l'espace des états discret et fini à un espace de croyance d'état continu et infini. Ceci rend la résolution exacte du problème d'optimisation difficile. Pour contourner ce problème, nous avons d'une part prouvé les propriétés structurelles du problème MDP sans changement de l'épaisseur de la chaussée puis construit un algorithme pour accélérer la résolution de cet MDP et d'autre part proposé une heuristique basée sur ces propriétés structurelles pour résoudre le modèle POMDP. Ce travail est présenté dans le Chapitre 7 et fait l'objet d'une soumission à *Operations Research* [78].

Nous avons, dans le Chapitre 8, étendu ces travaux pour la prise en compte de changements possibles d'épaisseur de chaussée après entretien. La difficulté ici est de définir une relation d'ordre quant à l'efficacité des actions de maintenance pour l'obtention de structures optimales pour la décision. Les résultats proposés dans ce chapitre feront l'objet d'une soumission dans *European Journal of Operations Research* [74].

Enfin, en termes de perspectives à mon travail de thèse centré jusqu'ici sur l'entretien d'une unique section de chaussée, nous proposons dans la partie annexe des travaux préliminaires pour l'optimisation de l'inspection et d'entretien d'un réseau routier, formé de plusieurs sections. Outre les aspects classiques d'économie d'échelle liée à la mutualisation des ressources, le problème de décision dans le contexte routier est lié à de très nombreuses contraintes comme le budget, la connectivité dans le réseau ainsi qu'à la qualité du trafic et la gêne aux usagers. Nous avons ici proposé une modélisation de type programmation stochastique pour lequel l'une des difficultés majeures est le problème d'incertitude endogène définie par la relation entre la première décision d'inspection avec la résolution de l'incertitude ainsi que l'effet des décisions de maintenance sur le processus stochastique de dégradation. La résolution d'un tel modèle est très complexe et reste une des perspectives immédiates à mes travaux de thèse.

I RÉSUMÉ DES TRAVAUX

1 Positionnement des travaux

1.1 Introduction

Les réseaux routiers représentent un des plus importants supports aux activités économiques dans les pays industrialisés. Ils doivent être régulièrement entretenus pour assurer les activités sociales et économiques ainsi que la sécurité et le confort des usagers. Par ailleurs, d'autres contraintes ont fortement motivé les agences de gestion des réseaux routiers à s'investir dans la complexe tâche de développer et améliorer des approches de gestion de la maintenance. On peut citer la limitation voire la réduction permanente des ressources budgétaires au vu de la taille et sollicitations des réseaux, les solutions de renouvellement très coûteuses non seulement en termes de budget mais aussi en termes de temps, de perturbation de trafic et de coûts usagers, des préoccupations environnementalles de plus en plus fortes ainsi qu'à la forte stochasticité des diverses dégradations due notamment à des charges de trafic croissantes et des agressions environnementales variées. Cette tâche est rendue encore plus compliquée par l'existence d'un panel d'actions de maintenance avec différents impacts sur l'état immédiat des chaussées ainsi que sur leurs futurs comportements et la difficulté de modélisation de ces effets.

Les recherches en optimisation de la gestion de la maintenance des réseaux routiers ont conduit au développement des PMS. Les PMS sont des outils d'aide à la décision qui fournissent aux gestionnaires des stratégies de maintenance efficaces et à moindre coût. Les PMS opèrent sur tous les niveaux de décision pour en assurer la consistance et la cohérence.

Les composantes d'un PMS comme définies par [51] sont : l'inventaire, l'évaluation de l'état de la chaussée, le modèle de dégradation de la chaussée et la méthode de planification des maintenances.

L'inventaire consiste à diviser le réseau routier en sections identiques (les plus petites unités de gestion). Notons qu'un PMS agit au niveau de la section et/ou au niveau du réseau. L'objectif économique dans le premier cas est de trouver les stratégies de maintenance qui minimisent les dépenses totales. Tandis que dans le deuxième cas, l'objectif est de trouver les stratégies les plus efficaces qui respectent certaines contraintes comme les contraintes budgétaires.

L'évaluation de l'état des chaussées consiste en une évaluation fonctionnelle et structurelle. L'évaluation fonctionnelle revient à évaluer certaines caractéristiques de la chaussée à travers la mesure d'indices de qualité. Par exemple, la rugosité est mesurée par l'indice de rugosité, la friction par l'indice de friction et la fissuration par la longueur des fissures. Quant à l'évaluation structurelle, elle se fait au travers de l'indice d'adéquation structurelle qui reflète la capacité d'une chaussée à supporter des charges données sans atteindre des niveaux de dégradation élevés. Cet indice permet d'évaluer les charges permises sur une chaussée donnée et de prédire sa durée de vie restante étant données ces charges.

Le modèle d'évolution de la dégradation permet de prédire le comportement de la chaussée en réponse à des charges de trafic et des agressions environnementales variables. D'après [51], les techniques les plus utilisées pour construire de tels modèles sont l'extrapolation linéaire, la régression, les techniques mécaniques-empiriques, les distributions de probabilité de variables aléatoires et les chaînes de Markov. Le modèle de dégradation peut aussi être construit à partir d'avis d'experts en cas de manque de données réelles.

La méthode de planification des maintenances fournit aux décideurs des politiques de maintenance optimales étant donné la condition courante de la chaussée et les possibles conditions futures prédites par le modèle de dégradation. La condition de la chaussée peut être décrite soit par son niveau de dégradation, soit par son âge effectif.

Afin de contribuer à l'amélioration des PMS existants et à développer des approches efficaces de gestion de l'entretien des réseaux routiers dans le cadre du projet SBaDFoRM, nous nous proposons de développer et améliorer simultanément les trois dernières composantes d'un PMS, à savoir : la méthode d'évaluation de la condition des chaussées, le modèle d'évolution de la dégradation et la méthode de planification des maintenances.

Avant de présenter l'ensemble de nos contributions, ce chapitre a pour objectif de préciser, après une brève description structurelle des chaussées, le problème de dégradation des chaussées et des pratiques de maintenance disponibles et leur effet sur l'état des chaussées. Le chapitre présente aussi une brève analyse bibliographique des modèles de dégradation ainsi que des approches d'optimisation de la maintenance existantes.

Notons que, bien qu'il existe différents modes de dégradation des chaussées comme la déformation, la fissuration, l'arrachement et le mouvement des matériaux, nous allons nous intéresser uniquement au mode de fissuration longitudinale appelé aussi fissuration de fatigue.

1.2 Présentation structurelle des chaussées

Une chaussée est une superposition de plusieurs couches de différents matériaux déposées sur un sol support [30] (Figure 1.1). La couche inférieure, appelée couche de forme, permet en plus de la circulation des engins pendant les travaux, de rendre le sol support plus homogène et de le protéger du gel. Quant au corps de la chaussée, il est constitué de deux sous-couches : la couche d'assise et la couche de surface. La couche d'assise est généralement formée d'une couche de

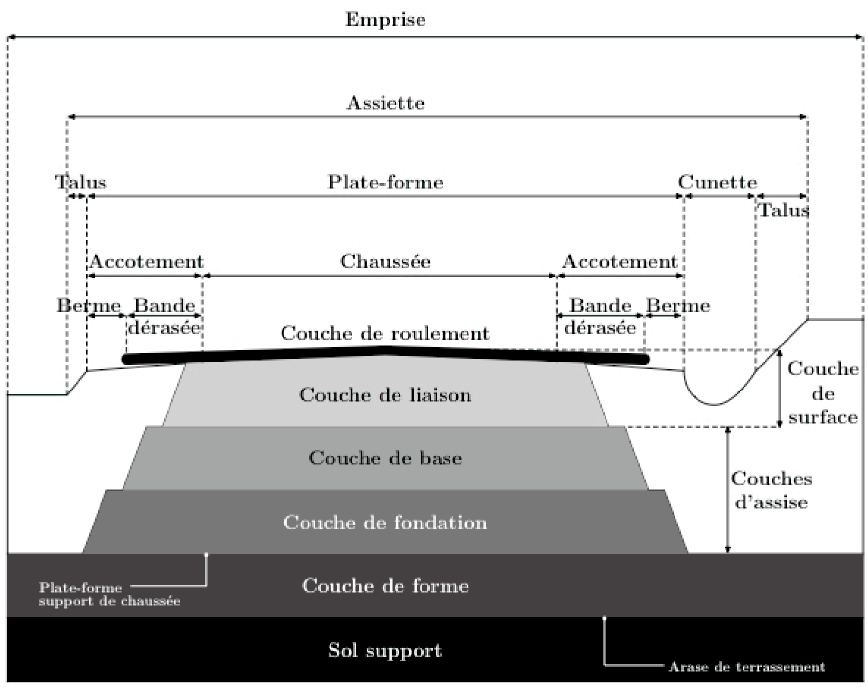

FIG. 1.1 – Coupe transversale d'une chaussées

fondation renforcée par une couche de base et qui permet à la chaussée de résister aux charges de trafic. La couche de surface est formée d'une couche de liaison liant l'assise à une couche de roulement, appelée aussi couche d'usure, qui subit directement les agressions de trafic et des variations climatiques. La couche de roulement assure la protection des couches inférieures, l'imperméabilisation de la chaussée, l'adhérence des véhicules ainsi que la sécurité et le confort des usagers. En particulier, pour assurer la sécurité des usagers, la couche de roulement doit posséder une bonne rugosité qui est la propriété anti-dérapante et qui s'apprécie par la profondeur moyenne de texture et par les coefficients de frottement longitudinal et transversal. Pour le confort des usagers, la couche de roulement doit présenter un bon uni évitant aux usagers les secousses brutales, les vibrations excessives ainsi que les bruits. Étant en contact direct avec les agents extérieurs (climat et pneumatique), la composition de la couche de roulement (béton bitumineux semi-grenu, béton bitumineux souple, béton bitumineux très mince) dépend essentiellement du type de voie considérée et des types et intensités des sollicitations.

Le mode et le niveau de dégradation observés dépendent de la composition de la chaussée et plus particulièrement de celle de la couche de roulement ainsi que des sollicitations dont elle est sujette.

1.3 Le problème de dégradation des chaussées

1.3.1 La fissuration longitudinale

Comme déjà mentionné, nous nous intéressons seulement au mode de dégradation par fissuration longitudinale. Ce choix est soutenu par le fait que les fissures, essentiellement les fissures longitudinales, sont les formes de dégradation les plus observées sur les chaussées de route. Par exemple, Meyers *et al.* [36] reportent qu'environ 90% des sections de route en Floride nécessitent des maintenances à cause des fissures longitudinales. De plus, les activités de maintenance pour cette forme de dégradation sont importantes, différentes et parfois lourdes pouvant aller jusqu'au décaissement total de la chaussée.

Les fissures longitudinales se développent parallèlement à l'axe de la chaussée. Elles apparaissent essentiellement à cause de la fatigue de la chaussée due à la répétitivité du trafic lourd. Ce phénomène est renforcée pour des structures insuffisantes vis-à-vis du trafic réel ou bien à une portance insuffisante du sol. La longueur des fissures est généralement considérée comme indicateur du volume et du degré d'évolution du phénomène de dégradation. Les fissures longitudinales peuvent par conséquent être considérées comme un bon indicateur du niveau de l'endommagement structurel d'une chaussée.

Bien que nos travaux ne cherchent pas à expliquer le processus de fissuration longitudinale, une description plus fine de ce processus est importante pour mieux le modéliser et formaliser les hypothèses de travail.

Chacune des différentes couches formant la chaussée (figure 1.1) est caractérisée par un ensemble de propriétés thermo-mécaniques liées aux matériaux qui la composent. Ces propriétés sont résumées dans des indicateurs tels que module d'élasticité, coefficient de Poisson et épaisseur. Ils permettent ensuite de caractériser l'effet des sollicitations cumulées par le trafic et les variations climatiques sur la chaussée. Avec le temps, la fatigue entraînée par le trafic et les variations thermiques causent le changement de ces caractéristiques thermo-mécaniques. Plus précisément, des flexions et des écrasements peuvent être créés sous l'effet de passages répétés des véhicules, ajoutés aux contraintes et déformations créées au sein des matériaux à cause des dilatations et contractions thermiques. Ceci conduit à l'apparition de micro-fissures de fatigue autour de points de faiblesse au niveau de la couche de base. Ces micro-fissures se propagent ensuite au travers des différentes couches de la chaussée jusqu'à atteindre la surface : c'est le phénomène de “remontée de fissures”. Les fissures se propagent ensuite en surface. La Figure 1.2 représente une coupe verticale d'une route présentant des fissures longitudinales et illustre, par les courbes en traits discontinus, leur propagation dans le temps.

La Figure 1.3 schématise les processus d'initiation d'une fissure en fond de couche et de remontée jusqu'à la surface. Le processus de fatigue est assimilé à la répétition dans le temps d'une force ponctuelle en surface d'intensité constante. Cette force se propage jusqu'au fond de couche tout en étant amortie grâce aux propriétés d'élasticité de la couche d'enrobé. Le caractère répétitif de la force tend, d'une part à faire perdre les qualités absorbantes de la couche et, d'autre part, à fragiliser les liaisons (traits pleins dans la loupe) entre les grains constituant le bitume jusqu'à leur rupture. Cette rupture entraîne l'initiation de la fissure en sous-couche.

Le processus de remontée de la fissure vers la surface est présenté sur les deux dernières dates de la Figure 1.3. Le volume fissuré représente une zone fragilisée de la structure qui permet ainsi de reproduire le processus d'initiation de micro-fissure en chaque point de discontinuité et ainsi remonter progressivement vers la surface. La vitesse de remontée d'une fissure est fonction de l'épaisseur totale de la chaussée ainsi que du niveau de fissuration sous-jacent.

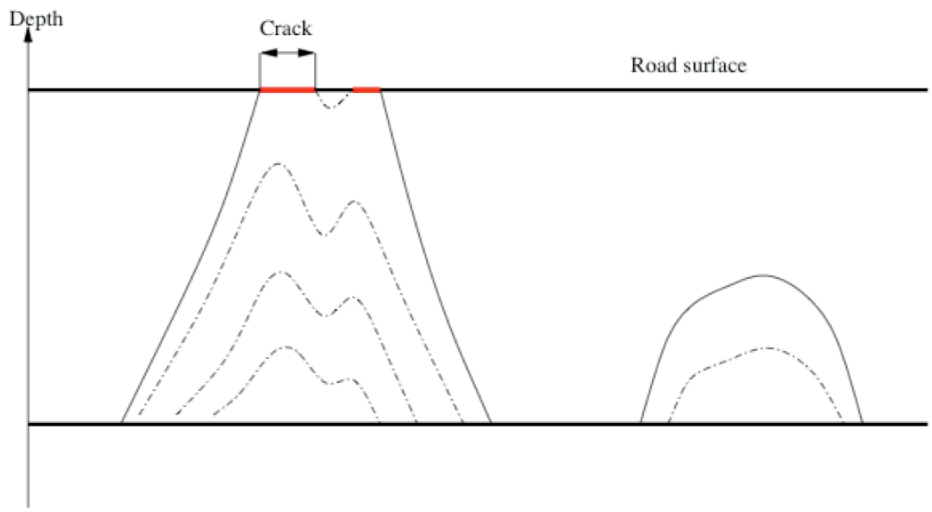

FIG. 1.2 – Illustration de l'évolution de la fissuration longitudinale - coupe verticale.

Ainsi, l'apparition de fissures au niveau de la surface et leur propagation sont précédées d'une phase d'initiation au niveau des sous-couches durant laquelle aucune dégradation ne peut être observée à l'œil nu. Nous supposons donc que le processus de fissuration longitudinale est un processus à deux étapes : initiation et propagation. Notons que pendant l'étape d'initiation, les fissures des couches inférieures, bien qu'elles ne soient pas observables, représentent un niveau de dégradation de la chaussée et un potentiel de dégradation de l'état de sa surface.

Il est à noter que les fissures qui apparaissent au niveau de la surface des chaussées peuvent être classées en deux familles selon leurs origines. Quand la fissure de surface résulte de la montée des micro-fissures initialement apparues dans la couche d'assise ou dans la base de la couche de roulement, on parle de fissures structurelles. Par contre, les fissures sont considérées superficielles quand elles ne concernent que la couche supérieure de la chaussée.

1.3.2 L'indicateur de fissuration

En France, la direction des routes a développé des méthodes d'évaluation des chaussées et d'ouvrages d'art afin de suivre l'évolution dans le temps de l'état du patrimoine routier et d'évaluer l'efficacité des stratégies adoptées. Les campagnes image qualité du réseau routier national (IQRN) pour les chaussées ont été mis

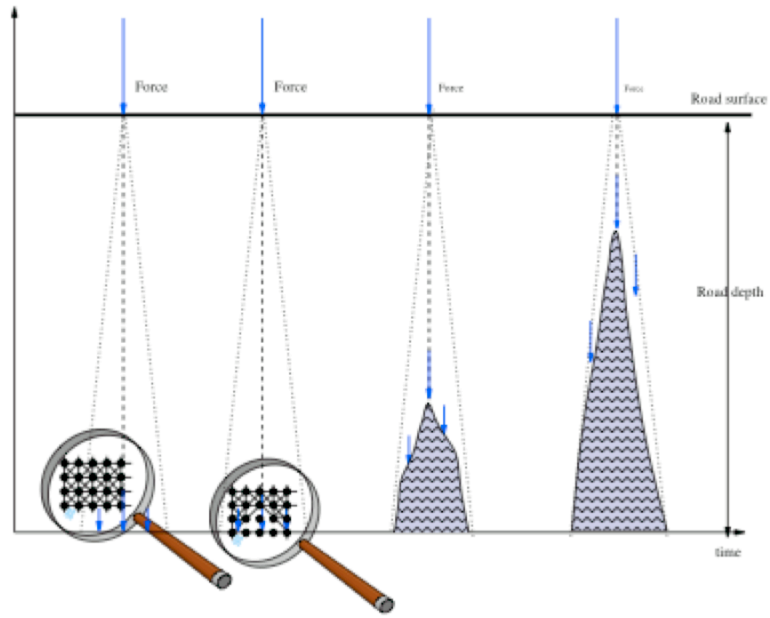

FIG. 1.3 – Schématisation du processus d'initiation et d'évolution d'une fissure longitudinale.

en place en 1992 et 1994 [30]. La démarche IQRN concerne annuellement un tiers du réseau des routes nationales et consomme environ 0.9% de la dotation de maintenance des chaussées allouée par la direction des routes. Les mesures et relevés de dégradation sont fournis pour des sections de 200 m de longueur. Ils sont ensuite analysés et synthétisés pour fournir, entre autres, un pourcentage de fissuration longitudinale (PFL) pour chaque section. Le PFL est une mesure unidimensionnelle qui représente le ratio de longueur fissurée sur la longueur totale de section comme illustré par la Figure 1.4. Ainsi, un PFL égal à zéro signifie qu'aucune fissure n'est encore apparue sur la surface de la section et un PFL égal à 1 signifie que toute la longueur est fissurée.

De plus, les relevés de PFL permettent de classer les sections selon différents niveaux de dégradation. Plus précisément, selon la gravité (significative ou grave) et l'extension ($10-50\%, > 50\%$, respectivement) des fissurations, la classification suivante est utilisée [30] :

- les chaussées présentant uniquement de la fissuration significative sont concernées par un problème de fissuration thermique ;
- les chaussées présentant uniquement de la fissuration grave sont concernées

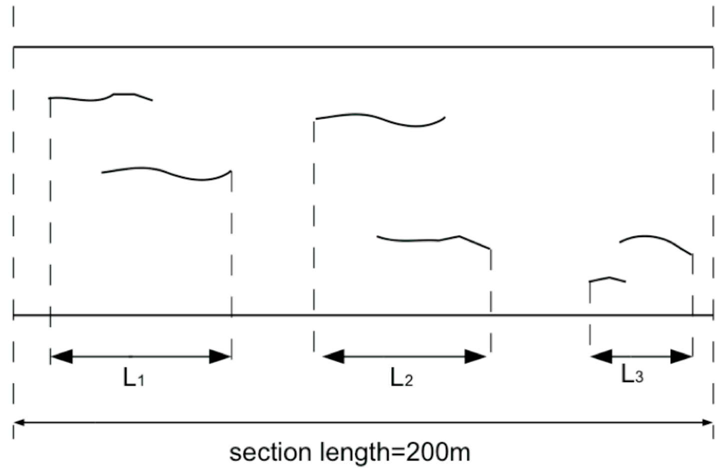

FIG. 1.4 – Pourcentage de fissuration longitudinale d'une section de longueur 200m : $\frac{L_1+L_2+L_3}{200}$

par un problème de fatigue du corps de chaussée.

Notons que cet indicateur ne permet pas de différencier les sections avec plusieurs ou peu de fissures puisqu'il ne prend pas en compte le nombre de fissures. La motivation du choix de cet indicateur est essentiellement liée aux techniques d'acquisition des mesures et donc à l'information actuellement disponible dans la base de données IQRN fournie pour ces travaux.

Bien que différentes classes de niveau de dégradation des sections puissent être distinguées selon leurs PFL, nous pensons que cet indicateur, seul, ne permet pas de rendre compte du niveau réel de dégradation aussi bien de la surface que des couches inférieures. En effet, (i) le PFL est inadéquat pour décrire la première phase de dégradation car l'absence de fissures de surface pendant cette phase ne veut pas dire que l'état de la chaussée est "aussi bon que nouveau" ; (ii) le PFL est insuffisant pour décrire la propagation des fissures dans les sous-couches pendant la deuxième phase comme il ne donne aucune information sur la distribution et densité des fissures de surface (e.g., un même niveau de PFL peut résulter d'une seule fissure marginale ou de plusieurs fissures assez développées mais superposées).

1.4 Les actions de maintenance

Cette section discute des actions de maintenance disponibles pour entretenir les chaussées souffrant de fissures longitudinales, ainsi que de leurs impacts sur la chaussée. D'une façon générale, les actions de maintenance peuvent être classées en actions correctives, entreprises suite à l'apparition de fissures, et préventives planifiées pour retarder leurs effets. Elles peuvent aussi être classées selon leurs impacts sur le système en actions parfaites, minimales et imparfaites.

Les approches classiques rencontrées dans le contexte de l'optimisation de la maintenance se concentrent essentiellement sur des actions dites de renouvellement à l'issue desquelles le système maintenu est considéré comme neuf et ses caractéristiques moyennes de dégradation se retrouvent à l'identique, ou encore sur des actions dites minimales qui remettent le système à l'état précédant la défaillance [45]. La maintenance imparfaite, remettant le système à un état entre l'état précédant la défaillance et l'état neuf, a été introduite plus tard et nécessite plus d'effort pour la modélisation de ses effets. Elle est généralement présentée dans un cadre de durées de vie [27, 17] et peu pour les approches conditionnelles où la décision est basée sur l'état du système.

Dans le contexte d'entretien des chaussées, il existe un panel diversifié d'actions de maintenance contenant des actions parfaites aussi bien qu'imparfaites. De plus, la décision est généralement précédée par une étape d'inspection des chaussées pour évaluer leurs niveau de dégradation. Les pratiques actuelles d'inspection ainsi que de maintenance sont décrites dans ce qui suit [30].

L'inspection : Les sections de routes sont inspectées au début de chaque période de décision, actuellement de trois ans, pour mesurer leurs niveaux de fissuration. Seul le PFL est actuellement fourni. On suppose que la technique non destructive visuelle utilisée pour prélever les mesures de fissuration des chaussées est parfaite dans le sens où les mesures fournies sont exactes.
Un coût unitaire fixe d'inspection est associé à chaque section inspectée.

Les actions d'entretien : Le rapport intermédiaire [30] du projet SBaDFoRM présente une classification des actions de maintenance des chaussées en quatre groupes en fonction de l'efficacité des techniques utilisées :
- *Actions ponctuelles* comme le colmatage des fissures. Ce type d'actions est effectué pour des fissures de longueurs faibles. Il permet de recouvrir ponctuellement

les fissures de surface d'une manière relativement temporaire.
- *Actions légères* consistant à mettre des couches d'enrobé (BBTM) de hauteur inférieure à 6 cm. Elles permettent aussi de recouvrir la totalité des fissures de surface d'une manière relativement temporaire mais plus efficacement que le colmatage dans le sens où le délai de réapparition des fissures est plus important.
- *Actions semi-lourdes* consistant à mettre des couches d'enrobé (BB) de hauteur supérieure à 6 cm. L'effet de ces actions quant aux fissures de surface est le même que les actions légères ou ponctuelles, mais les chaussées sont rendues plus rigides et la réapparition des fissures est encore plus retardée.
Notons que la répétition de ces actions peut empêcher la fissuration de surface. Cependant, des contraintes de dépassement d'épaisseur cumulée de la chaussée ne permettent pas de les multiplier à l'infini. Notons que les effets de recouvrement par des BB ne sont pas linéaires dans le sens où l'impact de plusieurs recouvrements d'épaisseur cumulée donnée répartis dans le temps est différent que celui d'un unique recouvrement d'une épaisseur équivalente.
- *Actions lourdes* telles que le décaissement. Ces actions reviennent à renouveler totalement ou partiellement la couche d'enrobée. Elles ne se limitent donc pas à éliminer les fissures de surface mais aussi à renouveler la totalité ou une partie de la sous-couche de la chaussée ce qui affecte les vitesses d'initiation et de propagation des fissures. Ces dernières seront fonctions du niveau de fissuration des sous-couches ainsi que du niveau d'entretien entrepris, i.e., l'épaisseur renouvelée. Notons aussi que le renouvellement partiel ou total de la chaussée sous-entend que l'épaisseur de la chaussée reste inchangée. Une telle hypothèse peut être restrictive dans le sens où le changement, i.e., augmentation ou diminution, de la hauteur des chaussées peut donner des politiques de maintenance beaucoup plus efficaces. En effet, l'épaisseur est, entre autres, un facteur très influant de la performance de la chaussée. De plus, son impact n'est pas linéaire, i.e., augmenter progressivement l'épaisseur de la chaussée jusqu'à une épaisseur donnée n'est pas équivalent à la construire initialement à cette épaisseur. Ceci peut être expliqué par le fait que le processus d'initiation des fissures est un processus relativement lent et que la vitesse d'évolution des fissures n'est pas constante mais dépend du niveau de dégradation des chaussées.

Par ailleurs, dans un contexte de maintenance conditionnelle, l'hypothèse classique considère que l'efficacité d'un entretien se mesure uniquement par son effet immédiat sur le niveau de dégradation de la chaussée. Cependant, d'après la description donnée ci-dessus des impacts des différentes actions de maintenance, nous pensons que la modélisation de l'effet de la maintenance sur la chaussée doit aussi pouvoir prendre en compte son effet sur l'évolution future du processus de dégradation. Cette considération, beaucoup plus réaliste, peut aussi être motivée par le fait que dans le cas de la maintenance des chaussées, limiter l'effet des actions à un effet immédiat sur son niveau de dégradation revient à considérer le même effet pour toutes les actions car elles assurent toutes le recouvrement des fissures et donc l'initialisation du PFL à 0.

Notons aussi que cette dernière remarque soutient le fait que seul le PFL comme indicateur de fissuration est insuffisant pour formaliser le problème de maintenance des chaussées.

Des coûts fixes et variables sont associés aux différentes actions de maintenance. La caractérisation de ces coûts n'est pas simple car ils doivent comprendre l'ensemble des coûts directs comme les coûts de matière et de main d'oeuvre ainsi que des coûts indirects comme les coûts usagers résultant de la gêne causée aux usagers des routes. Ils peuvent aussi contenir des coûts de pénalité pour prévenir la dégradation excessive des chaussées ainsi que de forcer la maintenance, préventive ou corrective (au lieu de choisir de ne rien faire pour minimiser les coûts de maintenance). L'hypothèse classique pour les coûts de maintenance est de considérer des fonctions de coût qui sont croissantes en fonction du type d'entretien, mais pas forcément linéaires.

Finalement, il sera supposé que la mise en place des différentes actions de maintenance est immédiate.

1.5 L'analyse bibliographique

Après avoir mis en évidence les points importants du problème de gestion des routes, nous proposons dans cette section de discuter des modèles existants. Du fait de la dépendance de la qualité des politiques fournies par une approche d'optimisation de la maintenance du modèle stochastique de dégradation utilisé, nous positionnerons cette discussion par rapport aux modèles de dégradation et aux modèles d'optimisation de la maintenance.

1.5.1 Modèles de dégradation stochastique

Nous ne cherchons pas dans cette section à cataloguer l'ensemble des modèles de dégradation fournis par la littérature mais nous concentrons notre analyse sur les modèles qui pourront répondre à notre problème de modélisation du processus de fissuration ainsi qu'à l'intégration des impacts de différentes actions de maintenance.

On distingue deux classes de modèles de dégradation pour l'optimisation de la maintenance : les modèles basés sur des variables aléatoires et ceux sur des processus stochastiques [43].

La première classe contient essentiellement des modèles de type régression ([33]), ou encore de type chaîne de Markov ([35, 11, 8, 32]). Ces derniers sont les modèles utilisés dans la majorité des approches de gestion de l'entretien des chaussées [68]. D'une façon générale, à cause de la dépendance de leur efficacité des données statistiques disponibles, la majeure limite de ces modèles est qu'ils ne permettent pas de prendre en compte l'incertitude liée au temps comme c'est le cas des processus stochastiques, ce qui est un aspect important spécialement dans un contexte de fatigue et d'agressions externes [53]. Particulièrement, la qualité d'un modèle de type chaîne de Markov dépend de la construction de matrice de probabilité de transition des états qui est généralement obtenue par le biais d'analyses statistiques de données ou par l'avis d'expert dans le cas où peu de données de dégradation est disponible. Wu et Ni [67] ont montré que pour le modèle de la chaîne de Markov stationnaire avec une matrice de probabilité de transition estimée à l'aide de données statistiques ne permet pas de prédire d'une manière efficace le processus d'accroissement d'une fissure.

On trouve aussi dans cette classe de modèles de dégradation des modèles paramétriques établis initialement dans des contextes déterministes et intégrant l'incertitude soit directement en considérant une partie stochastique dans la variable de dégradation ou bien en considérant les paramètres de la loi déterministe comme des variables aléatoires. De nombreux travaux développés dans ce cadre utilisent l'équation de Paris-Erdogan [44] pour la modélisation de la propagation d'une fissure. L'avantage d'une telle approche est qu'elle repose sur des considérations physiques de dégradation -l'équation de Paris-Erdogan considère que l'accroissement d'une fissure est une fonction de l'intensité du stress appliqué à la structure. Cependant, l'estimation des paramètres demande généralement une grande quantité de données expérimentales et dépend étroitement du type et de la composition de la structure.

Dans la classe des modèles basés sur des processus stochastiques, on trouve essentiellement les processus de Lévy comme le processus de Poisson composé ([55]), le processus brownien ([4, 5, 13, 39]) ou encore le processus gamma ([1, 22, 57, 60, 61, 50]). Une comparaison entre les processus brownien et gamma pour la modélisation de phénomènes de dégradation cumulative peut être trouvée dans [42].

L'intérêt porté aux processus de Lévy peut être justifié par la nature physique du processus de détérioration. En effet, l'évolution d'un système se dégradant suite à l'accumulation, durant sa durée de vie, de chocs à des occurrences et à intensités aléatoires peut parfaitement être modélisée par un processus de Lévy [42]. Cependant cette modélisation suppose la stationnarité du processus de dégradation, i.e., l'accroissement de la dégradation durant une durée de temps déterminée est indépendante de l'instant initial et donc de l'âge du système.

Une attention particulière a été portée pour l'application en fiabilité et maintenance des processus gamma qui ont montré une flexibilité pour la modélisation de dégradation essentiellement de structures en génie civil sous incertitude. Le lecteur pourra se référer à l'état de l'art proposé par van Noortwijk [59].

Bien que l'hypothèse d'accroissement de la dégradation linéaire dans le temps soit discutée par les experts, elle a le mérite d'être économe en termes de données pour l'estimation des paramètres (même si un grand nombre de données tend à en améliorer la qualité) tout en conservant une relative bonne qualité de prédiction. Cette propriété est particulièrement adaptée dans le cadre de systèmes pour lesquels les intervalles de mesure sont grands et tendent d'une part à réduire le nombre d'observations, d'autre part à lisser les impacts des phénomènes d'accélération des dégradations dans le processus de décision.

L'étude de l'application du processus gamma a aussi été étendu par [58] au processus gamma généralisé dont l'évolution moyenne n'est plus constante mais dépend bien de l'âge du système. Cependant, l'utilisation du gamma généralisé dans un contexte d'optimisation de la maintenance entraîne une inextricabilité mathématique du problème [58].

Par ailleurs, les processus de Lévy en général et les processus gamma en particulier ne permettent pas de modéliser et distinguer les deux phases de dégradation qui caractérisent le processus de fissuration étudié. L'utilisation d'un processus gamma pour modéliser le développement d'une fissure en conjonction avec un processus de Poisson pour son initiation a été proposée en premier lieu par van Noortwijk *et al.* [61] pour modéliser le développement de trous d'érosion dans la digue de protection Eastern-Scheldt contre les inondations aux Pays-Bas. van Noortwijk and Klatter [60] ont ensuite étendu ce travail pour pouvoir considérer des phénomènes d'érosion non constants en moyenne.

Nicolai *et al.* [40] présentent un modèle de dégradation à deux phases pour des structures en acier sujettes au phénomène de corrosion appelé *two-stage hit-and-grow* (TSHG). Dans ce modèle, la surface de la structure est divisée en une grille et l'apparition de points de corrosion sur chaque case de la grille pendant la phase d'initiation est modélisée par un processus de Poisson non-homogène. La phase de propagation des points de corrosion est modélisée en supposant qu'au bout d'une durée constante, toutes les cases voisines à une case corrodée le seront aussi. Cependant, les auteurs affirment que ce modèle ne peut être appliqué (analytiquement) qu'à des problèmes de petite taille et qu'il n'est plus efficace lorsque les points commencent à se superposer. Ils proposent alors d'utiliser un modèle de simulation qui devient aussi limité en termes de mémoire lorsque le nombre de points corrodés augmente.

1.5.2 Approches d'optimisation de la maintenance

Plusieurs approches d'optimisation des politiques de la maintenance sont proposées dans un contexte général de maintenance. Les références [18] et [63] présentent des revues récentes de ces approches. Ces dernières sont classées en deux catégories principales selon la nature des variables de décision. Dans chaque classe, on distingue les modèles de maintenance minimale des modèles de maintenance imparfaite.

Les politiques temporelles où la décision est basée sur l'âge du système ou sa durée de vie considérée comme variable aléatoire ([62, 23] et [69]). L'incertitude dans le temps de l'occurrence des défaillances est représentée par la fonction de taux de défaillance. Dans ces approches, le système est systématiquement remplacé à la défaillance, au bout d'un âge déterminé [64] ou bien remplacé périodiquement pour prévenir les défaillances [52, 2].

Dans le cadre des approches temporelles, l'impact de la maintenance imparfaite est modélisé au niveau de vieillissement du système. On trouve parmi les plus connus le modèle de Brown-Proschan [7] dans lequel la maintenance est supposée parfaite avec une probabilité p et minimale avec une probabilité $(1-p)$, ainsi que le modèle d'âge virtuel proposé par Kijima [27], où l'effet de la maintenance est de rajeunir le système. Plus récemment, Doyen *et al.* [17] proposent un modèle à réduction arithmétique d'âge et un modèle à réduction arithmétique d'intensité.

Ainsi, l'impact d'une maintenance est directement introduit au niveau du taux de défaillance du système. Les limites de ces approches se situent essentiellement à deux niveaux. Tout d'abord, il est extrêmement délicat de bien modéliser l'impact d'une maintenance sur le taux de défaillance au vu de son non observabilité. D'autre part, ces approches supposent que la forme de la loi de durée de vie reste inchangée après maintenance et identique à celle du système neuf, i.e., que le système vieillit selon le même mode. On ne peut pas alors prendre en compte des effets d'accélération du vieillissement.

Bien que non orientés vers les applications en maintenance, il est intéressant dans ce contexte de citer les modèles de durée de vie accélérée ou encore les modèles à hasard proportionnels [37] permettant de modéliser les variations des lois associées par le biais de facteurs d'accélération ou de l'intégration de différentes covariables.

Les politiques conditionnelles où la décision est basée sur une connaissance de l'état actuel du système. Cet état est généralement représenté par un indicateur observable de dégradation ([38, 6, 41, 26, 25] et [39]). La maintenance n'est pas systématique dans de telles politiques et peut dépendre dans certains cas d'une politique d'inspection. Les approches conditionnelles sont plus récentes que les approches temporelles et généralement plus efficaces car elles permettent une meilleure intégration dynamique (mise à jour) de l'information disponible sur la condition du système. Cependant, elles demandent un effort beaucoup plus important en termes de modélisation, de collecte et d'analyse des données de dégradation.

Les travaux développés dans le contexte de maintenance conditionnelle pour modéliser l'impact des entretiens sont plus rares.

Les premières approches développées [9, 10] sont à mettre en parallèle avec les premiers modèles de réduction de l'âge dans le sens où une maintenance dite imparfaite permet de diminuer le niveau de dégradation à une valeur pouvant être aléatoire comprise entre le niveau observé avant la maintenance et le niveau parfait sans changer la loi d'évolution de la dégradation. Les travaux de [10] présentent des politiques basées sur une caractérisation a priori de la structure décisionnelle (seuils de décision). De telles approches, bien qu'elles soient faciles à implémenter en pratique, n'assurent pas l'optimalité des politiques fournies. De plus, la loi stationnaire d'évolution du système maintenu est généralement difficile à évaluer et devient encore plus difficile avec plusieurs paramètres de dégradation.

Dans [42], l'auteur discute de l'utilisation d'approches type programmation dynamique pour la résolution du problème d'optimisation de la maintenance lorsque celle-ci permet de modifier les paramètres du processus gamma, processus modélisant l'évolution de la dégradation. Il souligne les difficultés numériques dues à la taille du problème.

Notons que l'étude des maintenances imparfaites dans un contexte de stratégies conditionnelles est une préoccupation d'actualité de la communauté scientifique, confirmant ainsi l'intérêt industriel sur ce problème et le manque d'outils et de modèles associés.

Dans le cas de la maintenance des routes, l'état des chaussées est régulièrement inspecté pour collecter des données de dégradation. Ces données peuvent être exploitées pour la prédiction des comportements des chaussées en réponse aux agressions extérieures, ainsi que pour décider en début de chaque période de la meilleure action qui minimise les coûts totaux, immédiats et futurs espérés, de maintenance. Ainsi, dans une approche conditionnelle, le problème peut être formulé sous forme de programme dynamique où les paramètres de décision sont représentés par l'état de la chaussée. Rappelons que dans une telle formulation, la décision de maintenance pour le présent est basée non seulement sur l'état présent observé du système, mais aussi sur tous les états futurs possibles. Dans le cas où l'état du système vérifie de plus la propriété markovienne, i.e., absence de mémoire, le programme dynamique peut être réduit à des processus de décision markovien (MDP). Cette formulation suppose par ailleurs que l'état du système est parfaitement observable. Par conséquent, la considération de nouveaux paramètres d'état doit vérifier cette condition. Dans le cas contraire, les modèles MDP peuvent être étendus en processus de décision markovien partiellement observable (POMDP) qui sont plus compliqués à modéliser et à résoudre numériquement.

1.6 Conclusions

Dans les approches d'aide à la décision dans l'incertain, une étape supplémentaire de modélisation des variables stochastiques et de leur évolution est nécessaire. En effet, l'efficacité des approches dynamiques d'aide à la décision dans l'incertain repose énormément sur la qualité du modèle de prévision utilisé pour modéliser l'évolution du processus stochastique dans le sens où les décisions optimales fournies pour le présent par de telles approches prennent en compte les possibles réalisations de l'incertain dans le futur à travers un modèle de prédiction.

A l'issu de la description et de l'analyse du processus de fissuration longitudinale d'une section de chaussée donnée, nous avons défini la fissuration comme un processus cumulatif à deux phases à savoir l'initiation pendant laquelle seules les sous-couches souffrent de dégradation et la propagation marquée par l'apparition de la première fissure en surface et pendant laquelle la dégradation évolue en sous-couche aussi bien qu'en surface.

Les bases de données actuellement disponibles pour le projet SBaDFoRM fournissent uniquement le PFL comme indicateur de niveau de dégradation d'une section de chaussée donnée. Bien que cet indicateur présente l'avantage d'être facilement et parfaitement observable, il est insuffisant pour une bonne description du processus de fissuration. Plus spécifiquement :

- En phase d'initiation comme de propagation, le PFL ne reflète pas le niveau de dégradation sous-jacent des couches inférieures de la chaussée, surtout en phase d'initiation en absence de fissures de surface.
- L'analyse du processus de fissuration montre que la vitesse d'évolution des fissures n'est pas constante et dépend du niveau de fissuration courant de la chaussée. Considérer le PFL comme seul paramètre d'état de la chaussée peut ainsi être restrictif et affecter la qualité des politiques de maintenance dégagées.
- Le PFL ne permet pas de modéliser les différents effets des actions de maintenance imparfaites car toutes les actions remettent le niveau du PFL à zéro.

Les pratiques actuelles en maintenance des chaussées permettent soit d'augmenter l'épaisseur de la chaussée par l'ajout de nouvelles couches d'enrobée tout en respectant une contrainte d'épaisseur maximale soit de renouveler partiellement ou totalement la chaussée en la maintenant à une épaisseur fixe. Cependant, plusieurs autres options de maintenance peuvent être efficaces comme l'option de changer (réduire ou augmenter) l'épaisseur de la chaussée en la renouvelant partiellement ou totalement. Une telle considération est motivée par le fait que l'impact d'une action de maintenance est un compromis entre épaisseur totale et épaisseur renouvelée et dépend du niveau de dégradation avant maintenance.

L'étude bibliographique des approches d'optimisation conditionnelle de la maintenance permet de conclure qu'à l'exception de [65] où un modèle de fiabilité conditionnelle est utilisé pour la maintenance conditionnelle, toutes ces approches sont basées sur des modèles de dégradation temporels. Un processus où l'évolution de la dégradation sur un intervalle de temps dépend aussi du niveau courant de dégradation nous semble plus adapté aux approches d'optimisation conditionnelles.

2 Contributions

2.1 Objectifs des travaux

Dans le chapitre précédant, l'identification de certaines limites des modèles de dégradation existants ainsi que des approches d'optimisation de la maintenance a permis de définir des opportunités pour contribuer au domaine de l'optimisation de la maintenance de systèmes sujets à des phénomènes de dégradation stochastiques à deux phases. Cette section présente les objectifs des travaux de thèse au vu des limites et besoins identifiés précédemment. Ainsi, nous proposons dans la suite de ce chapitre dans un premier temps de décrire brièvement chacune de nos contributions en situant l'objectif à atteindre. Dans un second temps, une description plus approfondie des modèles et contributions est fournie résumant ainsi les développements présentés dans la dernière partie de ce document sous la forme de publications.

Améliorer l'évaluation de l'état de la chaussée : extension de la définition de l'indicateur de dégradation Outre des données climatiques et d'environnement, les seules données actuellement disponibles dans l'IQRN sont les dates d'inspection, les niveaux de fissuration observés par section et la caractérisation d'éventuels entretiens. Elles ne permettent pas de bien rendre compte du niveau de fissuration des couches inférieures, propriété importante pour la construction d'un modèle prédictif de fissuration efficace. De plus, l'indicateur choisi doit permettre de différencier l'impact des entretiens sur l'évolution future de la fissuration et la différentiation des deux phases de dégradation. La seule considération du PFL est clairement limitative en ce sens.

Nous proposons d'enrichir l'information de dégradation en intégrant le *taux de croissance de dégradation (TCD)* pour chacune des sections. Physiquement, ce TCD rejoint l'idée de potentiel de fissuration en phase d'initiation, et d'une vitesse instantanée de dégradation en phase de propagation. Ce TCD doit être estimé à partir des données disponibles dans la base IQRN.

Définir les paramètres de décision : observabilité et non observabilité des paramètres Comme souligné précédemment, l'estimation de la variable TCD doit reposer sur les données fournies par l'IQRN. Nous proposons de traiter le problème sous deux volets. Le premier volet considère un problème approché pour lequel on assimile le TCD à la vitesse moyenne du PFL observée sur les deux dernières inspections. Ceci nous permet alors de formuler le problème sous la forme d'un processus de décision markovien (MDP). La deuxième approche consiste à rendre le problème partiellement observable en intégrant une loi de probabilité sur les observations. On rejoint ainsi une modélisation de type processus de décision markovien partiellement observable (POMDP).

Modéliser le processus de fissuration longitudinale : un processus stochastique bivarié Le PFL peut être assimilé à un processus stochastique de sauts croissant variant de 0 à 100% avec une vitesse moyenne non constante, dépendante de l'état courant. Ceci souligne les dépendances mutuelles entre PFL et TCD.

Nous proposons de construire un processus bivarié pour lequel les lois d'évolution des incréments sur un intervalle de temps donné sont fonctions des dernières valeurs observées du processus. La loi d'évolution des états est ainsi *conditionnelle à l'état courant*. Cette construction impose une extension des résultats classiques des processus de Lévy.

Modéliser l'efficacité de la maintenance : prise en compte du niveau de dégradation avant maintenance L'impact d'un entretien se définit en fonction du gain immédiat en termes de dégradation ainsi que de l'évolution future du système, ceci induisant la considération de nouvelles lois de dégradation après chaque nouvelle maintenance. Ces lois d'évolution doivent ainsi dépendre de la nature même de l'action entreprise et aussi de l'état du système avant maintenance.

Nous proposons de définir les fonctions *efficacité de maintenance* de deux manières. La première considère :

- Un effet déterministe sur l'état de la section immédiatement après la maintenance, fonction de l'état juste avant la maintenance et l'action sélectionnée ;

– l'introduction de facteurs d'accélération dans les lois d'évolution de la fissuration. Pour des raisons de facilité de mise en œuvre et l'obtention de la propriété markovienne des processus de dégradation, nous proposons de ne considérer que les effets de la dernière action de maintenance dans les lois d'évolution.

La seconde définition est liée à la décomposition d'une action d'entretien de chaussée en décaissement et chargement, cette hypothèse permettant directement de retrouver l'hypothèse markovienne sans l'introduction de facteurs d'accélération. À titre de discussion générale, un décaissement peut être vu comme une action de traitement de symptôme alors qu'un chargement est une action de protection ou de prévention de ce même symptôme.

Prendre en compte les améliorations des systèmes : évolution des caractéristiques de dégradation L'épaisseur totale de la chaussée est un des facteurs de rigidité importants pour la caractérisation du processus de fissuration. Ainsi, faire évoluer cette épaisseur entraînant des changements de modes de dégradation est une option intéressante tant au niveau de la conception et le dimensionnement des chaussées que dans leur phase d'exploitation. Classiquement peu abordés dans la littérature, ces problèmes sont apparentés à la définition de nouveaux problèmes de maintenance (ainsi une optimisation du problème de maintenance avec des paramètres actualisés est alors reconduite indépendamment de l'évolution antérieure) et rarement intégrés dans un unique modèle décisionnel.

La modélisation des effets des opérations de décaissement et de chargement discutée au paragraphe précédent offre un réel intérêt sur ce plan. La possibilité d'augmenter l'épaisseur de la chaussée permet d'améliorer les performances globales du système initial et, dans un contexte plus général que la gestion des routes, cette option est assimilable à un remplacement par un système de technologie plus avancée, ou alors une modification technologique du système existant. Cependant, elle entraîne une complexification au niveau décisionnel notamment liée à la non conservation de la relation croissante coût/efficacité. En définissant de nouvelles relations d'ordre, nous avons montré comment peuvent se généraliser les propriétés classiques de limite de contrôle à notre problème.

Résoudre les modèles d'optimisation : des propriétés structurelles pour des procédures de résolution Une première conséquence de l'extension du MDP au cas partiellement observable, i.e., POMDP, est la croissance de la taille du problème (passage d'un espace d'états discret à un espace de croyance continu).

La résolution exacte du problème étant alors impossible, nous cherchons à développer une heuristique qui prend avantage de propriétés spécifiques au problème. Pour ce faire, les conditions de transitions et de structures de coûts sous lesquelles les politiques optimales du problème complètement observable admettent des propriétés structurelles comme l'existence d'états de contrôle, se doivent d'être spécifiées. Ces propriétés doivent alors permettre d'accélérer l'algorithme classique de résolution du MDP.

Bien que ces propriétés ne soient pas nécessairement généralisables au problème partiellement observable, elles peuvent être utilisées pour définir une heuristique de résolution efficace et qui réduit considérablement le temps de calcul en comparaison de la méthode classique.

Notons que, d'un point de vue pratique, les propriétés structurelles sont utiles au gestionnaire dans le sens où elles dessinent une organisation de la décision proche des pratiques courantes.

Modéliser le problème d'optimisation de l'inspection et la maintenance du réseau : un modèle de programmation stochastique avec incertitude endogène Dans le cas de gestion d'un réseau de sections de routes, l'objectif est d'allouer les budgets de maintenance entre les différentes sections d'une façon optimale. Les problèmes d'optimisation globale de l'inspection et de la maintenance doivent alors être considérés. Cette combinaison des deux décisions entraîne une complexité au niveau du processus de décision dans le sens où l'incertitude (exogène sans optimisation de l'inspection) devient à caractère endogène.

Nous nous proposons aussi, dans une perspective d'amélioration de la qualité de services ainsi que de développement durable, de prendre en compte une contrainte qui assure un niveau acceptable des conditions de trafic lors des maintenances en plus de la contrainte budgétaire et de connectivité de tous les points du réseau routier. La modélisation d'un tel problème et l'expression du critère d'optimisation deviennent extrêmement complexes. Nous avons proposé, à titre de travail préliminaire, une formulation du problème sous la forme d'un programme mathématique multi-étapes.

2.2 Le modèle stochastique de dégradation à deux phases

Nous considérons une section de route dont le niveau de dégradation est décrit par son PFL et son TCD. Rappelons que le PFL représente le ratio de la longueur fissurée sur la longueur totale de la section, alors que le TCD mesure la dégradation sous-jacente et représente un potentiel de fissuration en phase d'initiation et d'une façon équivalente une vitesse instantanée de dégradation en phase de propagation.

Soit $\{\rho_t, t \geq 0\}$ et $\{\theta_t, t \geq 0\}$ les processus stochastiques qui représentent le PFL et le TCD, respectivement. Pour une section donnée, le PFL ρ est borné dans $[0, 1]$. Parce que le processus de dégradation des chaussées est un processus lent et cumulatif et qu'une section de route ne se fissure pas complètement d'une façon instantanée, nous supposons aussi que le TCD θ est borné dans $[\theta_0, \theta_{max}]$, θ_0 représentant le TCD d'une section nouvelle.

La phase d'initiation : Le PFL étant nul pendant la phase d'initiation, nous considérons le processus d'arrivée de la première fissure en surface pour la dégradation observable que nous modélisons par le processus de comptage de Poisson noté N. Le potentiel de fissuration étant non-décroissant pendant la période d'initiation (absence de relâchement par fissuration longitudinale), nous le modélisons par un processus gamma stationnaire.

Cependant, bien que nous supposons que le processus N est homogène dans le temps, nous estimons que la date d'apparition de la première fissure de surface dépend du niveau de dégradation des sous-couches, i.e., du niveau du TCD. Ainsi, pour un intervalle de temps de longueur τ, nous considérons une fonction du taux d'occurrence du processus Poisson qui dépend du niveau de TCD θ au début de cette période comme explicité par la fonction de densité du processus Poisson conditionnel suivante :

$$Pr\{N_{t+\tau} - N_t = n | \theta_t\} = \frac{\Lambda(\tau, \theta_t)^n}{n!} \exp\left(-\Lambda(\tau, \theta_t)\right), \tag{2.1}$$

où $\Lambda(\tau, \theta)$ est la fonction du taux cumulé d'occurrence du processus N.

La phase de propagation : Les processus de choc auxquels nous assimilons le processus de fissuration longitudinale sont classiquement modélisés par un processus gamma [1] qui possède les avantages d'être un processus de saut et non-décroissant. Cependant, le processus gamma est généralement supposé stationnaire dans le sens où sa loi d'évolution dépend uniquement de la longueur de l'intervalle de temps considéré. Cette hypothèse qui est loin d'être réaliste pour un grand nombre d'applications est relâchée par le processus gamma généralisé dont l'évolution dépend bien de l'âge du système. Cependant, son utilisation dans un contexte d'optimisation de la maintenance peut entraîner une inextricabilité du problème [58].

Dans ces travaux, nous proposons de relâcher l'hypothèse de stationnarité du processus gamma en état au lieu de l'âge. Plus spécifiquement, nous proposons de faire dépendre la loi d'évolution du processus gamma du niveau de dégradation au début de l'intervalle de temps considéré. Rappelons que le niveau de dégradation est donné par le PFL et TCD (ρ, θ). Nous appelons le processus résultant un processus gamma conditionnel (state-dependent gamma (SDG)). Ainsi, un processus gamma conditionnel de paramètre d'échelle β et de fonction de forme cumulative $\gamma(\tau, \rho_t, \theta_t)$ avec (ρ_t, θ_t) représentant le niveau du PFL et du TCD à l'instant t, respectivement, est le processus stochastique à incréments indépendants et ayant pour fonction de densité suivante :

$$g(x; \gamma(\tau, \rho_t, \theta_t), \beta) = \frac{x^{\gamma(\tau,\rho_t,\theta_t)-1} \, e^{-x/\beta}}{\beta^{\gamma(\tau,\rho_t,\theta_t)} \, \Gamma\left(\gamma(\tau, \rho_t, \theta_t)\right)}, \quad x > 0 \tag{2.2}$$

Notons que l'évolution du processus du PFL ne dépend pas seulement de son niveau courant mais aussi du niveau de la vitesse de dégradation. Cette dépendance du niveau du TCD permet de prendre en compte l'effet de la dégradation sous-jacente sur la dégradation observable, de différencier des comportements individuels d'un comportement de référence moyen et finalement de rendre au processus gamma conditionnel la propriété d'infini divisibilité perdue à cause de sa dépendance de lui même seulement, i.e., sa loi d'évolution dépend de son niveau courant (voir le Chapitre 4 pour les détails).

Quant au TCD pour lequel les incréments peuvent prendre des valeurs positives aussi bien que négatives, nous proposons de le modéliser par un processus gamma bilatéral (BG). Ce processus a été initialement introduit dans le domaine de la finance par Küchler et Tappe [28, 29] pour modéliser l'évaluation des options (option pricing) et, à notre connaissance, notre travail reste la première application de ce processus stochastique dans un contexte de maintenance. Le choix du BG est essentiellement motivé par sa propriété d'être la différence de deux processus gamma indépendants, un premier pour les variations positives et un deuxième pour ses variations négatives.

Nous proposons dans ce travail d'utiliser cette propriété (i.e., un processus BG est la différence de deux processus gamma indépendants) pour construire la loi conditionnelle du processus $\{\theta_t, t \geq 0\}$ en définissant son incrément $\Delta\theta(t, t+s)$ comme la différence de deux processus de fissuration de vitesses moyennes respectivement fonctions de ρ_t et ρ_{t+s} (on rappelle que le processus de fissuration est modélisé par un processus gamma). Cette construction permet ainsi de réduire le nombre de paramètres à estimer dans le modèle. Soulignons ici que nous n'avons pas, dans un soucis de simplicité, discuté des aspects troncatures et normalisation des lois de dégradation. Le lecteur pourra se référer au Chapitre 4 pour la construction complète de ces lois.

La dépendance entre l'évolution des deux processus de dégradation sous-jacente et observable est décrite par ce qui suit :

$$h(x, y; \rho, \theta, m) \quad = \quad g(x; \rho, \theta, m)\ f(y; \rho, x, m) \tag{2.3}$$

où m désigne le type de la dernière maintenance, $h(x, y; \rho, \theta, m)$ est la fonction densité jointe du processus bivarié $\{(\rho_t, \theta_t); t \geq 0\}$ conditionnelle à l'état courant (ρ, θ, m) et $g(x; \rho, \theta, m)$ et $f(y; \rho, x, m)$ sont les lois conditionnelles des processus $\{\rho_t, t \geq 0\}$ et $\{\theta_t, t \geq 0\}$, respectivement.

Notons que, d'après la relation (2.3), l'incrément du PFL dépend, pour un type de dernière maintenance m, du niveau courant en PFL et TCD ensemble alors que l'incrément du TCD ne dépend que du niveau de PFL et son incrément x (on suppose contenue l'information fournie par θ_t dans l'incrément observé de fissuration). Ceci est illustré par la Figure 2.1 : le processus ρ noté G commence à zéro à l'instant t_0 et passe immédiatement à G_1 suivant θ_0 à l'instant t_1 alors que le TCD passe à une nouvelle valeur θ_1, etc.

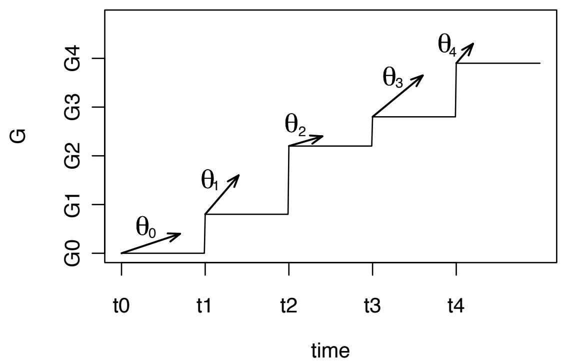

FIG. 2.1 – Évolution du processus de fissuration en fonction du TCD.

Comme déjà mentionné dans la Section 2.1, nous considérons que la section de route change de processus de dégradation après chaque maintenance. Ainsi, nous introduisons différents facteurs d'accélération dans les fonctions de taux d'occurrence du processus Poisson et de forme du SDG. Ces facteurs d'accélération sont fonctions de l'épaisseur renouvelée dans le cas d'actions de type 1 et des épaisseurs décaissées et rajoutées dans le cas d'actions de type 2. Pour assurer la propriété markovienne à l'état de la section (lorsque l'action "Ne rien faire" est choisie), nous supposons que la dépendance aux épaisseurs renouvelées se limite à la dernière intervention. On introduit pour cela à la définition de l'état le paramètre m dénotant le type de la dernière maintenance. Le Tableau 2.1 résume les processus utilisés pour les deux phases.

Les constructions des fonctions de taux d'occurrence du processus Poisson et des fonctions de forme des processus SDG et BG sont détaillées dans le Chapitre 4. Le Chapitre 5 présente aussi une discussion centrée essentiellement sur le choix de la fonction forme du processus SDG en justifiant et comparant une fonction linéaire et une fonction de forme gaussienne.

Finalement, le développement des outils statistiques pour l'estimation des paramètres des différents processus stochastiques présentés n'est pas intégré dans ce rapport. Cependant, il fait l'objet d'une communication soumise à la conférence *Transport Research Arena 2012 (TRA)*. L'estimation de ces paramètres est basée sur une approche type Maximum de Vraisemblance à données censurées à droite

TAB. 2.1 – Modèle de dégradation à deux phase pour une section de route

	Phase d'initiation	**Phase de propagation**
Description	Absence de fissures de surface : ρ=0 Potentiel de fissuration croissant	Fissures observables : $\rho > 0$ Détérioration sous-jacente
Variables clés	Arrivée de la 1^{ere} fissure de surface Potentiel de fissuration : θ	PFL : ρ TCD : θ
Processes	Processus de Poisson non homogène (NHPP) Processus gamma conditionnel (GP)	- State-dependent gamma process (SDGP) - Processus gamma bilatéral (BGP)

et à gauche lors de la phase d'initiation et avec la construction de classes définies en fonction des valeurs courantes (ρ, θ) pour la phase de propagation.

2.3 L'optimisation de la maintenance d'une section de chaussée

L'objectif est de fournir des politiques conditionnelles optimales pour la maintenance d'une section de route qui minimisent le coût total escompté de maintenance sur un horizon infini. Avec l'information fournie au début de chaque période de décision par l'inspection sur l'état de la section, le problème d'optimisation peut être formulé sous forme d'un programme linéaire où les paramètres de décision sont les paramètres d'état de la section. De plus, comme l'état de la section vérifie la propriété markovienne, le programme linéaire est alors équivalent à un processus de décision markovien (le processus de décision sur l'horizon fini ou infini est équivalent à celui sur une seule période, i.e., une seule transition à partir de l'état courant) (se référer à [48] pour les preuves).

Notons à ce niveau que nous supposons que pour un état noté $s = (\rho, \theta, m)$, seul le niveau de dégradation (ρ, θ) sert à évaluer la fonction gain du MDP et que le type de la dernière maintenance m intervient, avec le niveau de dégradation, dans la définition du processus de transition.

Bien que les processus de décision markoviens constituent un outil efficace pour de tels problèmes de décision dynamique, ils supposent que l'état est parfaitement observable, ce qui n'est pas le cas du TCD. Nous considérons dans un premier modèle que l'état de la section est parfaitement observable et résolvons le modèle MDP. Nous relaxons ensuite cette approximation en étendant la formulation au processus de décision markovien partiellement observable.

2.3.1 Le problème observable

Pour contourner la non-observabilité du TCD, nous approximons le TCD par la vitesse moyenne de fissuration en utilisant des observations successives du PFL. Ainsi, le problème peut être formulé comme un MDP [48]. La matrice de transition du MDP est donnée par le modèle stochastique de dégradation présenté dans la Section 2.2 et développé en détails dans le Chapitre 4.

Le problème est résolu avec un algorithme classique de programmation dynamique : Policy Iteration [48]. Le modèle fournit des politiques de maintenance pour chaque état du système, plus spécifiquement une matrice de décision qui précise la meilleure action pour chaque couple d'observations du PFL et du TCD sachant la dernière maintenance faite (voir les Chapitres 3, 4 et 5).

2.3.2 Le problème partiellement observable

Bien que l'approximation du TCD par la vitesse moyenne de fissuration permette de contourner le problème de non-observabilité du TCD, elle ne reflète pas la principale motivation qui nous a conduits à introduire le TCD comme second paramètre de dégradation, à savoir permettre de prendre en compte la dégradation sous-jacente non-observable.

Pour modéliser la non-observabilité de l'état de la chaussée, nous avons étendu la formulation MDP au cas partiellement observable (POMDP) [48]. Pour ce faire, l'approximation déterministe du TCD par la vitesse moyenne de fissuration est remplacée par une distribution de probabilité sur toutes les valeurs de TCD qui peuvent être observées. Ceci implique que les paramètres de décision du MDP donnés par l'état de la section sont remplacés dans le cas partiellement observable par un vecteur de croyance d'état associant une certaine probabilité à chaque état possible. L'espace des paramètres de décision passe ainsi d'un espace discret fini à un espace continu et infini (voir Chapitre 6).

Une conséquence immédiate de cette extension est l'augmentation de la taille du problème et la difficulté de le résoudre. En effet, une solution exacte pour les POMDPs est quasiment impossible, sauf pour des problèmes de très petite taille. Différentes méthodes numériques pour la résolution des modèles POMDP sont proposées dans la littérature comme les approches du type *grid-based approximations* ([31] et [70]) ou *point-based approximations* ([46, 12] and [47]). Ces approches sont basées sur la réduction du nombre de vecteurs de croyance et l'utilisation de méthodes d'interpolation pour estimer la valeur de la fonction objectif en différents vecteurs de croyance. Dans le Chapitre 6, nous utilisons l'algorithme Value Iteration avec la méthode *grid-based approximation* proposée par Lovejoy [31]. Cette méthode est basée sur (i) la construction d'une grille régulière de vecteurs de croyance caractérisée par un paramètre de résolution (ii) l'évaluation des valeurs de vecteurs de cette grille avec l'algorithme de Value Iteration et (iii) l'utilisation de cette grille comme une base pour évaluer tous les vecteurs de l'espace de croyance, en utilisant des approximations convexes. La limite de cette méthode est que la taille de la grille croit très vite en fonction de son paramètre de résolution et de la dimension de l'espace d'état. Dans les Chapitres 7 et 8, nous cherchons à définir une heuristique efficace pour la résolution du modèle POMDP en proposant une grille plus spécifique et adaptée.

2.4 Propriétés structurelles et procédures de résolution

Un des résultats les plus importants de la théorie des processus de décision markovien est l'existence, sous certaines conditions, de propriétés structurelles des politiques optimales. Ces résultats donnent les conditions que doivent vérifier les lois de transition d'état ainsi que les structures de coût considérées pour que les politiques optimales aient des propriétés remarquables comme la monotonie (i.e., l'efficacité des actions préconisées croît avec le niveau de dégradation) ou encore la propriété de limites de contrôle (i.e., l'existence d'états de contrôle qui définissent des limites entre les décisions de maintenance optimales).

Les propriétés structurelles des politiques optimales permettent une meilleure compréhension du processus de décision en fonction des rapports entre les paramètres et variables du problème comme le rapport entre investissement et utilité ou non investissement et risque. De plus, les propriétés structurelles peuvent permettre d'accélérer significativement les procédures classiques de résolution des problèmes MDP ou encore en définir de nouvelles, généralement spécifiques au problème étudié.

Les propriétés structurelles des politiques optimales fournies dans le cas parfaitement observable ne sont pas forcément valables pour le cas partiellement observable, sauf pour des cas particuliers d'information parfaite (par exemple quand la partielle observabilité provient de la non-inspection et que l'inspection est supposée parfaite quand elle a lieu) ou d'absence totale d'information [34]. Aucun de ces deux cas ne correspond à notre problème puisque une information partielle (observation du PFL) existe sur l'état alors qu'une autre partie (information sur TCD) est impossible à observer. Cependant, même si elles ne sont pas vérifiées, les propriétés structurelles dégagées pour le modèle MDP peuvent être exploitées pour proposer une heuristique efficace pour la résolution du modèle POMDP.

Nous étudions aussi deux cas différents selon les types d'actions de maintenance considérées et leur incidence sur l'évolution de l'épaisseur de la chaussée :

- **Type 1** Décaisser et recharger à la même hauteur. Dans ce cas, la hauteur de la chaussée est fixe. Une telle action réduit le PFL à zéro et le TCD à une valeur entre le TCD avant maintenance et le TCD minimum caractérisant une section aussi bonne que nouvelle et dénoté θ_0. Le niveau de TCD immédiatement après maintenance est fonction de l'état de la chaussée juste avant maintenance et de l'épaisseur renouvelée.

- **Type 2** Décaisser et recharger avec différentes épaisseurs. Ces actions remettent le PFL à zéro, et réduisent le niveau de TCD à une valeur fonction du niveau de dégradation avant maintenance, de l'épaisseur renouvelée ainsi que la nouvelle épaisseur totale qui peut être inférieure, égale ou supérieure à la hauteur avant maintenance. Ces actions de maintenance sont contraintes à une épaisseur maximale e_{max}. Nous définissons un état aussi bon que nouveau pour chaque épaisseur e caractérisé par un TCD $\theta_0(e)$. Avec de telles considérations, une action de maintenance peut donner une chaussée avec un TCD inférieur à $\theta_0(e)$ (par exemple, en renouvelant la totalité de la chaussée et augmenter sa hauteur).

Dans le Chapitre 7, nous considérons des actions de maintenance de Type 1 alors que dans le Chapitre 8, nous considérons des actions de maintenance de Type 2. Notons que bien que nous considérons spécifiquement le cas des actions de maintenance des chaussées, les résultats obtenus sont facilement généralisables à n'importe quel cas où les actions de maintenance peuvent être définies comme

Type 1 (systèmes inchangés) et Type 2 (systèmes évoluants).

Rappelons que dans les deux cas, une des particularités de notre problème est que le processus de transition n'est pas unique mais qu'il change après chaque action de maintenance.

2.4.1 Résultats pour des systèmes inchangés

Propriétés structurelles du problème observable : Les plus importantes propriétés structurelles obtenues pour le modèle MDP sont :

- Les monotonie et concavité de la fonction objectif en fonction de l'état.
- La monotonie des politiques optimales et l'existence d'états limites de contrôle.

La première propriété est une propriété classique généralement vérifiée sous condition que les fonctions de coût vérifient les mêmes propriétés (monotonie et concavité) et que les probabilités de transition d'état les vérifient aussi dans un sens stochastique [54].

La deuxième propriété, introduite initialement par [15, 49], assure que, sous les conditions de probabilités de transition qui sont du type taux de défaillance croissant (increasing failure rate), le système doit être renouvelé quand son état dépasse un état seuil appelé la limite de contrôle. Notons que l'existence d'états limites de contrôle est initialement définie et prouvée pour les problèmes d'arrêt, d'où le fait qu'elle soit définie pour l'action de renouvellement du système. Elle a été généralisée plus tard par Douer et Yachiali [16] à des problèmes avec des actions de maintenance imparfaite sous des conditions supplémentaires de structures de coût. Ils ont prouvé l'existence d'un état de contrôle au-dessous duquel il est optimal de ne rien faire et au-dessus duquel le système doit être maintenu, d'une façon parfaite ou imparfaite. Rappelons que dans les travaux de [15, 49] ainsi que [16], on suppose un processus de transition d'état unique, quelle que soit l'action de maintenance entreprise.

Nous étendons le résultat de Douer et Yachiali dans deux sens :

(i) Nous considérons différents processus de dégradation selon la dernière maintenance entreprise. Pour cela nous ajoutons une condition comparant l'évolution des transitions des différents processus de dégradation. Plus spécifiquement, cette condition assure que les probabilités de transition vers des états plus dégradés sont moins importantes après des actions plus fortes (Voir Condition 6, Chapitre 7).

(ii) Nous assurons que toutes les actions de maintenance sont sélectionnées par la politique optimale pour au moins un état possible du système. Cette extension est possible sous une condition additionnelle assurant que, pour n'importe quel état du système, la différence d'effets de deux actions successives sur le ralentissement des transitions devient moins importante quand les actions sont de plus en plus fortes (Voir condition de la Proposition 8, Chapitre 7).

Nous utilisons ces derniers résultats pour accélérer l'algorithme Policy Iteration. En effet, au cours de l'étape d'évaluation de politiques, les résultats obtenus nous permettent de réduire l'espace d'actions candidats pour la recherche de la politique optimale. Nous avons alors réalisé un gain de temps de calcul allant jusqu'à 60% (voir Tableau 1, Chapitre 7 pour les résultats numériques).

Cas du problème partiellement observable : Dans le cas du problème partiellement observable, les propriétés structurelles du MDP ne sont plus nécessairement vérifiées à cause de la probabilité d'observation, traduisant une information imparfaite, rajoutée au modèle. Cependant nous proposons d'utiliser certains résultats du MDP pour définir une heuristique de résolution du modèle POMDP. En effet, nous définissons la différence d'efficacité immédiate de deux actions différentes, incluant l'action ne rien faire, comme la différence entre, d'un côté, l'investissement supplémentaire nécessaire pour entreprendre une maintenance plus forte et d'un autre côté, le gain de risque (ici coût de qualité) réalisé en choisissant l'action la plus forte. Une action est dite immédiatement plus efficace qu'une autre action moins forte si le gain de risque qu'elle réalise est plus important que l'investissement supplémentaire nécessaire pour l'entreprendre. Nous avons démontré dans le cas parfaitement observable que si une action est immédiatement plus efficace qu'une autre qui est moins forte, alors elle le sera aussi pour l'horizon infini. Nous supposons que ce résultat reste vrai dans le cas partiellement observable.

Nous définissons une séparation entre deux actions sur l'espace de croyance par l'ensemble des vecteurs de croyance vérifiant l'investissement supplémentaire pour l'action la plus forte exactement égal au gain de risque qu'elle réalise. Nous utilisons alors ces séparations ainsi que leurs intersections, quand elles existent pour définir une grille pour l'approximation des valeurs des vecteurs de croyance dans l'algorithme Value Iteration. Une comparaison de la performance de notre heuristique avec celle de Grid-based approximation proposée par Lovejoy [31] dans le Chapitre 7 montre l'efficacité de notre heuristique en termes de politiques et de valeurs objectifs, mais surtout en termes de temps de résolution.

2.4.2 Résultats pour des systèmes évoluants

Dans le Chapitre 8, nous considérons des actions du type 2, i.e., des actions qui permettent de changer des caractéristiques du système initial le faisant évoluer vers des systèmes plus performants (qu'à l'état neuf). Plus spécifiquement, nous ajoutons la caractéristique épaisseur à la définition de l'état de la chaussée et introduisons une relation d'ordre entre les états. De plus, nous définissons la "force" d'une action de maintenance par son effet immédiat sur l'état de la chaussée (niveau de dégradation et caractéristique épaisseur). Une action est dite plus forte si elle réduit le système à un état inférieur.

Alors que classiquement les actions de maintenance (Type 1) vérifient un *ordre de force absolu*, i.e., indépendant de l'état de la chaussée (si une action est plus forte qu'une autre à un état donné, alors elle l'est à tous les états), des actions du Type 2 impliquent que l'ordre de force des actions est relatif à l'état auquel elles sont entreprises. De plus, l'ordre de force des actions à un niveau de dégradation donné ne correspond pas forcément au même ordre de coût (i.e., une action moins forte qu'une autre peut être plus chère).

Une conséquence immédiate de telles considérations est la possibilité d'avoir un état meilleur que nouveau qui est un nouvel aspect dans la littérature de la gestion de la maintenance.

Dans le Chapitre 8, nous présentons les conditions sous lesquelles les résultats structurels prouvés pour le cas d'actions de Type 1 sont étendus au cas de systèmes évoluants. Plus spécifiquement, nous avons défini la propriété de limites de contrôle conditionnels. Dans le cas classique, les limites de contrôle de la politique optimale sont fixes et ne dépendent que des paramètres du problème. Dans le cas de systèmes évoluants, ces limites de contrôle sont définies pour l'ordre d'actions conditionnel.

2.5 Optimisation de l'inspection et de la maintenance d'un réseau de sections

Jusqu'à présent, nous avons cherché à résoudre le problème de l'entretien d'une section sans considérer son appartenance à un réseau et donc l'incidence en termes de disponibilité lors de maintenance ainsi que les potentielles économies d'échelle par le regroupement d'entretiens sur des sections adjacentes. Outre ces deux points qui peuvent, comme nous l'avons déjà souligné, être déclinés sous forme de contraintes, nous proposons aussi d'aborder la problématique de l'inspection et son optimisation au niveau du réseau. On rappelle que d'ores-et-déjà une campagne d'inspection ne se déroule pas sur l'ensemble du réseau et donc ne permet que de rendre une vision partielle de son état, état défini par le niveau de dégradation de chacune des sections le constituant. Adapter l'inspection à l'état de la chaussée et prendre des décisions de maintenance sans connaissance précise de son état sont des challenges pour la réalisation de gains économiques significatifs assurant le respect des budgets alloués périodiquement.

Le travail que nous allons présenter ici n'est pas finalisé dans le sens où on ne présente que la construction du modèle mathématique que l'on va chercher à résoudre, le travail de résolution de ce modèle étant le sujet d'un contrat post-doctoral à venir.

À notre connaissance, les modèles s'attachant au problème de gestion des réseaux routiers ne traitent que de l'aspect optimisation de la maintenance en supposant connu le niveau de dégradation de l'ensemble des sections. Les approches proposées reposent généralement sur une classification des sections de route selon leur niveau de fissuration pour ensuite répartir au mieux les ressources en fonction des budgets disponibles. Ainsi, Worm *et al.* [66] proposent une approche de planification hiérarchique composée de quatre niveaux. Ainsi, ils commencent par s'intéresser à l'optimisation des politiques de maintenance pour le niveau section. Ensuite, ils définissent des regroupements de sections et attribuent une action de maintenance optimale à chaque groupe. Finalement, la contrainte budgétaire est ajoutée pour optimiser le problème d'affectation actions-groupe. Cette approche est modélisée comme un problème de programmation binaire sous contraintes de budget et de qualité et résolu avec une méthode du type *branch-and-bound*. La principale limite de ce modèle est qu'il dépend de la définition des groupes de sections et ne considère que la contrainte budgétaire.

Dahl *et al.* [14] modélisent le problème d'optimisation de la maintenance du réseau comme un problème de programmation en nombres entiers avec une structure dynamique sous-jacente. Ce modèle est résolu avec une méthode de relaxation Lagrangienne. Les éléments de réseaux considérés sont des routes et non pas des sections de route. En effet, le problème d'une route est tout d'abord considéré et modélisé comme un problème de plus court chemin. Cette approche est ensuite généralisée au niveau réseau. La limite majeure de ce modèle est que la maintenance des routes entières est obligatoire, même si seulement quelques parties nécessitent un entretien.

Nous avons noté que l'objectif de la majorité des approches rencontrées est d'assurer la meilleure allocation des ressources budgétaires entre les différentes parties du réseau routier. Certains modèles comme [3], considèrent aussi la maximisation de la qualité des chaussées comme objectif.

Les contributions visées de notre modèle sont détaillées ci-après :

1. Optimiser les politiques d'inspection simultanément à celles de maintenance, et d'une façon non-périodique ;
2. Optimiser les politiques de maintenance non seulement du point de vue gestionnaire mais aussi du point de vue usager, en considérant des coûts usagers et environnementaux engendrés par l'ouverture de chantiers de maintenance. Ce dernier point devient une réelle préoccupation pour les gestionnaires ;
3. Considérer, en plus de la contrainte budgétaire, une contrainte de qualité de trafic en considérant des demandes et des capacités de trafic et un seuil acceptable de qualité de trafic ;
4. Pour assurer les deux derniers objectifs (2 et 3) et ne pas concentrer tous les travaux de maintenance en début de la période de décision, nous considérons la possibilité de diviser cette dernière en sous-périodes de maintenance que nous appelons périodes pour planifier efficacement les travaux de maintenance.

La considération du problème d'optimisation de l'inspection en plus de celui de maintenance rend le processus de décision plus compliqué du fait de la dépendance de la résolution des incertitudes (i.e., inspection) des décisions. Tout d'abord, alors que les décisions de maintenance sont basées sur les observations d'états, ces dernières ne sont réalisées que lorsqu'une inspection est programmée. De plus, comme dans la suite des travaux de modélisation de la dégradation des chaussées, nous supposons que les processus d'évolution de l'état des chaussées (processus de dégradation) dépendent de l'action de maintenance entretenue, les décisions de maintenance affectent les distributions des probabilités (par exemple, en rendant certaines réalisations plus probables que d'autres).

Dans la littérature liée à l'optimisation stochastique, le cas d'incertitude où le temps de réalisation des paramètres aléatoires (ici états de la section) ou la distribution des réalisations dépendent des décisions (inspection ou maintenance) est connu sous l'expression d'incertitude endogène (par rapport à l'incertitude exogène où les réalisations des paramètres aléatoires sont indépendantes des décisions).

Seules quelques études sur des problèmes avec incertitudes endogènes sont disponibles [56]. De plus, dans la plupart de ces modèles ([19, 20, 21] and [56]), l'incertitude est endogène seulement à cause des décisions d'inspection qui précèdent celles d'investissement. Par ailleurs, dans notre problème, l'incertitude revient à chaque période après la maintenance, même si elle a été résolue au début de cette période.

En plus de l'objectif d'optimiser l'allocation des ressources budgétaires disponibles d'un point de vue gestionnaire, nous prenons en compte des considérations comme le coût et le confort des usagers, les coûts environnementaux. En effet, nous considérons dans la fonction objectif à minimiser des coûts usagers et de qualité engendrés par la fermeture de certaines routes, les distances supplémentaires supportées par les usagers ainsi que les embouteillages et les conditions de trafic inacceptables, etc.

Nous formulons le problème d'inspection et de maintenance du réseau routier comme un *multi-stage stochastic program* (MSSP). Nous illustrons avec un exemple les contraintes de non-anticipativité.
Avec cet exemple, nous démontrons que même si l'incertitude n'a pas été résolue avec l'inspection, une information additionnelle peut être disponible après la décision de maintenance, ce qui peut conduire, dans certains cas, à résoudre l'incertitude. Nous pensons qu'il est possible de tirer avantage de cette structure particulière pour déterminer l'application des contraintes de non-anticipativité. Nous avons aussi linéarisé la majorité des variables et expressions du modèle.

Pour générer des scénarios, nous proposons d'utiliser le modèle conditionnel de dégradation que nous avons développé.

La programmation stochastique avec incertitude endogène est l'un des problèmes de programmation mathématique les plus difficiles [56], en termes de modélisation et de réalisation. Avec ces travaux sur l'optimisation de l'inspection et de la maintenance d'un réseau de sections, nous pensons ouvrir des perspectives de recherche très intéressantes en programmation stochastique avec incertitude endogène.

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II LES PUBLICATIONS

3 A Preliminary Condition-Based Approach

A Condition-Based Imperfect Maintenance Model with Action-Dependent Deterioration

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Abstract

We propose a maintenance decision framework where the state of the system is based on a percentage of total degradation and the rate of change of the degradation. At each decision interval, the degradation percentage may be observed perfectly as well as the rate of change with respect to the last observation. We consider fve possible actions based on these two observable phenomenon: Do nothing, three imperfect maintenance actions, and a complete renewal. Prior imperfect maintenance models do not consider a scenario where imperfect maintenance not only restores the system to a state less than "as good as new" but also under a new deterioration law. In this case, it is necessary to consider that the rate of degradation after imperfect maintenance will be altered based on the action performed and the state of the system prior to the maintenance. We propose to develop an imperfect maintenance optimization model based on a Markov decision process framework to determine the optimal maintenance actions given the state of the system at each decision interval and motivate the problem through road maintenance.

Keywords: Imperfect maintenance, Markov decision process, Road maintenance.

Notations

A	set of actions
a	action
a_q	unit quality cost
c_0	inspection cost
c_a	action cost
$c_q(\cdot)$	quality cost
ct_m	θ-reduction factor
$f_m(\cdot)$	joint density function of ρ and θ
$f_{\Delta\rho,m}$	density function of ρ
$f_{\Delta\theta,m}$	density function of θ
$g_k(\cdot)$	maintenance effect function on θ of action k
i_ρ	discrete values of ρ
i_θ	discrete values of θ
l_ρ	length of discrete intervals of ρ
l_θ	length of discrete intervals of θ
m	type of the last performed action
N_ρ	number of discrete values of ρ
N_θ	number of discrete values of θ
$p_{ss'}^{m}$	transition probability given the last maintenance m
P_m	potential release probability in the initiation phase
r_s^a	immediate reward after action a in state s
s	section state
s^+	section state immediately after action
$S_s(a)$	set of possible states at the end of the decision epoch after action a
V	Total expected discounted cost-to-go over the infinite horizon
ρ	longitudinal cracking percentage
θ	deterioration growth rate
θ_{max}	maximum deterioration growth rate
θ^+	deterioration growth rate immediately after action
$\hat{\theta}$	average deterioration speed
$\Delta\rho$	increment of ρ
$\Delta\theta$	increment of θ
λ	discount factor
τ	inter-inspection period
π	maintenance policy

1 Introduction

A wide range of preventive maintenance optimization models have been developed in the literature. For a survey, see [10, 13]. The maintenance decision is based on different system characteristics in order to reduce the associated costs and maximize the availability and safety of the considered system [11]. Condition-Based Maintenance (CBM) approaches [13], where the decision is directly driven by a measurable condition variable that reflects the system deterioration level, have proven their efficiency compared to classical time- or age-based policies [6] in terms of economical benefits and also in terms of system safety performance. Nevertheless, moving from classical age- or time-based policies to CBM requires significant efforts in degradation modelling, associated data collection, and additional assumptions to ensure the mathematical tractability of the decision criterion.

Few works developed in the CBM context are devoted to model the efficiency of maintenance actions: repair is usually supposed perfect and restores the system to the "as good as new" state. First introduced by Barlow and Proschan [1], imperfect maintenance models have been proposed in the context of age-based maintenance [9, 7, 5, 2]. These models are based on the characterization of the failure intensity process after a repair and most of them are discussed in the virtual age reduction model introduced by Kijima [8]. CBM approaches that consider imperfect maintenance only adjust the deterioration level after maintenance, leaving the deterioration evolution characteristics unchanged [3].

One objective of this work is to develop an imperfect maintenance model for cumulative-deteriorating systems when a maintenance action also impacts the behaviour of the future deterioration. Hence, the maintenance decision after system inspection to select the most appropriate maintenance action among a set of actions is based on the current system state and the impact on the future behaviour of the deterioration process with respect to costs. We propose to model this problem as a Markov decision process (MDP) and will motivate its application to the maintenance of a road section subject to cracking.

The remainder of this paper is organized as follows. Section 2 formally states the general imperfect maintenance model and the construction of the cost criterion. Section 3 outlines the road maintenance problem and the motivation for our assumptions. Section 4 discusses the model assumptions with respect to cracking and our solution procedure. Section 5 proposes a numerical experiment to illustrate the performance of the model. Finally, Section 6 is devoted to conclusions and perspectives.

2 Model Description

Consider a repairable system that operates continuously and is stochastically deteriorating. The system degradation can be described by a deterioration percentage $\rho(t) \in [0,1]$ and let $\theta(t)$ be its instantaneous growth rate. We suppose that the degradation percentage $\rho(t)$ is a non decreasing process in time whereas its instantaneous growth rate $\theta(t)$, is not necessarily a monotone process in time.

To maintain the system, we consider a periodic inspection problem with a decision epoch of length τ and both perfect and imperfect maintenance actions. In any epoch, the decision maker is forced to inspect the system at a cost c_0. The inspection is supposed perfect and instantaneous and yields the values of both the total degradation percentage $\rho(t)$ and its growth rate $\theta(t)$. After inspection, the decision maker may opt to do nothing (DN), to initiate one of four maintenance actions MX_k, $k \in \{1,2,3,4\}$ where MX_3, MX_2 and MX_1 are imperfect action with increasing impact in k and (MX_4) is a perfect action that renews the system. Let $A = \{DN, MX_1, MX_2, MX_3, MX_4\}$ denote the action set, and a denote an action in A. The cost of any maintenance action is given by c_k such that $c_{MX_1} < c_{MX_2} < c_{MX_3} <<< c_{MX_3}$. Moreover, a quality cost $c_q(s,s')$ is incurred. The quality cost which a function of the cumulated deterioration of the system moving from state s to s' within a decision interval.

Let $s = (\rho, \theta, m)$ denote the state of the system, where ρ and θ are the current levels of total degradation percentage, its growth rate and $m \in \{1,2,3,4\}$ is the type of the last performed maintenance action, respectively. S denotes the system states set. We assume that the effects of a decision taken in a state s depend only on that state and not on the prior history. Thus, the process $\{(\rho(t), \theta(t), m), t > 0\}$ is a Markov process.

Let (ρ^+, θ^+) denote the values of the degradation percentage and its growth rate just after a decision is taken. We suppose that when DN is performed the system begins the next epoch in the same state (ρ, θ, m) yielded by inspection. After both perfect and imperfect maintenance actions, the total degradation percentage ρ is reset to zero whereas θ^+, the deterioration growth rate immediately after maintenance, is assumed to be a deterministic function $g_m(\rho, \theta)$ of the maintenance level and the deterioration level before maintenance. The dependence of the growth rate after maintenance on the road state before maintenance is due to imperfect maintenance and is suggested by data.

Let $S_s(a)$ be the set of states to which the system can move at the end of the decision epoch after action a is performed in state s.

$$S_s(a) = \begin{cases} \{(\rho', \theta', m);\ \rho' \geq \rho,\ \theta' \geq 0\} & \text{if } a = DN \\ \{(\rho', \theta', a)\ ;\ \rho' \geq 0,\ \theta' \geq 0\} & \text{if } a = MX_m,\ m \in \{1, 2, 3, 4\} \end{cases}$$

The one-step transition probability from a state s to s' given the last maintenance action m is denoted $p^m_{ss'}$. A maintenance policy π is a mapping from the system states set to the set of actions to be taken when in those states. Therefore, the minimum expected discounted cost-to-go over the infinite horizon which is the value of taking action a when the system state is $s = (\rho, \theta, m)$ under the policy π, denoted by $V^\pi(s)$ can be expressed as

$$V^\pi(\rho, \theta, m) = c_0 + \min \left\{ \begin{matrix} DN(\rho, \theta, m), MX_1(\rho, \theta, m), MX_2(\rho, \theta, m), \\ MX_3(\rho, \theta, m), MX_4(\rho, \theta, m) \end{matrix} \right\} \tag{1}$$

where

$$\begin{aligned} DN(\rho, \theta, m) &= \sum_{s' \in S_s(DN)} p^m_{ss'}(c_q(\rho, \rho') + \lambda\, V^\pi(\rho', \theta', m)) \\ MX_k(\rho, \theta, m) &= c_{MX_k} + DN(0, g_k(\rho, \theta), k), \qquad k \in \{1, 2, 3, 4\}. \end{aligned}$$

where $\lambda \in [0, 1]$ is the discount factor.
Equation (1) states that following the DN action, if state s incurs the expected cost-to-go to the state $s' \in S_s(DN)$ by the end of the decision epoch with the respective probability $p^m_{ss'}$. Whereas the (MX_k) action, $k \in \{1, 2, 3, 4\}$ restores intantaneously the degradation percentage to zero and its growth rate to $\theta^+ = g_k(\rho, \theta)$, $k \in \{1, 2, 3, 4\}$ and then incurs, in addition to the action cost c_{MX_k}, the expected cost-to-go associated with starting in the state $s^+ = (0, \theta^+, k)$.

Let now r^a_s denote the reward function for the single decision epoch given the current state s and the performed action a,

$$r^a_s = \sum_{s' \in S_s(a)} p^m_{ss'}\ \{c_0 + c_a + c_q(s, s')\} = \sum_{s' \in S_s(a)} p^m_{ss'}\, r^a_{ss'}$$

where c_a, $a \in A$ is the action cost and c_{DN} =0.

Equation (1) can be formulated as

$$V^{\pi}(s) \;=\; r_s^a + \lambda \sum_{s' \in S_s(a)} p_{ss'}^m \, V^{\pi}(s') \;=\; \sum_{s' \in S_s(a)} p_{ss'}^m (r_{ss'}^a + \lambda \, V^{\pi}(s')), \qquad (2)$$

where a is the action performed under the policy π when the system state is s. Equation (2) is used for the policy iteration algorithm [12] that will be discussed in section 4.

3 Application to the road maintenance problem

We are interested in the application of the MDP model presented in the previous section to the maintenance of a road section subject to longitudinal cracking. A longitudinal crack appears on the surface of the road because of fatigue mainly due to repetitive traffic.

Cracks are not continuously observable in time and an inspection must be performed to reveal the current state of the road. Currently in practice, road sections of 200m in length are inspected and the deterioration is measured as the longitudinal cracking percentage (LCP), denoted ρ which is the ratio of the fissured road section length on the total length of the section (Figure 1). The two-phase of cracking process of $\rho(t)$, initiation and propagation, is illustrated in the different paths presented in Figure 2. If $\rho(t) = 0$, i.e, no cracks have appeared, $\theta(t)$ represents the potential initiation of cracks where $\theta(t)$ is increasing in time.

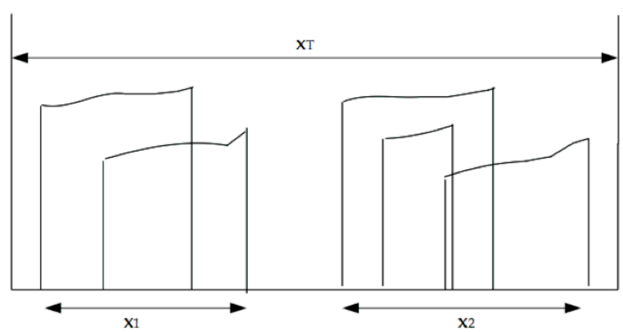

Figure 1: Total percentage of degradation $\rho = (x_1 + x_2 + x_3)/x_T$.

Castanier and Yeung [4, 14] have proposed a CBM policy with imperfect maintenance where the decision rules are based on both the total observed crack size and the cumulated repaired crack size before a complete renewal. Their model is not directly applicable here because in practice, individual cracks are not measured. In this paper, the evolution of $\rho(t)$ is a complex function of the existing cracks growth and the appearance rate of new ones.

Based on the LCP observation at instant t, different alternatives are proposed under our policy. If the road deterioration level (combination of ρ and θ) is considered as acceptable, no maintenance is required and the road is left as is until the next inspection at time $t + \tau$. If the road deterioration is considered too large, a maintenance action or a complete renewal is performed. A maintenance action consists of a complete resurfacing of the road section

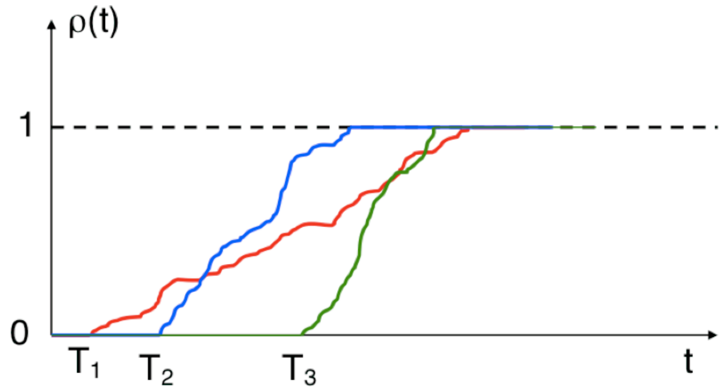

Figure 2: Different paths of $\rho(t)$ for non maintained road sections

with a given thickness; the cracks will temporally disappear from the surface but still exist in the road foundation. Hence, we can consider that after maintenance $\rho(t) = 0$ but its future evolution will depend on the deterioration state before maintenance and the efficiency of the maintenance action which is a function of the the new thickness. As a renewal action either rebuilds the road or places a layer so thick that it is not impacted by the underlying cracks, the road can be considered as good as new after this action.

In the road maintenance case, only the longitudinal cracking percentage ρ is actually observable, thus the Markov decision process is only partially observable. The growth rate θ will be estimated by the average speed $\hat{\theta}$ given by two successive observations of ρ. We justify this estimation in the road maintenance case because the evolution of degradation is slow and the inspection period is considered small compared to the life cycle of the road. To solve this MDP, a classical dynamic programming (DP) algorithm, policy iteration, is used. For numerical purposes and because, in practice, the growth rate θ is limited and the cracking process can be considered continuous in time, we suppose that a maximum speed θ_{max} can be given so that $\theta \leq \theta_{max}$.

4 Crack evolution and solution procedure

Let $\Delta\rho$ and $\Delta\theta$ be the respective dependent changes in the cracking percentage and in the growth rate in a decision period, i.e. $\rho(t+\tau) = \rho(t) + \Delta\rho$ and $\theta(t+\tau) = \theta(t) + \Delta\theta$. $\Delta\rho$ and $\Delta\theta$ are naturally correlated. Moreover, because the evolution of both ρ and θ depends on the last maintenance action, the joint probability density of $(\Delta\rho, \Delta\theta)$ given a state $s = (\rho, \theta, m)$ is:

$$f_m(\Delta\rho, \Delta\theta \mid \rho, \theta) = f_{\Delta\rho,m}\left(\Delta\rho \mid \rho, \theta, \Delta\theta\right)\ f_{\Delta\theta,m}\left(\Delta\theta \mid \rho, \theta\right),$$

where $f_{\Delta\rho,m}\left(\Delta\rho \mid \rho, \theta, \Delta\theta\right)$ and $f_{\Delta\theta,m}\left(\Delta\theta \mid \rho, \theta\right)$ are the conditional probability density of $\Delta\rho$ and the marginal density of $\Delta\theta$, respectively. The choice in the conditional probability functions should be conducted by the analysis of the mechanical degradation process described in Section 3.

For the use of the policy iteration algorithm, we propose to discretize the problem as follows. Let $s = \big(i_\rho, i_\theta, m\big)$ be the system state at the begining of current decision epoch where i_ρ and i_θ represent the first values in N_ρ and N_θ equal size discrete intervals of length l_ρ and l_θ on $[0,1]$ and $[0,\theta_{max}]$, respectively such that N_ρ and N_θ divide evenly into 100, i.e: $i_\rho \in \big\{[0,l_\rho[,[l_\rho,2l_\rho[,\cdots,[(N_\rho-1)l_\rho,1]\big\}$ and $i_\theta \in \big\{[0,l_\theta[,[l_\theta,2l_\theta[,\cdots,[(N_\theta-1)l_\theta,\theta_{max}]\big\}$.

The probability of transition from a state $s = (i_\rho, i_\theta, m)$ to $s' = (i'_\rho, i'_\theta, m)$ given the performed maintenance m can then be expressed as:

$$p^m_{ss'} = \begin{cases} \displaystyle\int_{i'_\rho-i_\rho}^{i'_\rho-i_\rho+l}\int_{i'_\theta-i_\theta}^{i'_\theta-i_\theta+l} f_{\Delta\rho,m}(x \mid \rho,\theta,y) f_{\Delta\theta,m}(y \mid \rho,\theta)\, dxdy & \text{if } a = DN. \\ \\ \displaystyle\int_{i'_\rho}^{i'_\rho+l}\int_{i'_\theta-i^+_\theta}^{i'_\theta-i^+_\theta+l} f_{\Delta\rho,a}(x \mid \rho,\theta,y) f_{\Delta\theta,a}(y \mid \rho,\theta)\, dxdy & \text{otherwise} \end{cases}$$

where $i^+_\theta = g_a(i_\rho, i_\theta)$.

5 Numerical example

We propose here to discuss a numerical example of the maintenance policy, π. We utilize the following functions for the purposes of our numerical example:

(a) The conditional probability density $f_{\Delta\rho,m}\big(\Delta\rho \mid \rho,\theta,\Delta\theta\big)$ is the truncated probability density function of a Beta-distributed random variable with the expected mode $b_m(\theta + \frac{\Delta\theta}{2})\tau$ where b_m is a given decreasing factor in the maintenance type m. The choice of the Beta distibution is based on the its flexibility and the fact that it is in $[0,1]$. The Beta parameters α and β could be easily derived with further assumptions. The truncation is due to the constraint $\Delta\rho \leq 1-\rho(t)$, $\forall\theta$ and $\Delta\theta$.

(b) Given a state s and an action a, because of the lack of information on $\Delta\theta$ distribution, we choose for $\Delta\theta$ the uniform distribution on the possible values $\Delta\theta \leq \theta_{max} - \theta(t)$. Moreover, if $\rho(t) = 0$, $a = DN$ and $\rho(t+\tau) = 0$, we assume that the potential is non decreasing, hence $\Delta\theta \geq 0$.

(c) The growth rate after a maintenance m is reduced such that $g_m(\rho,\theta) = ct_m\rho\cdot\tau$, $m \in \{1,2,3\}$ and $g_4(\rho,\theta) = \theta_0$;

(d) The quality cost per unit of time is a function of the expected deterioration in the next epoch. Because it is not possible to monitor continuously the road, we define an average quality cost on a decision epoch by $c_q(\rho,\rho') = a_q\cdot(\rho^+ + \rho')\cdot\tau/2$ where ρ^+ and ρ' are the respective deterioration level at the beginning and the end of the period.

We also assume a discount factor $\lambda = 0.9$ and an inspection period $\tau = 1.4$. We use the following parameters for the probability distributions, costs, and effects of maintenance (the values of these parameters are arbitrary and are not estimated by field data):

- the Beta function parameters for the deterioration process: $\beta = 2.4$, and the associated mode is controlled by the respective values $b_m = \{0.7, 0.55, 0.4, 0.2\}$, with $m \in \{1, 2, 3, 4\}$;
- the efficiency of the maintenance described by the respective probabilities of releasing the potential if $\rho(t) = 0$ given the maintenance m$\in \{1, 2, 3, 4\}$, $P_m = \{0.75, 0.6, 0.4, 0.1\}$ and the corresponding θ-reduction factor $ct_m = \{0.8, 0.65, 0.35\}$. $\theta_0 = 0.01$ when the road is "new";
- the cost of inspection $c_0 = 30$;
- the associated unit costs of maintenance $c_{MX_k} = \{40, 50, 60, 100\}$;
- the cost of quality per unit of time and degradation $a_q = 70$;
- the range of the uniform distribution $\theta_{max} = 1$.

In Table 1, are the proposed decision matrices, each matrix is for the decision at each state of the system. Each matrix represents the last maintenance action ($m = \{1, 2, 3, 4\}$) and contains the action for each deterioration vector (ρ, θ). For the first matrix, $m = 1$, if at time t, the deterioration level ρ belongs to $[0.2, 0.3[$ and the observed degradation speed $\hat{\theta}$ in the last time interval belongs to $[0.6, 0.7[$, the advocated decision is *Do a type-2* maintenance. The presence of MX_4 in at least of the decision matrices, ensures that the road will be renewed at some state.
The resulting expected cost starting from the new state is $V^{\pi}(0, \theta_0, 4) = 544$.

6 Summary

In this paper, we have formulated an imperfect maintenance policy for cumulative deteriorating systems as a Markov decision process. The maintenance decision is based on the observation of the current system state and its expected evolution in the next interval. The efficiency of the maintenance action is not restricted to the classical "as good as new" assumption: the deterioration law is a function of the type of the last maintenance action and its associated efficiency. The efficiency of the maintenance action also depends on the system deterioration level before the maintenance. The maintenance model has been applied to optimize the maintenance decision for a road section subject to longitudinal cracking. A discussion based on a numerical example has been presented.

Although the main assumptions of the model have been discussed with civil engineering experts, the applicability of the proposed methodology should be aligned with the individual practices of a given locale. Future work should consider a partially observed Markov decision problem instead of the approximation of the instantaneous growth rate by the average.

m=1

ρ \ $\hat{\theta}$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	**0**	**2**	**2**	**2**	3	3	3	3	**4**	**4**
0.8	**0**	**2**	**2**	**2**	3	3	3	3	3	**4**
0.7	**0**	**2**	**2**	**2**	3	3	3	3	3	**4**
0.6	**0**	**2**	**2**	**2**	**2**	3	3	3	3	**4**
0.5	**0**	1	**2**	**2**	**2**	3	3	3	3	3
0.4	**0**	1	**2**	**2**	**2**	3	3	3	3	3
0.3	**0**	1	**2**	**2**	**2**	**2**	3	3	3	3
0.2	**0**	1	1	**2**	**2**	**2**	3	3	3	3
0.1	**0**	1	1	**2**	**2**	**2**	3	3	3	3
0.0	**0**	**0**	**0**	**0**	**2**	**2**	**2**	3	3	3

m=2

ρ \ $\hat{\theta}$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	**0**	**2**	**2**	**2**	3	3	3	3	**4**	**4**
0.8	**0**	**2**	**2**	**2**	3	3	3	3	3	**4**
0.7	**0**	**2**	**2**	**2**	3	3	3	3	3	**4**
0.6	**0**	**2**	**2**	**2**	**2**	3	3	3	3	**4**
0.5	**0**	1	**2**	**2**	**2**	3	3	3	3	3
0.4	**0**	1	**2**	**2**	**2**	3	3	3	3	3
0.3	**0**	1	**2**	**2**	**2**	**2**	3	3	3	3
0.2	**0**	1	1	**2**	**2**	**2**	3	3	3	3
0.1	**0**	1	1	**2**	**2**	**2**	3	3	3	3
0.0	**0**	**0**	**0**	**0**	**0**	**2**	**2**	3	3	3

m=3

ρ \ $\hat{\theta}$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	**0**	**2**	**2**	**2**	3	3	3	3	**4**	**4**
0.8	**0**	**2**	**2**	**2**	3	3	3	3	3	**4**
0.7	**0**	**2**	**2**	**2**	3	3	3	3	3	**4**
0.6	**0**	**2**	**2**	**2**	**2**	3	3	3	3	**4**
0.5	**0**	1	**2**	**2**	**2**	3	3	3	3	3
0.4	**0**	1	**2**	**2**	**2**	3	3	3	3	3
0.3	**0**	1	**2**	**2**	**2**	**2**	3	3	3	3
0.2	**0**	1	1	**2**	**2**	**2**	3	3	3	3
0.1	**0**	1	1	**2**	**2**	**2**	3	3	3	3
0.0	**0**	**0**	**0**	**0**	**0**	**0**	**2**	3	3	3

m=4

ρ \ $\hat{\theta}$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	**0**	**2**	**2**	**2**	3	3	3	3	**4**	**4**
0.8	**0**	**2**	**2**	**2**	3	3	3	3	3	**4**
0.7	**0**	**2**	**2**	**2**	3	3	3	3	3	**4**
0.6	**0**	**2**	**2**	**2**	**2**	3	3	3	3	**4**
0.5	**0**	1	**2**	**2**	**2**	3	3	3	3	3
0.4	**0**	1	**2**	**2**	**2**	3	3	3	3	3
0.3	**0**	1	**2**	**2**	**2**	**2**	3	3	3	3
0.2	**0**	1	1	**2**	**2**	**2**	3	3	3	3
0.1	**0**	1	1	**2**	**2**	**2**	3	3	3	3
0.0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	3	3	3

Table 1: Optimal decision matrices. $0 = DN$, $1 = MX_1$, $2 = MX_2$, $3 = MX_3$, $4 = MX_4$

Further, the decision interval length τ should be considered as a decision variable and better algorithms for larger scale problems with a more detailed discretization should be constructed . Nevertheless, we are convinced that our proposal is a real contribution in maintenance optimization, especially in condition-based imperfect maintenance models and road maintenance practice.

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4 A Two-Phase, State-Dependent Deterioration Process

A Two-Phase State-Dependent Deterioration Model for Maintenance Optimization

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Abstract

We consider the inspection and maintenance optimization problem for cumulative deteriorating systems with multiple and imperfect maintenance actions. Maintenance not only results in a perfect or partial rejuvenation of the system state but also changes the future deterioration law. We construct a new two-phase, state-dependent degradation model based on two correlated variables: the observable degradation indicator and the deterioration growth rate. The latter variable provides information on the intrinsic ability of the system to deteriorate in future periods. Its introduction allows modeling the underlying, non-observable deterioration and the effect of different imperfect actions that have the same effect on observable deterioration. We model the initiation phase of deterioration using Poisson and gamma processes and the propagation phase as a new state non-stationary gamma process and a bilateral gamma process for the deterioration growth rate. The use of the bilateral gamma process in a maintenance context is novel and allows direct derivation of the growth rate. We formulate a Markov decision process for minimizing cost and utilize our state-dependent deterioration model model to derive its probability transition matrix. Finally, we illustrate the benefits of our approach on a real-world road maintenance problem.

Keywords: two-phase deterioration process, state-dependent gamma process, Bilateral gamma process, state-non-homogeneous Poisson process, imperfect maintenance, multi-action policies, Markov decision process.

Notations

$c(k)$	action cost	T	random variable of the first observable degradation arrival time
c_i	inspection cost	V	Total expected discounted cost-to-go over the infinite horizon
$c_q(\cdot)$	quality cost	θ_t	non-observable deterioration process
G_t	observable deterioration process	θ_0	minimum deterioration growth rate
$f(\cdot)$	density function of the bilateral gamma distribution	θ_{max}	maximum deterioration growth rate
$\hat{f}(\cdot)$	density function of the truncated bilateral gamma distribution	ρ_t	deterioration growth rate process
$g(\cdot)$	density function of the gamma distribution	$\gamma(\cdot)$	SDG shape function
$\hat{g}(\cdot)$	density function of the truncated gamma distribution	γ_n, γ_p	shape functions of the bilateral gamma process
$h(\cdot)$	joint density function of ρ and θ	β	SDG scale parameter
i_ρ	discrete values of ρ	τ	inter-inspection period
i_θ	discrete values of θ	$\Psi_{X_t}(\cdot)$	characteristic function of process X_t
l	length of discrete intervals of ρ and θ	α_p, α_n	shape parameters of the bilateral gamma process
m	type of the last performed action	β_p, β_n	scale parameters of the bilateral gamma process
N_t	Poisson process	π^*	optimal maintenance policy
N	number of actions	$\phi(\cdot)$	maintenance effect function on θ
N_ρ	number of discrete values of ρ	$\Delta\rho$	increment of ρ
N_θ	number of discrete values of θ	$\Delta\theta$	increment of θ
$p^m_{ss'}$	state transition probability given the last performed action m	$\Lambda(\cdot)$	intensity function of the Poisson process
r_m	cracking deceleration factor of action m	$\Lambda'(\cdot)$	intensity rate function of the Poisson process
s	system state	λ	discount factor

1 Introduction & literature review

The challenge for maintenance optimization is to propose the best decision framework as a function of a complex decision criterion. The complexity of the criterion is due to the diversity of the objectives (availability maximization, cost and risk minimization, etc.), the diversity of the maintenance actions and the stochastic nature of deterioration.

Recent surveys ([9] and [34]) present many maintenance models for optimizing decisions. There are two main approaches to maintenance optimization: (i) age-based maintenance (ABM) approaches ([33, 13] and [37]) where the decision parameter is directly modeled by lifetime distributions, and (ii) condition-based maintenance (CBM) policies ([22, 3, 25, 16, 15] and

[23]) where the decision is based on an observable deterioration indicator. CBM approaches are more effective than age-based approaches in that they integrate more information on the failure mechanism. However, they require more effort in modeling and data collection.

In this paper, we develop a condition-based approach for optimizing the maintenance of stochastic, cumulative deteriorating systems and derive an optimal inspection and maintenance policy. We motivate our approach through a real-world road maintenance application. Our optimization model has the advantage to consider multi-imperfect maintenance actions with different effects on the deterioration evolution at different costs. To obtain the optimal multi-action policy, we use a Markov decision process (MDP) approach [27]. The transition probability matrix for the MDP is derived by developing a new two-phase "state-dependent" deterioration model for cumulative deteriorating systems that we incorporate into an MDP framework in order to eliminate the need for large statistical data and expert judgement.

Our deterioration model is comprised of (i) an initiation phase where, although no observable degradation is recorded, an underlying deterioration is cumulating and (ii) a propagation phase where both the underlying deterioration and an observable one are present. In order to take into account the underlying deterioration process, we introduce into the system state definition a the deterioration growth rate (DGR) in addition to the observable deterioration metric.

From the literature, most CBM policies only consider minimal or perfect repairs as possible actions. Perfect repairs rejuvenate the system to an as-good-as new condition whereas minimal ones reset the system to the state just prior to failure. Imperfect maintenance is usually modeled in CBM approaches by a random reduction of the deterioration level but does not allow for the deterioration law to change. Such an assumption prevents distinguishing maintained systems from new ones. In more recent work, Nicolai *et al.* [25] assume that imperfect maintenance activities reduce the deterioration level and the system may have a new evolution law that depends on the performed action. We extend this concept further by also making the deterioration law depend on the state before maintenance.

The quality of maintenance optimization models is strongly correlated to the relevance and the mathematical tractability of the deterioration model. Three types of deterioration models are commonly used. The models of the first class explain the deterioration process in accordance with the stresses and external forces applied on the system such as the Paris-Erdogan model ([26], [20, 38] and [32]). The construction of such models requires a great deal of knowledge on both the degradation process and the material properties.

The second class of degradation models deals with discrete-time Markov chains ([16, 7, 5] and [19]). They are strongly dependent on the definition of the discrete degradation states and the construction of the associated transition probability matrix. In practice, this matrix is obtained from a large amount of statistical data or expert judgment. Other limits for the use of these models are the difficulties to model some continuous degradation processes and the dependence of the length of the decision epoch on the one-step degradation transition interval, especially for slow-degradation processes. Their advantage is the resolution of complex maintenance criterion without imposing an a priori decision-making structure.

The third class of deterioration models is based on Lévy processes that are widely used to model accumulation phenomenon such as deterioration ([29]). The most used Lévy processes for deterioration modeling are Brownian motion (BM) ([36, 23] and [3]) and gamma processes ([12, 28, 30, 31, 29, 2] and [32]). BM has the advantage to be a continuous Lévy process, but it is not monotonic. However, the gamma process is monotone which makes it a good candidate to model strictly increasing cumulative deterioration ([2]). In more recent works for maintenance applications, the gamma process has been generalized to a non-stationary process ([24, 15] and [13]). The resulting process belongs to the class of additive processes [8] that allow for more accurate deterioration modeling.

In this work we combine the latter two degradation models discussed previously, i.e., Markov chains and Lévy processes. We extend the classical definition of the gamma process to what we define as a "state-dependent gamma process" (SDGP) to model observable deterioration and utilize a bilateral gamma process in order to model the non-observable deterioration growth rate of the GDP. The BGP comes from finance literature and its use in a maintenance context is novel. Moreover, it allows direct derivation of the growth rate. The combination of these two Levy processes is used to derive the transition probability matrix for a Markov decision process. This eliminates the need for large amount of statistical data.

The remainder of this paper is structured as follows: In Section 3, we give the model assumptions and the maintenance optimization formulation. In Section 2, we formally define the state-dependent deterioration model for the initiation and propagation phases. We propose an application of the model on a real-world road maintenance problem in Section 4, followed by a numerical example in Section 5 and conclusion in Section 6.

2 A state-dependent deterioration model for a two-phase deteriorating system

We model the cumulative deterioration process as a two-phase process: an initiation and propagation phases. Most of the existing stochastic deterioration models used in maintenance optimization do not take into account the underlying initiation phase that caracterizes such deterioration processes. In [30], Noortwijk and Klatter use a two-phase deterioration model for the problem of inspecting the block mats of the Eastern-Scheldt storm-surge barrier. The stochastic processes of scour-hole initiation and scour-hole development are modeleded by a homogenous Poisson process and a gamma process, respectively. We extend this model by introducing the deterioration speed parameter and make the processes state-dependent and thus non-stationnary. Moreover, in our model, we show the continuity and the relationship between the two deterioration phases. In more recent work, Nicolai *et al.* [24] use a two-stage hit-and-grow (TSHG) process to model the random deterioration of a coating layer. In this model, the initiation phase is modeled by an non-homogenous Poisson process (NHPP) that counts the number of new "infection" arrivals and the propagation process by deterministic interexpansion times. A disadvantage of the TSHG process is that the probabilistic characteristics cannot be determined analytically but must be approximated by simulation. Nicolai *et al.* [24] concluded that a BM process or a gamma process can replace the TSHG.

In this work, we model the level of deterioration in the initiation phase as a potential of initiation. We call it potential with respect to potential energy as it can be stored or transformed in another energy form (e.g. kinetic or mechanical). This transformation is the arrival of the first observable deterioration that initiates the propagation phase. We assume that the observable deterioration is non decreasing and its growth is due to the underlying deterioration process which we now detail. Note that θ refers to the initiation potential before observable deterioration and to the growth rate after.

We present in this section the state-dependent gamma process and introduce its growth rate process as a bilateral gamma process. Because we model the initiation phase as a specific case of the propagation phase, we first present the deterioration model in the propagation phase and then show how to derive the initiation phase model.

In the remainder of the paper, for any stochastic process $\{X_t, t \geq 0\}$, we denote X as the process and X_t as the realization of X at time t.

2.1 The propagation phase: the state-dependent gamma process and its associated growth process as a bilateral gamma process

In this section, we define a non-stationary gamma process by relaxing the stationarity constraint in state and term the resulting process a "state-dependent gamma process". However, this relaxation makes the process lose the infinite divisibility property. We associate to the state-dependent gamma process its expected growth rate process that we model as a BGP and define the system state by the state of the two processes. The resulting bi-process has the advantage to be an infinite divisible process which allows us to optimize the decision interval as well.

2.1.1 The state-dependent gamma process

Let G be a non-decreasing stochastic process and assume that its evolution depends on its current level G_t in the following way. Let $\gamma(\tau, G_t) = \int_t^{t+\tau} \alpha(s, G_t) ds$ where $\alpha(s, G_t)$ is a non-decreasing function in s.

Definition 1 *A state-dependent gamma process G is a stochastic process that verifies*

(i) $G_0 = 0$*;*

(ii) $G_1 - G_0,\ G_2 - G_1,\ G_3 - G_2, \cdots$ *are independent;*

(iii) For a time interval of length $\tau > 0$, the distribution of $G_{t+\tau} - G_t$ depends only on G_t and is a gamma distribution with scale parameter β and a state-dependent shape parameter $\gamma(\tau, G_t)$.

Lemma 1 *The probability distribution of the state-dependent gamma process (Definition 1) is not infinitly divisible.*

The following definition of the infinite divisibility property is necessary for the proofs of Lemma 1 and Lemma 2.

Definition 2 *If $\{X_t, t \geq 0\}$ is a stochastic process and $\Psi_{X_t}(z)$, $z \in R$ its characteristic function, then, for every t, X_t has an infinitly divisible distribution if and only if for $n \in N$, $\Psi_{X_t}(z) = \left[\Psi_{X_{t/n}}(z)\right]^n$.*

Proof Let G be a state-dependent gamma process as in Definition 1. For $t > 0$, let $\Psi_t(z) = \Psi_{G_t}(z)$ denote its characteristic function. For $t > s$, we write $G_{t+s} = G_s + (G_{t+s} - G_s)$. As $(G_{t+s} - G_s)$ and G_s are not independent from (iii) of Definition 1 of the state dependent gamma process, then $\Psi_{t+s}(z) \neq \Psi_t(z)\Psi_s(z)$. This implies that the distribution of G_t, $t > 0$ is not infinitly divisible by Definition 2. □

We associate to the state-dependent gamma process defined in Definition 1 a growth rate process that we denote θ. The growth rate θ characterizes the expected right-variations of the state-dependant gamma process caused by its current state (Figure 1). Figure 1 illustrates the state-dependent gamma process. At time t_0, the process begins at zero and immediately and instantaneously jumps to G_1 according to θ_0 at t_1 and takes on a new DGR θ_1, etc.

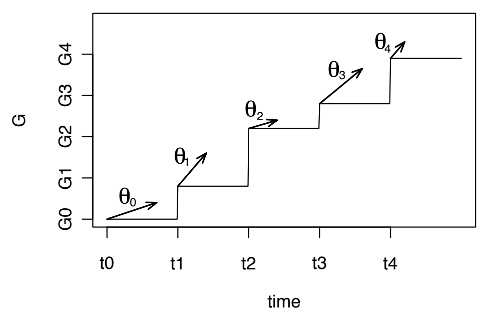

Figure 1: The growth process of state dependent gamma process

We define now the state of the system using the resulting biprocess (G, θ). Let G be a Lévy gamma process, θ its growth rate process and γ the function given by $\gamma(\tau, G_t, \theta_t) = \int_t^{t+\tau} \alpha(s, \theta_t, G_t)ds$ for $t > 0$ where $\alpha(s, \theta_t, G_t)$ is a non-decreasing function in s.

Definition 3 *The state-dependent gamma process with scale parameter $\beta > 0$ and cumulative shape function $\gamma(\tau, G_t, \theta_t)$ is a stochastic process with independent increments $G_{t+\tau} - G_t$ having the following density function for $x > 0$*

$$g(x; \gamma(\tau, G_t, \theta_t), \beta) = \frac{x^{\gamma(\tau, G_t, \theta_t)-1}\, e^{-x/\beta}}{\beta^{\gamma(\tau, G_t, \theta_t)}\, \Gamma\left(\gamma(\tau, G_t, \theta_t)\right)} \tag{1}$$

Lemma 2 *Let θ be a stochastic process and G be the state-dependent gamma process defined by Definition 3 with growth rate process θ. Then, the probability distribution of G is infinitly* *divisible.*

Proof Definition 3 implies that the increments of the state-dependent gamma process are entirely characterized by its growth rate process θ. Therefore, the increments of the state-dependent gamma process are independent and identically distributed, i.e, for t and $s > 0$, $(G_{t+s} - G_s)|\theta_s$ and $G_s|\theta_t$ are independent and identically distributed (i.i.d.). Moreover, as the sum of gamma distributed variables is a gamma distributed variable, then the SDGP is infinetly divisible. □

Introducing the deterioration growth process in a system reliability and maintenance context has the following two advantages: (i) The deterioration process is usually described by an observable metric which does not provide complete information. The growth process models the effects of the non-observable deterioration process on the observable one. (ii) In condition-based approaches, the effects of imperfect maintenance on system behaviour can be modeled through the deterioration growth process.

2.1.2 The gamma growth rate process as a bilateral gamma process

We choose to model the state-dependent gamma growth rate process by a bilateral gamma process (BGP). In fact the BGP is the difference of two independent gamma processes. This property can be used to derive directly the deterioration growth rate. The BGP is a class of Lévy processes that has been developed in the finance literature, specifically option pricing. It was first introduced by Küchler and Tappe [17] to model financial market fluctuations. To our knowledge, we are the first to use the BGP in a maintenance context.

The BGP is an extention to generalized hyperbolic process [4] and the variance gamma process (VGP) [18] proposed to replace classical Brownian motion to model price movements in financial markets. Unlike the BM process commonly used to model option pricing, the BGP has the advantage to explain both the jumps observed in stock movements on small time scales and the fact that the total amount of up and down moves is finite rather than infinite [14]. The BGP is defined as the difference of any two independent gamma processes [17]: $BG \sim G_p - G_n$ where G_p and G_n model the positive and the negative variations of the process. Thus, the BGP allows for positive and negative variations. It is a four-parameter process $\alpha_p, \beta_p, \alpha_n, \beta_n > 0$ with bilateral gamma distributed increments. The BG distribution density is given in [17] in terms of the Whittaker function studied in [11]. An advantage of the BGP over the VGP is that it is defined by the difference of any two independent gamma processes unlike the VGP which is constructed by subordinating a gamma Brownian motion and can be represented by the difference of two gamma processes with the same shape parameters necessarily.

In this work, the time horizon is discretized into equal intervals of length τ. The state-non-stationnary gamma process (Definition 3) is represented as a series of state-stationnary gamma processes in each time interval. Thus, the growth rate of the deterioration process can be seen as the process resulting from the gamma process variations from one time interval to another. More specifically, consider two consecutive epochs of length τ with (G_t, θ_t) and

$(G_{t+\tau}, \theta_{t+\tau})$ the system state at the beginning of each time interval. Let G_1 and G_2 be the cracking processes over these two decision intervals with state-dependent shape functions $\gamma(\tau, G_t, \theta_t)$ and $\gamma(\tau, G_{t+\tau}, \theta_{t+\tau})$, respectively. Without loss of generality, we assume that G_1 and G_2 have the same scale parameter β. Thus, we can then define the deterioration growth rate process by the SDGP changes over two consecutive time intervals and therefore, as the difference of two gamma processes G_n and G_p with respective shapes γ_n and γ_p that verify

$$\begin{cases} \Delta G \sim SDG(\gamma(G_t, \theta_t, \tau), \beta) \\ \Delta\theta \sim BG(\gamma_n(G_t, \tau), \gamma_p(G_t, \Delta G, \tau), \beta) \end{cases} \tag{2}$$

To use the BGP to model the gamma growth rate process, the independence assumption of the two gamma processes of negative and positive variations should be justified. If G_1 a state-dependent gamma process, its growth process θ tells us how G_1 is going to evolve to a different state-dependent gamma process G_2, more specifically $G_2|G_1$. From the following Lemma, G_1 and $G_2|G_1$ are independent.

Lemma 3 *Let G and D be two dependent processes, then G and $D|G$ are independent.*

Proof For any $t > 0$, let $Z_t = D_t|G_t$

$$\begin{aligned} F_{G_t,Z_t}(x,z) &= f_{Z_t}(z|x)\, f_G(x) \\ &= f_{Z_t}(z)\, f_{G_t}(x) \end{aligned}$$

Therefore, G_t and $D_t|G_t$ are independent. □

2.2 The initiation phase: a state-non-homogenous Poisson process

The initiation phase can be regarded as a specific case of the propagation phase. Therefore, we can apply the same deterioration model with two natural modifications: replacing (i) the BGP for the deterioration growth rate by a gamma process and (ii) the state-dependent gamma process by a conditional Poisson process.

(i) In the initiation phase, θ is an expected instantaneous growth rate of observable degradation given that the deterioration current level is zero. Therefore, similar to the deterioration growth rate in the propagation phase, we model the initiation potential in the first phase as a BGP. Using the fact that the BGP is a difference of two gamma processes and given that the gamma process of negative variations G_n is equal to zero as no degradation is observed before initiation, the BGP is equivalent to a single stationnary gamma process. This result has the advantage to ensure the initiation potential θ to be strictly increasing process as long as no initiation is observed.

(ii) In the propagation phase, the gamma process is used to model the evolution of observable degradation ρ as the result of an infinite number of jump arrivals. However, in the initiation phase, we are interested in a finite number of shock arrivals, namely the time of the first observable deterioration arrival. As the gamma process is an infinite activity process near zero, we can not use it to derive the distribution of the time of the first observable degradation

arrival. Using both the property that gamma process can be regarded as a limit of a compound Poisson process [8] and the fact that we are interested in the first arrival, we replace the state-dependent gamma process defined in the propagation phase by a θ-conditional Poisson process which is a non-homogeneous process N with a θ-dependent intensity function.
For a time interval of length $\tau > 0$, the density of the Poisson process N is modified by the addition of θ as follows

$$Pr\{N_{t+\tau} - N_t = n|\theta_t\} = \frac{\Lambda(\tau, \theta_t)^n}{n!} \exp\left(-\Lambda(\tau, \theta_t)\right), \tag{3}$$

where θ is the initiation potential at the beginning of the time interval τ and $\Lambda(\tau, \theta)$ is the cumulative intensity function of the Poisson process. It is obvious that the Poisson Process as in (3) has an infinitely divisible distribution.
Let T denote the random variable of the time of the first observable degradation arrival. Therefore, the distribution of the first observable degradation occurrence within a time interval $[t, t+\tau]$ given that the initiation potential at the beginning of this interval is θ_t is given for $t \leq s \leq t+\tau$ by

$$\begin{aligned} F_T(s; \theta_t) &= Pr\{T < s|\theta_t\} \\ &= 1 - Pr\{N_s - N_t = 0|\theta_t\} \\ &= 1 - \exp\left(-\Lambda(s-t, \theta_t)\right). \end{aligned} \tag{4}$$

3 Maintenance Optimization Model

Let ρ_t be the positive measure of the degradation at time t and θ_t be the associated instantaneous deterioration growth rate. During the propagation phase, θ_t can be assimilated to the random instantaneous speed of deterioration whereas, during the initiation phase, because no degradation is observable, we assimilated θ_t to the expected instantaneous growth rate. We assume that the observable degradation ρ as well as the initiation potential of the first phase θ are non-decreasing in time, whereas the degradation growth rate θ is not necessarily monotone in time during propagation, i.e, θ may increase or decrease.

To maintain the system, we consider a periodic inspection problem with both perfect and imperfect maintenance actions. At the beginning of any decision epoch of length $\tau > 0$, the system is perfectly and instantaneously inspected at a fixed cost c_i and only an observation of ρ is yielded. Let N be the total number of available maintenance actions. After inspection, the decision-maker may opt to do nothing (DN), perform one of (N−1) available imperfect maintenance actions $(MX_k)_{k \in \{1,2,\cdots,N-1\}}$, that have different impacts on the system state and its future deterioration rate, or a perfect action (MX_N) that resets the system to an as-good-as-new (AGAN) state. The cost of any maintenance action does not depend on the system state and is given by $c(k), k \in \{1, 2, \cdots, N\}$, an increasing function of the maintenance level. Moreover, a cost of quality $c_q(\cdot)$ which is an immediate expected cost cumulated over the decision epoch is incurred and function of the deterioration level.

A DN-action has no impact onto the current system state or the future deterioration process. After both perfect and imperfect maintenance actions, the observable degradation level ρ is reset to zero but the system is still endowed with an initiation potential. We assume that the immediate initiation potential after a maintenance is performed is a deterministic function of both the maintenance level and the amount of deterioration before maintenance denoted $\phi(\rho, \theta, k)$.

We define the system state by $s = (\rho, \theta, m)$ where ρ is the current deterioration level, θ is the initiation potential if $\rho = 0$ and the deterioration growth rate otherwise and m is the last performed maintenance. m ensures the Markovian property of the state. Let S and A denote the set of the system states and the set of available actions, respectively.

We model both the total deterioration percentage and the deterioration growth rate as dependent stochastic processes. Let $h(x, y; \rho, \theta, m)$, $(x, y) \in \mathbb{R}^+ \times \mathbb{R}$ be the joint density of the processes $\{\rho_t, t \geq 0\}$ and $\{\theta_t, t \geq 0\}$ when the current state at observation is (ρ, θ, m) and the DN-action is selected. We assume that transition from the state s occurs at the end of the decision interval. Let $s' = (\rho' = \rho + x, \theta' = \theta + y, m)$ be the state of the system at the end of the current decision epoch.

The objective is to find the optimal policy π^* that maps the system states set S to the set of actions A and determines the optimal inspection interval τ^* to minimize the expected discounted cost-to-go over the infinite horizon, expressed as,

$$V^{\pi^*}(\rho, \theta, m) = c_i + \min_{\tau \in \mathbb{R}^+} \{DN(\rho, \theta, m), \min_k MX_k(\rho, \theta, m)\}, \tag{5}$$

where

$$\begin{aligned} DN(\rho, \theta, m) \ =& \ \int_x\!\!\int_y h(x, y; \rho, \theta, m) \times \\ & \left[c_q(\rho, \rho + x, \tau) + \lambda V^{\pi^*}(\rho + x, \theta + y, m)\right] dydx, \end{aligned} \tag{6}$$

$$MX_k(\rho, \theta, m) = c(k) + DN\big(0, \phi(\rho, \theta, k), k\big), \quad k \in \{1, 2, \cdots, N\} \tag{7}$$

where $\lambda \in [0, 1]$ is the discount factor. Equation (6) states that following the DN action when the current state is s incurs an immediate quality cost plus the cost-to-go to state s' by the end of the decision epoch. The MX_k action (7), $k \in \{1, 2, \cdots, N\}$ incurs maintenance cost $c(k)$ plus the DN-cost of a system with ρ reset to zero and θ to $\phi(\rho, \theta, k)$.
In section 4.2, we solve equation (5) by discretizing the state space and the decision interval length τ and using the policy iteration algorithm [27].

4 Application to road maintenance

4.1 Road deterioration and maintenance activities

Road network maintenance management represents an important economic issue. It is, for example, it is reported in [35] that dramatic cost savings have been achieved by the Pavement Management System (PMS) developed by Golbai *et al.* [10] for the Arizona Department of Transportation and Woodward-Clyde Consultants.

In this section we apply the maintenance decision framework presented in Section 3 with the state-dependent deterioration model of section 2 to roads subject to a single deterioration mode, namely longitudinal cracking. Longitudinal wheel-path cracks in the pavement are the main cause of road maintenance [6]. According to [21], 90% of road sections in Florida are in need of repair because of longitudinal cracks. We first give a brief description of the cracking process as a two-phase process. We then apply our two-phase deterioration model to a given road section and use this deterioration model to solve the road section maintenance planning as a Markov decision problem. A numerical example is given to illustrate our model.

Repeated traffic loads applied to the pavement surface generate tensile stresses at the bottom of the asphalt layer [6] and lead micro-cracks to be initiated overtime in the road foundation when the stress applied by traffic load exceeds the asphalt resistance. These micro-cracks then propagate through the asphalt layer until reaching the road suface where they give way to observable longitudinal cracks. Thus, the road cracking process is a two-phase deterioration process: (i) the crack initiation phase during which the road surface remains crack-free and (ii) the crack propagation phase that begins with the first surface crack arrival.

To maintain the road, an inspection is first performed to evaluate its deterioration level by measuring a total crack length L_c (Figure 2): L_c is the length of the projection of all the cracks on the road axis. Figure 2 illustrates a deteriorated road section and the crack length metric as the total longitudinal projection of cracks. Note that overlaping cracks are not taken into account with this metric. The road is then divided into equal sections of length

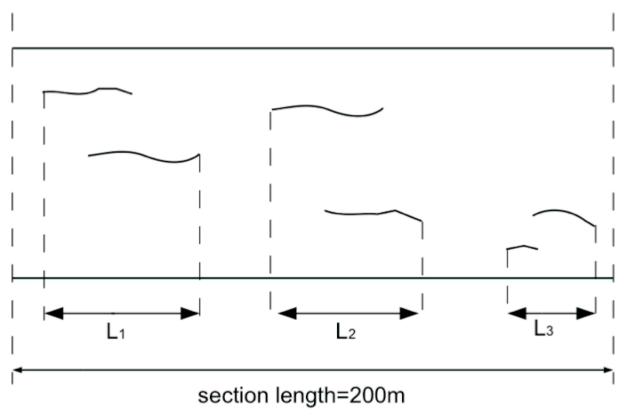

Figure 2: Total longitudinal cracking: $L_c = L_1 + L_2 + L_3$

$L_s = 200m$ and a longitudinal cracks percentage (LCP), $\rho \in [0, 1]$, is defined in the french IQRN (Indices image Qualité du Réseau routier National) database [1] as $\rho = L_c/L_s$. We also consider the deterioration growth rate that we denote θ such that $\theta \geq 0$. The deterioration growth rate is bounded by a maximum growth rate θ_{max}.

For $\rho \in [0, 1]$ and $\theta \in [0, \theta_{max}]$, their increments over a time interval verify $\Delta\rho \in [0, 1 - \rho]$ and $\Delta\theta \in [0, \theta_{max} - \theta]$ in the initiation phase and $\Delta\theta \in [-\theta, \theta_{max} - \theta]$ in the propagation phase. Hence, their probability densities within an interval of length τ are truncated as follows to accomodate the bounds defined above.

When the current state is $(\rho, \theta) \in [0, 1] \times [0, \theta_{max}]$, the truncated density of the state-dependent gamma process, denoted $\hat{g}$ is given by:

$$\hat{g}(x;\gamma(\tau,\rho,\theta),\beta)=\begin{cases} g(x;\gamma(\tau,\rho,\theta),\beta) & x\in[0,1-\rho[\\ \\ 1-\int_0^x g(u;\gamma(\tau,\rho,\theta),\beta)du & x=1-\rho \end{cases} \quad (8)$$

where g is the density function of the gamma distribution.

The density function of the truncated BG distribution, denoted $\hat{f}$, is given in the propagation phase for $x\in]0,1-\rho]$,$y\in[-\theta,\theta_{max}-\theta]$ by

$$\hat{f}(y;\gamma_n(\tau,\rho),\gamma_p(\tau,\rho,x),\beta) = \frac{f(y;\gamma_n(\tau,\rho),\gamma_p(\tau,\rho,x),\beta)}{\int_{-\theta}^{\theta_{max}-\theta} f(u;\gamma_n(\tau,\rho),\gamma_p(\tau,\rho,x),\beta)du} \quad (9)$$

For the initiation phase, $\hat{f}$ is given for $y\in[0,\theta_{max}-\theta]$ by

$$\hat{f}(y;0,\tilde{\gamma}_p(\tau,\theta_0,m),\beta) = \frac{g(y;\tilde{\gamma}_p(\tau,\theta_0,m),\beta)}{\int_0^{\theta_{max}-\theta} g(u;\tilde{\gamma}_p(\tau,\theta_0,m),\beta)du} \quad (10)$$

where $f(y;\gamma_n(\tau,\rho),\gamma_p(\tau,\rho,x),\beta)$ denotes the density of the bilateral gamma process and $\tilde{\gamma}_p(\tau,\theta_0,m)$ is the shape function of the stationnary gamma process that models the DGR.

After inspection, a decision-maker must select a maintenance action among a set of available actions, e.g., $A=\{DN,MX_1,MX_2,MX_3,MX_4\}$ where MX_4 is the complete renewal of the road section and MX_k, $k\in\{1,2,3\}$ are three imperfect maintenance actions that consists in renewing different thicknesses of the road section.

The different imperfect maintenance actions correspond to different layer thicknesses that can be applied to the road surface. Their impact depend on the thickness of the added layer and are such that $MX_3>MX_2>MX_1$. Both perfect and imperfect actions reset the value of the cracking percentage to zero as the cracks are concealed, but only the perfect action restores the road to an AGAN condition represented by $(0,\theta_0,m)$ where θ_0 is the cracking growth rate of a new road section. The imperfect maintenance effects are modeled through the deterioration growth rate process θ as it is not reset to zero but to a new value depending on the performed action and the prior state. Figure 3 illustrates the decision procedure and the effect of different decisions on the road state and its future behavior. (ρ_τ,θ_τ) and (ρ_k,θ_k) are the road state just before and after maintenance, respectively. After the $DN-$action (Figure 3-a), the state is unchanged as an the deterioration process depends on (ρ_k,θ_k). Whereas, after a maintenance (Figure 3-b), ρ is reduced to zero, the DGR changes to θ_k, and the road deteriorates according to a new law that depends on both (ρ_k,θ_k) and the type of the maintenance performed.

4.2 Derivation of transition probabilities for the MDP

We now derive the transitions over a decision epoch of length τ from an observed state $s=(\rho,\theta,m)$ at the beginning the decision epoch. The joint density of the LCP and the DGR

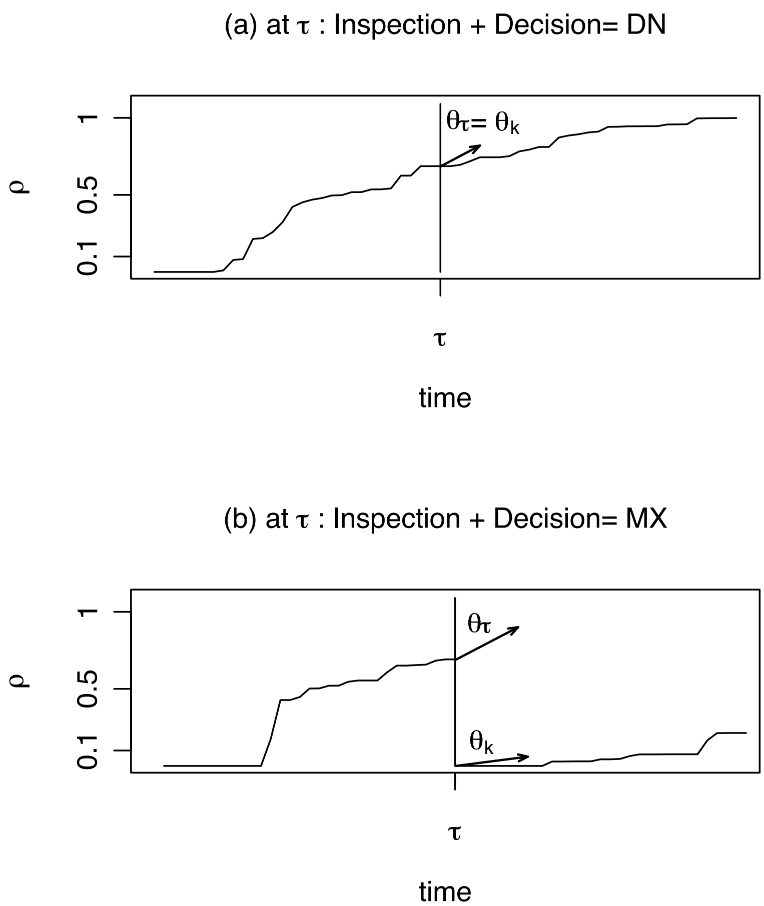

Figure 3: Maintenance effects on the LCP ρ and the deterioration growth rate θ.

processes is:

$$h(x, y; \rho, \theta, m) = \hat{g}(x; \gamma(\tau, \rho, \theta, m))\ \hat{f}(y; \gamma_n(\tau, \rho, m), \gamma_p(\tau, \rho, x, m), \beta).$$

To compute the transition probabilities, we propose to discretize the problem as follows. Let $s = \big(i_\rho, i_\theta, m\big)$ be the system state at the begining of the current decision epoch where i_ρ and i_θ represent the first values in N_ρ and N_θ equal size discrete intervals of length l on $[0, 1]$ and $[0, \theta_{max}]$, respectively such that N_ρ and N_θ divide evenly into 100. As $\rho = 0$ is a key-state, we consider it as a single state , i.e: $i_\rho \in \big\{0,]0, l[, [l, 2l[, \cdots, [(N_\rho - 1)l, 1]\big\}$ and $i_\theta \in \big\{[0, l[, [l, 2l[, \cdots, [(N_\theta - 1)l, \theta_{max}]\big\}$. Let $p^m_{ss'}$ denote the transition probability from the state s to $s' = \big(i_{\rho'}, i_{\theta'}, m\big)$ when the last performed action is m. Three cases are possible: (i) the road section remains free of cracks during the decision epoch, i.e, $\rho = 0$ and $\rho' = 0$, (ii) the cracking process is initiated during the current decision epoch, i.e. $\rho = 0$ and $\rho' > 0$ and (iii) the cracks are already initiated at the beginning of the current decision epoch, i.e. $i_\rho > 0$ and $i_{\rho'} > 0$.

(i) $\rho = 0$ and $\rho' = 0$

$$\begin{aligned} p^m_{ss'} &= Pr\big\{s' \mid s\big\} \\ &= Pr\big\{\rho'{=}0, \theta' \in [i_{\theta'}, i_{\theta'}{+}l] \mid \rho{=}0, \theta{=}i_\theta, m\big\} \\ &= Pr\big\{\rho'{=}0 \mid \rho{=}0, \theta{=}i_\theta, m\big\}\ Pr\big\{\theta'{\in}[i_{\theta'}, i_{\theta'}{+}l] \mid \rho'{=}0, \theta{=}i_\theta, m\big\} \end{aligned} \tag{11}$$

The first factor in the right-hand side of equation (11) is the probability that no crack appears in the current period and is given by the Poisson process (3) of the initiation phase:

$$Pr\{i_{\rho'} = 0 |\ i_\rho = 0, i_\theta, m\} = \exp(-\Lambda(\tau, i_\theta, m)).$$

The second factor is given by the density of the gamma process equivalent to the BGP in the initiation phase (10)

$$Pr\{\theta'{\in}[i_{\theta'}, i_{\theta'}{+}l] | \rho'{=}0, m\} = \int_{i_{\theta'}-i_\theta}^{i_{\theta'}-i_\theta+l} \hat{f}(y; 0, \tilde{\gamma}_p(\tau, \theta_0, m), \beta) dy$$

Therefore, when no crack is initiated in the current decision epoch of length τ.

$$p^m_{ss'}{=} \exp(-\Lambda(\tau, i_\theta, m))\int_{i_{\theta'}-i_\theta}^{i_{\theta'}-i_\theta+l} \hat{f}(y; 0, \tilde{\gamma}_p(\tau, \theta_0, m), \beta) dy \tag{12}$$

(ii) $\rho = 0$ and $\rho' > 0$

$$\begin{aligned} p^m_{ss'} &= Pr\big\{\rho'{\in}[i_{\rho'}, i_{\rho'}{+}l], \theta'{\in}[i_{\theta'}, i_{\theta'}{+}l] \mid \rho{=}0, \theta{=}i_\theta, m\big\} \\ &= Pr\big\{\rho'{\in}[i_{\rho'}, i_{\rho'}{+}l] \mid \rho = 0, \theta = i_\theta, m\big\} \times \\ &\quad Pr\big\{\theta'{\in}[i_{\theta'}, i_{\theta'}{+}l] \mid \rho'{\in}[i_{\rho'}, i_{\rho'}{+}l], \rho{=}0, m\big\} \end{aligned} \tag{13}$$

Let T be a random variable representing the first crack time arrival. Using (4), we have

$$\begin{aligned} Pr\{T < t \mid 0 \le T \le \tau\} &= \frac{Pr\{0 \le T \le t\}}{Pr\{0 \le T \le \tau\}} \\ &= \frac{1 - \exp(-\Lambda(t, i_\theta, m))}{1 - \exp(-\Lambda(\tau, i_\theta, m))} \end{aligned} \tag{14}$$

The density function of the first crack arrival time distribution is then derived from (14)

$$h_T(t) = \Lambda'(t, i_\theta, m) \; \frac{\exp(-\Lambda(t, i_\theta, m))}{1 - \exp(-\Lambda(\tau, i_\theta, m))} \; ,$$

where $\Lambda'(t, i_\theta, m) = d\Lambda(t, i_\theta, m)/dt$ is the intensity rate function of the Poisson process. As the initiation may happen at any time $t \in [0, \tau]$, the transition probability (13) is then

$$\begin{aligned} p^m_{ss'} \;\; = \;\; & \int_0^\tau h_T(t) \int_{i_{\rho'}}^{i_{\rho'}+l} \hat{g}(x; \gamma(\tau - t, i_\rho, i_\theta, m), \beta) \times \qquad (15) \\ & \int_{i_{\theta'} - i_\theta}^{i_{\theta'} - i_\theta + l} \hat{f}(y; \gamma_n(\tau - t, 0, m), \gamma_p(\tau, 0, x, m), \beta) dy dx dt. \end{aligned}$$

Note that, in (15), we assume that the level of the GDR remains constant even after crack initiation.

(iii) $i_\rho > 0$ and $i_{\rho'} > 0$

$$\begin{aligned} p^m_{ss'} \;\; = \;\; & Pr\big\{\rho' \in [i_{\rho'}, i_{\rho'} + l] \mid \rho = i_\rho, \theta = i_\theta, m\big\} \times \\ & Pr\big\{\theta' \in [i_{\theta'}, i_{\theta'} + l] \mid \rho' \in [i_{\rho'}, i_{\rho'} + l], \rho = i_\rho, m\big\} \,. \end{aligned}$$

The first factor is given by the state dependent gamma process with density (8) and the second one is given by the density (9) of the BGP.

$$\begin{aligned} p^m_{ss'} \;\; = \;\; & \int_{i_{\rho'} - i_\rho}^{i_{\rho'} - i_\rho + l} \hat{g}(x; \gamma(\tau, i_\rho, i_\theta, m), \beta) \\ & \int_{i_{\theta'} - i_\theta}^{i_{\theta'} - i_\theta + l} \hat{f}(y; \gamma_n(\tau, i_\rho, m), \gamma_p(\tau, i_\rho, x, m), \beta) dy dx \qquad (16) \end{aligned}$$

5 Numerical example

The following numerical experiment is based on generic data. Although the different functions as well as the values of the associated parameters are chosen for highlighting the implementation process and the associated maintenance policy, their constructions are motivated by expert knowledge on cracking and road maintenance practices.

The shape function of the state-dependent gamma process is proportional to the average crack growth given the current state:

$$\gamma(\tau,\rho,\theta,m)=\left(r_m+\frac{a_1\theta}{(a_2-a_3\theta)+\big(\rho-(1-\theta)\big)^2}\right)\tau, \tag{17}$$

where a_1 and a_2 are constants and r_m is a factor that reflects the decreasing acceleration of cracking in the last performed action.

Note that the shape function in (17) is the sum of a first term that represents the time effect

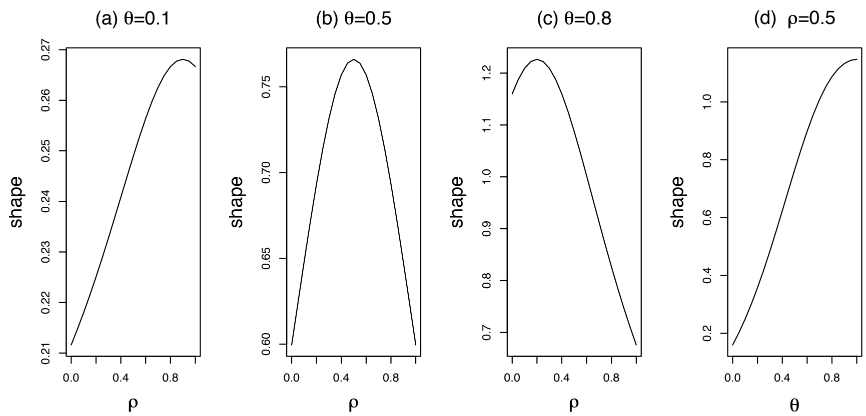

Figure 4: Different shapes of the shape function γ in ρ for different fixed values of θ (a,b,c) and in θ for a fixed value of ρ (d), with $a_1 = 0.5,\ a_2 = 0.76,\ a_3 = 0.2$

and depends on the last performed action, and a second term that represents the state effect on the deterioration process. Moreover, note from Figure 4 that the shape function is non decreasing in the DGR θ but not necessarily monotone in ρ. Depending on the level of θ, the expected deterioration increases first then decreases in ρ. The choice of such a shape function can be motivated by the fact that newly-appearing cracks have more chance to modify the value of the LCP when its initially small than when it is large (as overlaping cracks do not have an effect on the LCP). Moreover, from the three cases ((a)-(c)) in Figure 4, the expected $\Delta\rho$ increases in θ but decreases faster when θ is high as the road deteriorates faster and so it becomes harder for new cracks to change the LCP.

The shape functions used for the BGP are

$$\begin{aligned}\gamma_n(\tau,\rho,m) &= \frac{\gamma(\tau,\rho,\frac{\rho}{\tau},m)}{\tau}\\ \gamma_p(\tau,\rho,x,m) &= \frac{\gamma(\tau,\rho+x,\frac{\rho+x}{\tau},m)}{\tau}\end{aligned}$$

so that the expected variation in θ which is proportional to $(\gamma_p(\tau,\rho,x,m)-\gamma_n(\tau,\rho,m))$ varies as a function of LCP variations x as shown in Figure 5. Note that expected variations in θ increase for relatively (i.e., relatively to the level of ρ) low levels of x, and this increase is

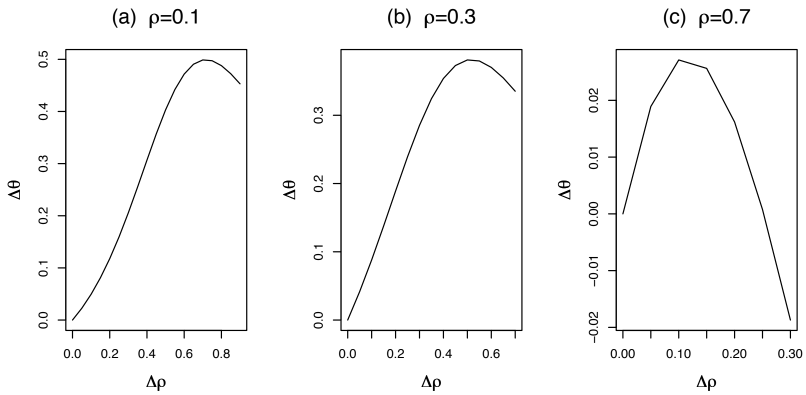

Figure 5: Expected variations in θ for different levels of ρ with $a_1 = 0.5,\ a_2 = 0.76,\ a_3 = 0.2$.

accelerated then decelerated. For relatively large levels of x, expected variations in θ decrease and can even be negative for high levels of ρ, i.e., for highly cracked sections.

The cumulated intensity function of the Poisson process:

$$\Lambda(\tau, \theta, m) = (b_{_1} + r_{_m} + \theta)\ \tau,$$

Here, we also choose an intensity function that is the sum of a state effect term and a time effect one.
To update the initiation potential at the end of the decision interval when no cracks appear, we use the following shape function which reflects the increase in potential during the first phase as a function of its current state:

$$\tilde{\gamma}_p(\tau, \theta_0, m) = \Lambda(\tau, \theta_0, m)\ .$$

The effect of a maintenance action k on the deterioration growh rate for a given state (ρ, θ) is modeled by

$$\phi(\rho, \theta, k) = \theta_0 + (0.65\rho + 0.35\theta) r_k$$

where r_k is a reduction factor that ensures that $\phi(\rho, \theta, k) \le \theta_{max}$.
The cost of imperfect and perfect actions is given by: $c(k) = (200, 450, 1080, 2950)$
The quality cost is a function of the LCP level at the beginning and the end of a decision epoch

$$c_q(\tau, \rho, \rho') = (c_1\rho + c_2\rho')\ \tau$$

where c_1 and c_2 are constants.
We also consider: $\theta_{max} = 1,\ \theta_0 = 0.03,\ \beta{=}1.5,\ \lambda = 0.97$.

To solve this numerical example, the possible values of the decision interval length τ are restricted to the range $[1.2, 1.4, 1.6, 1.8, 2, 2.2, 2.4, 2.6]$. Using the numerical approximations described in Appendix A, the optimal policy given by the policy iteration algorithm applied to the range of values τ is given by the four decision matrices presented in Tables 1-4 with an optimal decision interval length $\tau* = 1.8$.

Table 1: Optimal maintenance decisions for $m = 1$

	0.90	4	4	4	4	4	4	4	4	4	4
	0.80	3	3	3	3	3	3	3	3	3	3
	0.70	2	2	2	2	2	2	2	2	2	2
	0.60	2	2	2	2	2	2	2	2	2	2
	0.50	2	2	2	2	2	2	2	2	2	2
ρ	0.40	2	2	2	2	2	2	2	2	2	2
	0.30	2	2	2	2	2	2	2	2	2	2
	0.20	2	2	2	2	2	2	2	2	2	2
	0.10	2	2	2	2	2	2	2	2	2	2
	0.03	1	1	1	1	1	1	1	1	1	1
	0.00	0	1	1	1	1	1	1	1	1	1
		0.03	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
						θ					

Table 2: Optimal maintenance decisions for $m = 2$

	0.90	4	4	4	4	4	4	4	4	4	4
	0.80	3	3	3	3	3	3	3	3	3	3
	0.70	2	2	2	2	2	2	2	2	2	2
	0.60	2	2	2	2	2	2	2	2	2	2
	0.50	2	2	2	2	2	2	2	2	2	2
ρ	0.40	2	2	2	2	2	2	2	2	2	2
	0.30	2	2	2	2	2	2	2	2	2	2
	0.20	2	2	2	2	2	2	2	2	2	2
	0.10	0	0	1	1	1	1	1	1	1	1
	0.03	0	0	1	1	1	1	1	1	1	1
	0.00	0	0	0	1	1	1	1	1	1	1
		0.03	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
						θ					

Table 3: Optimal maintenance decisions for $m = 3$

	0.90	4	4	4	4	4	4	4	4	4	4
	0.80	3	3	3	3	3	3	3	3	3	3
	0.70	2	2	2	2	2	2	2	2	2	2
	0.60	2	2	2	2	2	2	2	2	2	2
	0.50	2	2	2	2	2	2	2	2	2	2
ρ	0.40	2	2	2	2	2	2	2	2	2	2
	0.30	1	1	1	1	1	1	2	2	2	2
	0.20	0	1	1	1	1	1	1	1	1	1
	0.10	0	0	1	1	1	1	1	1	1	1
	0.03	0	0	0	1	1	1	1	1	1	1
	0.00	0	0	0	1	1	1	1	1	1	1
		0.03	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
						θ					

Table 4: Optimal maintenance decisions for $m = 4$

	0.90	4	4	4	4	4	4	4	4	4	4
	0.80	3	3	3	3	3	3	3	3	3	3
	0.70	2	2	2	2	2	2	2	2	2	2
	0.60	2	2	2	2	2	2	2	2	2	2
	0.50	2	2	2	2	2	2	2	2	2	2
ρ	0.40	2	2	2	2	2	2	2	2	2	2
	0.30	1	1	1	1	1	1	1	1	1	1
	0.20	0	1	1	1	1	1	1	1	1	1
	0.10	0	0	0	1	1	1	1	1	1	1
	0.03	0	0	0	1	1	1	1	1	1	1
	0.00	0	0	0	1	1	1	1	1	1	1
		0.03	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
						θ					

The four decision matrices above provide the optimal maintenance action for each deterioration vector (ρ, θ, m). For example, for an observed $(\rho, \theta) = (0.2, 0.5)$, the best maintenance action to perform is MX_2 when the last performed action is $m = 1$ or $m = 2$ (Tables 1 and 2), and MX_1 when $m = 3$ or $m = 4$ (Tables 3 and 4), respectively.

Note that the difference between the four matrices is especially clear for the small values of ρ and θ. The stronger the last maintenance action is, the weaker the selected action is when the deterioration level is low. For example, for a deterioration level $(\rho = 0.2, \theta = 0.1)$, the optimal action is MX_2 when $m = 1$ (Table 1), MX_1 when $m = 3$ (Table 3) and DN when $m = 4$ (Table 4). For higher deterioration levels, the effect of the last performed action is not important, e.g., optimal actions for states with $\rho \geq 0.4$ are the same for all last maintenance types.

The presence of the MX_4-action in the optimal policy ensure that the road will be renewed at some time. Numerical examples have shown that our policy is sensitive to the state-dependent deterioration model parameters especially the choice of the shape function for the state-dependent gamma process and the intensity function of the Poisson process. However, the policy is quite robust with respect to the cost parameters. Moreover, numerical examples have shown the potential convexity of some maintenance regions and the possible existence of some structural properties with respect to conditions upon cost functions.

6 Conclusion

In this paper, we have developed a condition-based maintenance optimization approach for cumulative deteriorating systems based on a Markov decision process. A special feature of our MDP is that its transition matrix is obtained using a state-dependent deterioration model.

We introduce a state-dependent deterioration process for cumulative deteriorating systems based on a state-dependent gamma process and its growth rate process as a bilateral gamma process. We use the property of the BGP of being the difference of two independent gamma processes in order to derive its parameters eliminat the need for large amount of statistical data. One contribution of this model is that it takes into account not only the observable deterioration but also the underlying one. Moreover, having the deterioration depend on the system state at inspection as well as the effectiveness of the last maintenance action allows for a new deterioration law in every decision epoch. Using this model, we have formulated a multi-action imperfect maintenance policy for inspection and maintenance decision. This maintenance model has been applied to road degradation problem to optimize the maintenance decision for a road section subject to longitudinal cracking.

As future work, we hope to derive conditions under which some structural properties with respect to the two deterioration parameters exist. Moreover, this model can be extended, in the case of road maintenance problem, by introducing a road thickness constraint and the possibility to remove road before adding a new layer.

A Numerical approximations

To make the Policy Iteration algorithm faster, we make the following numerical approximations:

- The integral over the time t in (15) is numerically approximated by

$$\frac{\tau}{8}\sum_{k=1}^{8} h_T(t_k)\int_{i_{\rho'}}^{i_{\rho'}+l} g(x;\gamma(\tau-t_k,i_\rho,i_\theta,m),\beta)\times$$
$$\int_{i_{\theta'}-i_\theta}^{i_{\theta'}-i_\theta+l} f(y;\gamma_n(\tau-t_k,i_\rho,m),\gamma_p(\tau,i_\rho,x,m),\beta)dy\,dx$$

  where $t_k = \frac{(2k+1)\tau}{16}$, $k \in \{0,1,2,\cdots,7\}$

- The double integral in (15) and (16) as follows :

$$\int_{i_{\rho'}-i_\rho}^{i_{\rho'}-i_\rho+l} g(x;\gamma(\tau,i_\rho,i_\theta,m),\beta)\times\int_{i_{\theta'}-i_\theta}^{i_{\theta'}-i_\theta+l} f(y;\gamma_n(\tau,i_\rho,m),\gamma_p(\tau,i_\rho,x,m),\beta)dydx =$$
$$\frac{1}{2}\int_{i_{\rho'}-i_\rho}^{i_{\rho'}-i_\rho+l} g(x;\gamma(\tau,i_\rho,i_\theta,m),\beta)dx\times\begin{bmatrix}\int_{i_{\theta'}-i_\theta}^{i_{\theta'}-i_\theta+l} f(y;\gamma_n(\tau,i_\rho,m),\gamma_p(\tau,i_\rho,x_1,m),\beta)dy+\\ \int_{i_{\theta'}-i_\theta}^{i_{\theta'}-i_\theta+l} f(y;\gamma_n(\tau,i_\rho,m),\gamma_p(\tau,i_\rho,x_2,m),\beta)dy\end{bmatrix}$$

  where $x_1 = i_{\rho'} - i_\rho$ and $x_2 = i_{\rho'} - i_\rho + l$

- The Whittaker function $W_{a,b}(z)$ studied in [11] can be written in terms of the Whittaker function $M_{a,b}(z)$ and it holds that:

$$W_{a,b}(z)=\frac{\Gamma(-2b)}{\Gamma(\frac{1}{2}-b-a)}M_{a,b}(z)+\frac{\Gamma(2b)}{\Gamma(\frac{1}{2}+b-a)}M_{a,-b}(z)$$

  where the Whittaker function $M_{a,b}(z)$ verifies [11]

$$M_{a,b}(z) = z^{b+\frac{1}{2}}e^{-\frac{z}{2}}\Phi(b-a+\frac{1}{2},2b+1;z)$$

  with $\Phi(c,d;z)$ denoting the confluent hypergeometric function [11]

$$\Phi(c,d;z) \;=\; 1+\frac{c}{d}\frac{z}{1!}+\frac{c(c+1)}{d(d+1)}\frac{z^2}{2!}+\cdots \tag{18}$$

  To approximate the Whittaker function $W_{a,b}(z)$, we use the series representation (18) to the level 9.

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5 Optimizing Milling and Resurfacing Decisions

Optimizing Road Milling and Resurfacing Actions

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Abstract

A condition-based maintenance optimization approach is developed for the road-cracking problem in order to derive optimal maintenance policies that minimize a total discounted maintenance cost. The approach is based on a Markov decision process that takes into account multiple actions with varying effects on future road performance. Maintaining the road consists of adding a new asphalt layer; however, as resurfacing actions are constrained by a maximum total road thickness, the maintenance decision is not only how thick a layer to apply, but also how much old road to remove. Each combination of these actions leads to different maintenance costs and different future degradation behaviours. The road state is modelled by a dependent bivariate deterioration variable (the longitudinal cracking percentage and the deterioration growth rate), for taking these different changes in the cracking patterns into account. Moreover, the sensitivity to cracking for existing roads can be reduced with the addition of new layers, and thus actions that can lead to states better than good-as-new have to be considered. A numerical analysis is provided to illustrate the benefits of the introduction of the deterioration speed in the decision framework, as well as the belief that initially building a road to its maximum thickness is not optimal. The trade-offs in the design decisions and the exploitation/maintenance costs are also explored.

Keywords: road cracking, stochastic deterioration, maintenance optimization, Markov decision processes

Notations

c_0	inspection cost	$\alpha(\cdot)$	cumulative hazard function of the Poisson process
c_1	fixed costs of milling	ℓ	section thickness
c_2	fixed costs of resurfacing	ℓ_{max}	maximum section thickness
$c_a(\xi)$	variable cost of adding thickness ξ	$\gamma_1(\cdot)$	linear-form shape function of the state-dependent gamma process
$c_r(\zeta)$	variable cost of removing thickness ζ	$\gamma_2(\cdot)$	gaussian-form shape function of the state-dependent gamma process
$c_q(\cdot)$	quality cost	Δ_θ	length of discrete intervals of θ
$f(\cdot)$	density function of θ in both phases	Δ_ρ	length of discrete intervals of ρ
$g(\cdot)$	density function of ρ in both phases	ζ	milling thickness
$h(\cdot)$	joint (ρ, θ)density function in both phases	θ	deterioration growth rate
i_θ	discrete value of θ	θ_{max}	maximum growth rate
i_ρ	discrete value of ρ	λ	discount factor
N_θ	number of possible values of θ	ξ_m	last added thickness
N_ρ	number of possible values of ρ	π	maintenance policy
S	set of states	ρ	longitudinal cracking percentage
s	state of the section	τ	decision epoch length
V	Total expected discounted cost	ξ	resurfacing thickness
		$\phi(\cdot)$	maintenance effect function on the deterioration growth rate

1 Introduction

During recent decades, transportation administrations of several countries have made continuing efforts to develop and implement efficient pavement management systems (PMS) in order to define the best maintenance policies that ensure acceptable use and safety conditions of the road in the most cost-effective way possible. A survey of existing PMSs is presented in reference [2]. According to reference [9], a PMS consists of four main components: the inventory, the pavement condition evaluation, the performance prediction models, and the planning method. Definition of the deterioration model is one of the most important components, as it determines the quality and the efficiency of the optimal policy.

From a survey of maintenance optimization approaches [1, 12], deterioration models can be classified into three classes as a function of the consistency of information and knowledge on the deteriorating process. The first class consists of models based on a resistance-load relationship, such as the Paris-Ergodan equation [6]. They have proved their efficiency for reliability purposes, but because of their numerical complexity they become intractable in more maintenance decision frameworks. The second class consists of models based on Markov processes [3, 4], which are widely used in existing PMSs. These models require the determination of a transition probability matrix, usually obtained by statistical data analysis techniques [2] when large amounts of deterioration data are available, or by expert judgements when only a few data are available. The third class contains models based on Lévy processes, such as the Brownian motion and gamma processes that are widely used for cumulative deterioration modelling [11]. Zouch *et al.* [13] proposed a deterioration model that is a combination of the two last model classes.

Moreover, existing cracking models are based on observable deterioration indexes that only model the observable deterioration through quality indexes such as the surface distress index, the distress manifestation index, or the pavement condition index [2], and do not take into account the underlying deterioration processes. In fact, most of the existing condition-based approaches utilize a restrictive definition of the system state classically defined by an observable and measurable metric such as the size of the road cracks. This can be restrictive, especially when modelling the effects of imperfect maintenance. Different imperfect actions can have the same effects on the observable deterioration, i.e. concealing the observable damage, but different effects on the level of both the underlying and future deterioration.

Few maintenance planning methods, as in references [8] and [10] consider imperfect maintenance actions in addition to minimal and perfect repairs. In these approaches, the effects of imperfect maintenance consist of a partial reduction in the deterioration level, but they do not allow for the change of the system deterioration law, as in reference [13]. In the latter paper, Zouch *et al.* present a condition-based maintenance optimization approach for road maintenance that takes into account multiple imperfect actions with different effects on the immediate state of the road, as well as its future deterioration law. This is made possible by considering a second deterioration parameter that models the underlying deterioration process in addition to the observable parameter. The two parameters are then used to define a "state-dependent" deterioration model, where the deterioration law in each decision period depends on the deterioration level just before maintenance and the performed action.

To derive the optimal maintenance policy that minimizes the discounted total cost over the infinite horizon, a classical dynamic programming formulated as a Markov decision process (MDP) [7] is solved. The state-dependent deterioration model is incorporated into the MDP framework to derive action-dependent transition matrices.

In reference [9], maintenance decisions relate to the road thickness to renew, i.e. remove and add. The total road thickness is therefore kept fixed. The objective of this paper is to extend the model presented in reference [13] to take into account a constraint of a maximum road thickness, in order to make the model more realistic and more applicable. The maximum thickness constraint renders the decision more complex, as it consists in determining not only the resurfacing layer thickness to add, but also the thickness to remove prior to resurfacing. A special feature of this model is that consecutive changes in road thickness and composition, i.e. in new and old layers, may result in a better-than-new road performance. The possibility to restore a system to a state better than as-good-as-new (AGAN) is a new aspect in the maintenance literature. Our model accounts for the reality that two roads of the same thickness are not equal, and that in general the road with more layers will be superior.

The remainder of the paper is organized as follows. Section 2 briefly presents the state-dependent deterioration model for the road longitudinal cracking process. The optimization problem is formulated as a Markov decision process in Section 3. Finally, Section 4 is dedicated to the solution procedure and sensitivity analysis from numerical examples.

2 The road cracking process

Road deterioration is essentially due to harsh environmental conditions and traffic loads. One of the most important deterioration modes is longitudinal cracking, as it represents the structural health of the road. Meyers *et al.* [5] reported that over 90 percent of road sections in Florida that are in need of repair have such cracks. The longitudinal cracking process operates as follows. A repetitive tensile stress is generated at the bottom of the asphalt layer, and leads the road's tensile strength to deteriorate over time. When the stress applied by the traffic load exceeds the tensile strength of the road, micro-cracks appear at weak spots in the base of the road. These micro-cracks then propagate through the inferior layers of the road until they reach the road surface to give way to surface cracks that continue to propagate. The road-cracking process is therefore a two-phase cumulative process. The first phase is the initiation phase, during which no deterioration can be observed on the surface, but the road cannot be considered to be in perfect condition. The second phase is the propagation phase, which begins with the arrival of the first observable crack.

The current metric used in France to measure the cracking level of a road section is the longitudinal cracking percentage (LCP), represented in Figure 1. Note that the LCP met-

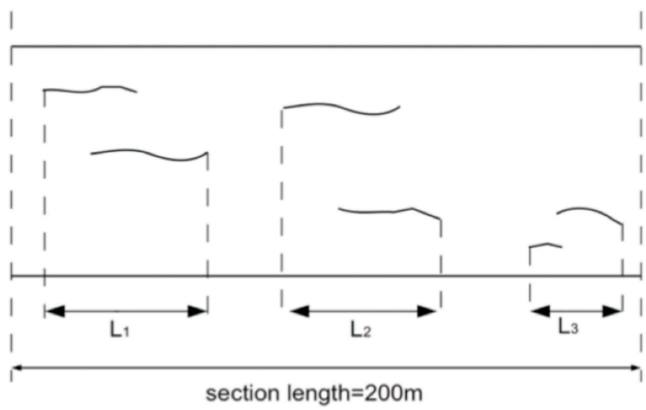

Figure 1: Longitudinal cracking percentage metric for 200m road section LCP=$\frac{L_1+L_2+L_3}{200}$

ric represents only the total longitudinal cracking, and does not take into account the number of cracks or overlapping cracks. Moreover, it is not suitable for the initiation phasewhere no cracks are observable. In reference [13], Zouch *et al.* have highlighted the benefits of introducing a new decision parameter, the deterioration growth rate (DGR), which models the underlying cracking process. More specifically, the DGR models a cracking potential in the first phase when the road is crack-free, and an instantaneous speed of deterioration in the second phase.

In this paper, the two-phase deterioration model developed in reference [13] is used. More specifically, the LCP and DGR are modelled as stochastic processes, denoted by $\{\rho_t\}_{t\geq 0}$ and $\{\theta_t\}_{t\geq 0}$, respectively. In the first phase, the cracking potential u and the first observable crack arrival are considered and modelled as a gamma process and a θ-dependent Poisson process, respectively. In the second phase, the DGR θ and the LCP ρ are modelled using a bilateral gamma (BG) process and a state-dependent gamma (SDG) process, respectively. The proposed gamma process in reference [13] is called state-dependent, since its shape function in each decision epoch depends on the level of deterioration at its start, given by (ρ, θ) and on the last performed action. For more details of the deterioration model and the SDG process, the reader should refer to reference [13].

3 Model Formulation

We consider a road section, characterized by an initial thickness ℓ, that is continuously and stochastically deteriorating under the two-phase deterioration process [13] presented in the previous section. Periodic inspections are performed at the beginning of the decision epoch of length τ to yield perfect observations of the LCP ρ. Since θ cannot be measured, the problem is partially observed. However, the DGR θ can be approximated in the propagation phase using successive observations of ρ, and estimated in the initiation phase [13]. The problem can therefore be solved as a fully observable problem.

Based on the information yielded by inspection, the decision-maker should decide whether to do nothing (DN) and let the system deteriorate until the next decision epoch, or to maintain the section (MX). As the road thickness is constrained by a maximum thickness l_{max}, the maintenance consists of milling a thickness ζ before adding a new resurfacing layer of thickness ξ. Resurfacing the road section with a new layer conceals the longitudinal cracks, so that ρ is reset to zero, but also changes the thickness and the composition of the section, which makes it more rigid and reduces its DGR θ.

A maintenance decision (ξ, ζ) incurs an action cost $c_1 + c_2 + c_r(\zeta) + c_a(\xi)$, where c_1 and c_2 are the fixed costs of removing and adding, respectively, whereas $c_r(\zeta)$ and $c_a(\xi)$ are the variable costs of removing a thickness ζ and adding a thickness ξ.

Moreover, a quality cost $c_q(\cdot)$ is incurred for beginning a decision epoch in a given state (ρ, θ, ℓ). This quality cost represents a penalty or a risk function for leaving the road in a deteriorated state for the current decision epoch.

Both decisions and maintenance actions are performed instantaneously at the beginning of each decision epoch, as decision and maintenance times are assumed to be negligible compared with the decision epoch length. The state of the road section is defined by $(\rho, \theta, \ell, \xi_m)$, where ρ and θ are the current LCP and DGR yielded by inspection, respectively; ℓ is the current thickness of the road section; and ξ_m is the last thickness added to the road section. Considering the last-performed maintenance action as a state parameter allows the decision-maker to know the current composition of the road, i.e. the old and new layer thicknesses. Therefore the state of the section is Markovian.

Let $S = [0, 1] \times [\theta_0, \theta_{max}] \times [0, \ell_{max}] \times [0, \ell_{max}]$ denote the state space. A DN action is assumed to have no impact onto the state of a road section. If a maintenance action $(\xi > 0, \zeta > 0)$ is selected, the state is transformed into $(0, \phi(s, \xi, \zeta), \ell + \xi - \zeta)$, where $\phi(s, \xi, \zeta)$ is the function that models the deterministic effect of the maintenance action on the DGR. Note that as maintenance actions may change the road section thickness and composition, it is possible to obtain a road that has better performance than the initial one. Thus it is possible to obtain a state better-than-new.

The objective is to derive an optimal maintenance policy p that minimizes the total discounted cost-to-go over the infinite horizon, denoted by V. Before presenting the MDP formulation, let $g(x;\rho,\theta,\ell,\xi_m)$ and $f(y;\rho,x,\theta,\ell,\xi_m)$ denote the density distributions of the LCP and DGR processes in the two phases, respectively, given the current levels of ρ and θ as well as the current section thickness ℓ and the last performed action type given by ξ_m.

More specifically, g represents the density of the Poisson process in the first phase and the density of the SDG process in the second phase, whereas f represents the density of the BG process in both phases. The joint density of the two deterioration parameters in both phases, denoted $h(x,y;\rho,\theta,\ell,\xi_m)$ can be expressed as follows:

$$\begin{aligned} h(x,y;\rho,\theta,\ell,\xi_m) &= g(x;\rho,\theta,\ell,\xi_m)f(y;\rho,x,\theta,\ell,\xi_m), \\ & \quad x \in [0,1-\rho], y \in [-\theta, \theta_{max}-\theta] \end{aligned}$$

Explicit expressions of the density distribution functions g and f are given in reference [4]. Hence the MDP formulation of the maintenance optimization problem can be expressed as follows.
For $s = (\rho,\theta,\ell,\xi_m)$,

$$V^*(\rho,\theta,\ell,\xi_m) = c_0 + \min\{DN(\rho,\theta,\ell,\xi_m), MX(\rho,\theta,\ell,\xi_m)\} \tag{1}$$

where

$$\begin{aligned} DN(\rho,\theta,\ell,\xi_m) &= \lambda\int_0^{1-\rho}\int_{-\theta}^{\theta_{max}-\theta} h(x,y;\rho,\theta,\ell,\xi_m) \\ & \qquad V(\rho+x,\theta+y,\ell,\xi_m)dydx \end{aligned} \tag{2}$$

and

$$MX(\rho,\theta,\ell,\xi_m) = \min_{(\xi,\zeta)} \left\{ \begin{array}{l} c_1 I_{(\zeta>0)} + c_2 I_{(\xi>0)} + c_r(\zeta) + c_a(\xi) + \\ DN(0,\phi(\rho,\theta,\ell,\xi_m,\xi,\zeta),\ell+\xi-\zeta,\xi) \end{array} \right\} \tag{3}$$

subject to

$$\ell + \xi - \zeta < \ell_{max} \tag{4}$$

where $\lambda \in [0,1]$ is the discount factor.

Note that the last added thickness jm is directly introduced as a parameter of the joint density because of its influence in the cracking process, and thus the associated definition of the road state. Equation (1) states that following the DN action when the current state is s incurs a quality cost plus the expected cost-to-go of the system deterioration from the state s to all possible states s'. The MX action in equation (2) incurs a maintenance cost, plus the quality cost and the cost-to-go of the the system beginning in the resulted state $(0,\phi(\rho,\theta,\ell,\xi_m),\ell+\xi-\zeta,\xi)$. The thickness constraint is formulated in (4).

4 Solution procedure and numerical examples

4.1 Solution procedure

We solve the MDP problem formulated above using the policy iteration algorithm (PIA) [7]. The following discretization of the decision problem is proposed. Assume that ρ and θ represent the first values in N_ρ and N_θ equal-sized discrete intervals of length Δ_ρ and Δ_θ on $[0,1]$ and $[0,\theta_{max}]$, respectively, such that N_ρ and N_θ divide evenly into 100. As $\rho = 0$ is a key state, we consider it as a single state. Let N be the cardinality of the state space S, and $p^m_{ss'}$ denote the transition probability from state s to state s' when the last performed maintenance type is m.

$$\begin{aligned}
p^m_{ss'} &= Pr\big\{s' \mid s\big\} \\
&= Pr\big\{\rho' \in [i_{\rho'}, i_{\rho'}{+}\Delta_\rho], \theta' \in [i_{\theta'}, i_{\theta'}{+}\Delta_\theta] \mid \rho{=}\, i_\rho, \theta{=}i_\theta, \ell, \xi_m\big\} \\
&= \int_{i_{\rho'}-i_\rho+\Delta_\rho}^{i_{\rho'}-i_\rho} g(x; i_\rho, i_\theta, \ell, \xi_m) \int_{i_{\theta'}-i_\theta+\Delta_\theta}^{i_{\theta'}-i_\theta} f(y; i_\rho, x, \ell, \xi_m) dy dx \qquad (5)
\end{aligned}$$

The set of possible maintenance actions in order to use the PIA is discretized as follows:

$$A = \{(\xi, \zeta) \mid \xi \in \{5, 10, 15, 20\}, \zeta \in [\min(0, {}_{max} - \ell - \xi), \ell]\}$$

4.2 Characterization of the deterioration functions

We shall focus in this section on the construction of the deterioration function f for the ρ-process. The direct derivation of g for the BG process (θ_t) from the SDG law is proposed in reference [13]. The construction of these laws is expert-based knowledge on the cracking process rather than field data; one of the main motivations is the lack of confidence in the current French database for these longitudinal cracks. Recall that the ρ-process is modelled by an SDG process [13], where the deterioration function f on a time interval t is a gamma density function with a shape function, a function of the current state (ρ, θ), and a given scale parameter β. Two choices of shape function for f are proposed here.

4.2.1 The shape function of the SDG process

For the cracking process (ρ_t), two examples of the SDG process shape function are considered. The first shape function, γ_1, has a linear form of the state parameters, whereas the second function, γ_2, has a form inspired by the well-known Gaussian function:

$$\gamma_1(\tau; \rho, \theta, \ell, \xi_m) = (a_0 + a_1\rho + a_2\theta + \frac{a_3}{\ell} + \frac{a_4}{\xi_m})$$

$$\gamma_2(\tau; \rho, \theta, \ell, \xi_m) = \frac{b_1(\ell_{max} - \xi_m)^2}{\ell_{max} - \ell}\theta^2 \; \tau \; \exp\left\{\tfrac{(\rho-(1-\theta))^2}{b_2(\ell_{max}-\xi_m)}\right\}$$

Note that the two shape functions reflect both time and state effects, as they depend on the decision epoch length t as well as on the state parameters. However, they vary differently as the state varies. Figure 2 shows the variations of the expected deterioration in ρ for different levels of θ given by the Gaussian form shape function. Note that at the beginning of the propagation phase the expected increase in deterioration increases as ρ increases, but with decelerated variation. However, when the DGR level becomes higher, the expected increase in cracking level increases and then decreases quickly, as the DGR is influential. This reflects the fact that, for a single road section, the probability of increasing the cracking level of the section is higher at the beginning of the propagation phase, and decreases as the section becomes more cracked (since overlapping cracks do not account for ρ). The main objective in considering these two shape functions is to highlight the state-dependent character of the deterioration model, as well as the effect of introducing the DGR as a deterioration parameter.

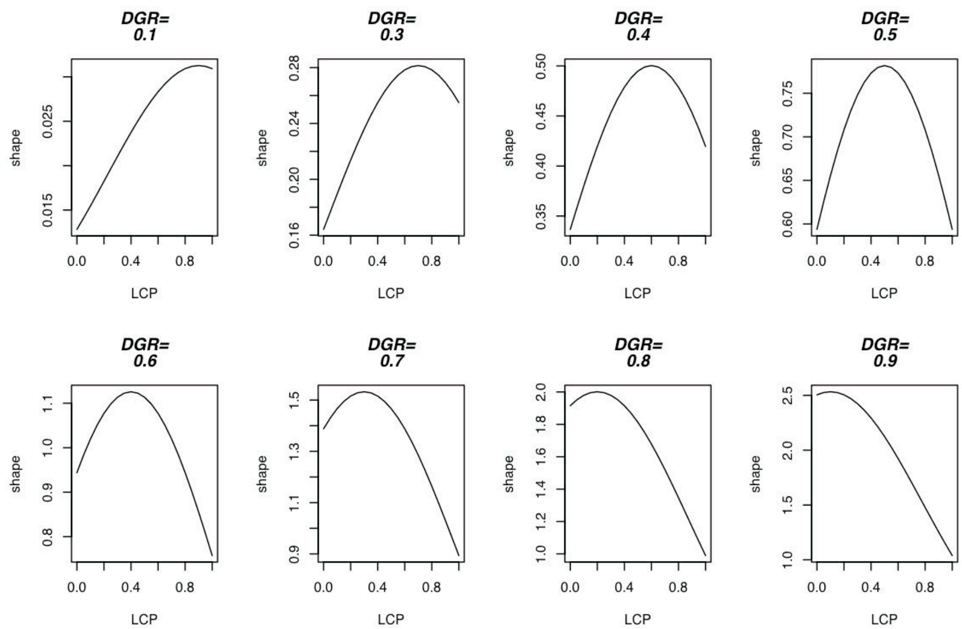

Figure 2: Variations of Gaussian form shape function in ρ for different levels of θ

Figure 3 reflects the fact that the expected crack growth obtained with the Gaussian form shape function increases with the DGR (a), and decreases as the total road thickness and the latest added thickness increase ((b) and (c)).

Figure 4 illustrates the BG process shape functions that represent the expected variation in the DGR θ as a function of the LCP. Note that the cracking process is accelerated for low LCP levels, i.e. the beginning of the cracking process, and is decelerated when the cracking level is high. This reflects the same behavior shown by the shape function γ_2 of a non-increasing probability of cracking propagation as the section becomes more cracked.

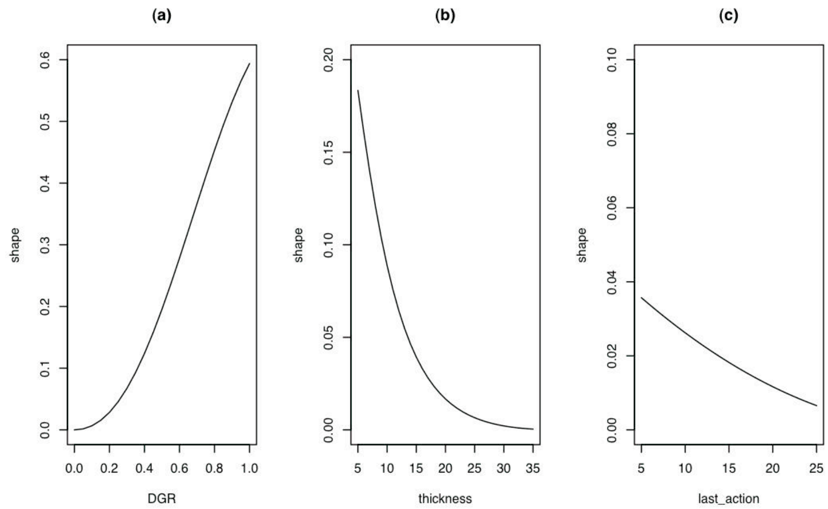

Figure 3: Variations of the Gaussian form shape function in θ, ℓ and ξ_m

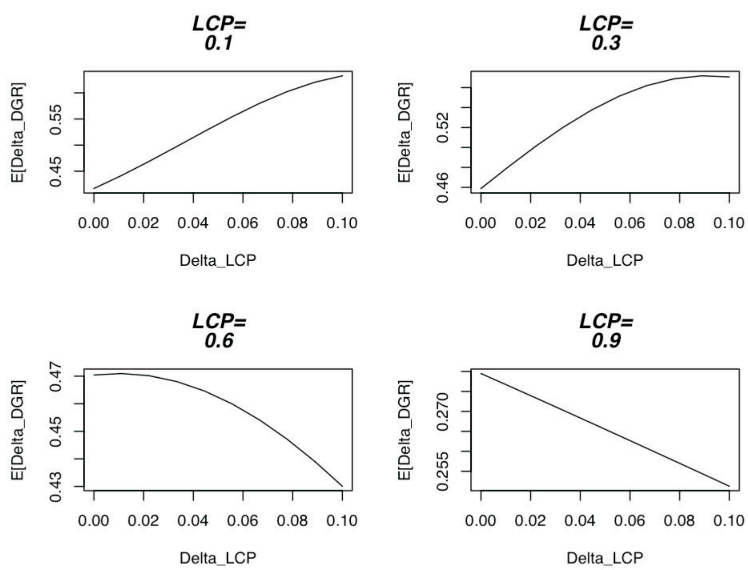

Figure 4: Variations of the DGR in increments of LCP for different levels of ρ

4.2.2 Comparison of the linear and Gaussian form shape functions

The main motivation behind comparing the linear and Gaussian form shape functions is to highlight the state-dependent character of the road-cracking process. In fact, in this model, the deterioration law changes in each decision epoch according to the road state at its beginning and the last performed maintenance action. Both linear and Gaussian form shape functions track this change, since their parameters are state and action dependent. The main difference between the two functions is the fact that, for a given DGR level, the linear shape function has the same variation rate for all LCP levels (the same slope), whereas the Gaussian form function has a variation rate that depends on the cracking level (the slope changes as the LCP changes) (Figure 2).

Moreover, as illustrated in Figure 2, for different values of θ, the linear shape function has a constant variation rate in ρ, whereas the Gaussian form function allows for different variation rates in ρ when the DGR level changes. These properties will be illustrated with the numerical analysis to show that the Gaussian form shape function takes future deterioration effects into account better, since it allows changing deterioration speeds for different cracking levels.

4.2.3 The crack occurrence rate function

The cumulative hazard function of the Poisson process of the initiation phase is given by the θ-dependent function $\alpha(\tau;\theta,\ell,\xi_m)$. Note that α is non-decreasing in the cracking potential θ and non-increasing in the total and new added thicknesses.

$$\alpha(\tau;\theta,\ell,\xi_m) = \left(d_0 + d_1\theta + \tfrac{d_2}{\ell} + \tfrac{d_3}{\xi_m}\right)$$

4.3 Numerical examples

In the following section, we present some numerical examples, and provide some analysis of the structural properties of the optimal policies.

4.3.1 Description of the case studies

We derive optimal policies for two types of section - sections with the current thickness less than the maximum thickness, and sections at maximum thickness - in order to show the effect of the thickness on the road performance as well, as to analyse the trade-off between thicker and newer roads. We consider three sections, which we denote Sec1, Sec2 and Sec3, and which have the following characteristics: $(\ell = 10, \xi = 10)$, $(\ell = 20, \xi = 10)$ and $(\ell = 30, \xi = 5)$, respectively, where each pair denotes the total section thickness ℓ and the newer layer thickness ξ.

For $\ell_{max} = 30$, consider the following set of available actions for each section:

$$\begin{aligned}
A(Sec1) &= \{(\xi,\zeta) = \left\{\begin{matrix}(0,0);(10,5);(10,10);(0,5);(0,10);(10,15);\\ (0,15);(10,20);(0,20);(5,25);(10,30)\end{matrix}\right\} \\
A(Sec2) &= \left\{\begin{matrix}(0,0);(20,5);(10,5);(20,10);(10,10);(0,5);(15,15);\\ (0,10);(20,20);(5,15);(10,20);(15,25);(20,30)\end{matrix}\right\} \\
A(Sec3) &= \left\{\begin{matrix}(0,0);(25,5);(15,5);(5,5);(30,10);(20,10);(25,15);\\ (10,10);(30,20);(15,15);(20,20);(25,25);(30,30)\end{matrix}\right\}
\end{aligned}$$

Note that an action can decrease, increase or leave unchanged the total thickness of the road. The efficiency of a maintenance action is defined by a maintenance effect function that reflects the effect of different road compositions (i.e. new and old thick composition) on its performance. An example of a maintenance effect function is given by the following, where $(\rho, \theta, \ell, \xi_m)$ represents the state parameters just before maintenance, and (ζ, ξ) is the selected maintenance action.

$$\phi(\rho,\theta,\ell,\xi_m) = \begin{cases} 0.85A\,\theta & ,\xi < \zeta, \xi \geq \xi_m \\ 0.75A\,\theta & ,\xi < \zeta, \xi < \xi_m \\ 0.55A\,\theta & ,\xi = \zeta, \xi \geq \xi_m \\ 0.45A\,\theta & ,\xi = \zeta, \xi < \xi_m \\ 0.25A\,\theta & ,\xi \geq \zeta, \xi \geq \xi_m \\ 0.1A\,\theta & ,\xi \geq \zeta, \xi < \xi_m \end{cases}$$

where $A = \left(\rho + \frac{1}{(\ell+\xi-\zeta)^2} + \frac{2}{\xi}\right)$

Note that the maintenance actions in the proposed sets A(Sec1), A(Sec2) and A(Sec3) are ordered in terms of efficiency with respect to this function. We use different quality cost functions that are non-decreasing in deterioration parameters ρ and θ, and non-increasing in the total section thickness and in the last added layer thickness.

4.3.2 Numerical analysis

In the following, we present some optimal policies derived by the PIA for the three considered sections (Sec1, Sec2 and Sec3) using the different input functions presented above.

The deterioration parameters used for the numerical examples are given by
$(a_0, a_1, a_2, a_3, a_4) = (1, 1.5, 1.2, 1, 1)$, $(b_1, b_2) = (1.2, 2)$
$(d_0, d_1, d_2, d_3) = (1, 1.2, 1, 1)$

The set-up maintenance costs are

$$c_1 = 900, c_2 = 1700$$

$c_r(\zeta) = c_{ur}\zeta$ with $c_{ur} = 90$ the milling cost per thickness unit,
$c_a(\xi) = c_{ua}\xi$ with $c_{ua} = 210$ the resurfacing cost per thickness unit.

The policies presented in Tables 1-6 state, for each possible road section, the optimal maintenance action to perform. The policies are called Gaussian form and linear form with respect to the shape function used. For example, if the observed section state is ($\rho = 0.4, \theta = 0.7, \ell = 20, \xi_m = 10$), then the action advised by the Gaussian-form policy (Table 1(a)) is resurfacing the road section with a 10 cm layer without milling, whereas the policy advised by the linear-form policy (Table 1(b)) is resurfacing the section with a 5 cm layer without milling.

The discount factor used for the following numerical examples is 0.95. However, in order to highlight the difference between the two shape functions, some policies with different discount factors are compared. It is found that Gaussian-form policies with a low discount factor (0.3) may converge to linear-form policies obtained with higher discount factors (0.8). This shows that the Gaussian-form shape function takes into account the effects of future deteriorations better than the linear-form function does.

Tables 1-6 present decision matrices that detail the optimal maintenance action for each possible observed state. For example, from Table 1(a), if after inspection Sec2 is found with $\rho = 0.3$ and $\theta = 0.5$, then the optimal action given by the decision matrix for this state is to resurface the section with a thickness 5.

The obtained policies present some monotonic properties. More specifically, the efficiency of the recommended action is increasing in the deterioration level and the optimal policy is of the control limit type. For example, in Table 2(a), the DN action is the optimal decision for weak degradation levels (bottom left-hand side of the matrix). When degradation is growing (from the bottom left-hand to the upper right-hand side of the decision matrix), the recommended actions become stronger in terms of costs and efficiency.

The Gaussian-form policies promote predictive policies, whereas the linear form policies do not advise maintenance when the crack level is zero (see the first line of the two matrices in Table 1). This proves that the Gaussian-form shape function takes the effects of the DGR as a deterioration parameter into account better. To show the sensitivity of the optimal policies to the section thickness as well as to its composition, the policies of Tables 1 and 2 are compared, as follows.

1. For thinner sections such as Sec1, note that only actions that increase the section thickness are optimal (Table 2(a)).

2. When the sections are at their maximum thickness, it is not optimal to reduce their thickness (Table 2(b)). In fact, in this example (i.e. given the maintenance effect and cost parameters), actions that completely or partially renew sections with maximum thickness and reduce their thickness are less efficient than actions that keep the section thicknesses unchanged.

3. When the road section has a 'medium' thickness, there is a trade-off between making the section thicker or making it newer.

Results 1 and 2 confirm expert judgement stating that although the performance of a road is increasing in its thickness, it is not optimal to build the road at it is maximum thickness from the beginning; it is better to add new layers gradually. Moreover, results 1 and 2 indicate

Table 1 Optimal policies with linear quality cost function for Sec2

(a) Gaussian form

ρ	θ									
	0.03	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.8	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.7	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.6	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.5	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)
0.4	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)
0.3	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)
0.2	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)
0.1	(0, 0)	(0, 0)	(0, 0)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.03	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.0	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 5)	(0, 5)	(0, 5)	(0, 5)

(b) Linear form

ρ	θ									
	0.03	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)
0.8	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.7	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.6	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.5	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.4	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.3	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.2	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.1	(0, 0)	(0, 0)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.03	(0, 0)	(0, 0)	(0, 0)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.0	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)

Table 2 Optimal Gaussian-form policies with linear quality cost function

(a) Sec1

ρ	θ									
	0.03	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.8	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.7	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.6	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.5	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.4	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)
0.3	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)
0.2	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)
0.1	(0, 0)	(0, 0)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)
0.03	(0, 0)	(0, 0)	(0, 0)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)
0.0	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)

(b) Sec3

ρ	θ									
	0.03	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(30, 30)
0.8	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(10, 10)
0.7	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(10, 10)
0.6	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(10, 10)
0.5	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)
0.4	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)
0.3	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(10, 10)	(10, 10)	(10, 10)
0.2	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(10, 10)	(10, 10)
0.1	(0, 0)	(0, 0)	(0, 0)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(10, 10)
0.03	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)	(5, 5)
0.0	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(5, 5)	(5, 5)	(5, 5)	(5, 5)

that a control limit in thickness exists under which it is never optimal to reduce the section thickness.

Note that for given unit resurfacing and removing costs, some actions are never selected. This is because action efficiency is determined by the maintenance effect function. If the efficiency order is different from the total cost order, more expensive and less efficient actions will never be selected.

The trade-off between thicker and newer roads (3) is determined by the cost rates. Note that, especially for medium thickness sections, the policies are very sensitive to both maintenance and quality cost variations. The sensitivity of the model to the cost functions is illustrated by Tables 3 and 4, which present policies obtained using the Gaussian-form (a) and linear-form (b) shape functions with different quality cost functions. The policies in Table 3 are obtained using a quality cost function that increases more quickly in θ, whereas the policies in Table 4 are obtained using a quality cost function that increases more quickly in ρ.

From the numerical results, the policies are very sensitive to the variations of the quality cost function, especially the Gaussian-form policies. It is found that the Gaussian-form policies are much more sensitive to the quality cost variations in θ than the linear-form ones. For example, in Table 3, for $\rho > 0$, when θ increases from 0.4 to 0.5, the Gaussian-form policy (a) advises a stronger action than the linear-form one (b), which gives the same action. This confirms the fact that the Gaussian-form shape function takes the effects of the DGR into account better. Note that the actions in Table 3 are ones that make the road thicker, whereas most of the actions in Table 4 keep the section at the same thickness. This reflects the fact that to make roads more rigid, i.e. with lower DGR levels, it is better to make them thicker (Table 3). However, if costs are more sensitive to the LCP variations, it is better to conceal cracks by reducing or keeping the same thickness (Table 4). When the quality cost is not very important compared with the action costs, we note that DN action may become a better choice than maintenance actions that reduce the section thickness, especially for low levels of DGR.

Moreover, Table 4 shows that Gaussian-form policies are more sensitive to the quality cost variations in ρ than linear-form policies. For example, in Table 4 for $\theta > 0.5$, when ρ increases from 0.1 to 0.2, the Gaussian-form policy (a) advises stronger actions whereas the linear-form one (b) gives the same action. This can be explained by the fact that the Gaussian-form shape function represents the effect of the LCP variations on the evolution of the deterioration process better, and therefore reflects the deterioration evolution risk better. Thus the quality cost is a very important function in this model, as it represents a measure of cumulative risk for the decision epoch.
In Tables 5 and 6, the extended model is compared with the model in reference [13]. However, for the model of reference [13], only resurfacing actions that respect the total maximum thickness constraint are considered. Tables 5 and 6 present policies for low- and high-quality costs compared with maintenance costs, respectively. Note that for quality costs that are less important than maintenance costs, the two approaches converge to the policies presented in Table 5. These two policies have almost the same average values, the extended model allows the section to be renewed to the maximum thickness for maximum deterioration levels. When the deterioration risk is high, i.e. the quality cost is much more important than maintenance

costs, the two approaches give different policies (Table 6), but the difference in the policy average values is not very significant, i.e. $V(a) = 1.05V(b)$. This shows that the main interest from extending the possible maintenance actions to removing as well as adding different thicknesses is to improve prevention of the cracking risk.

5 Conclusion

This paper presents a condition-based maintenance optimization approach for the road-cracking problem based on a Markov decision process. This model extends a previous work [13] by taking into account the constraint of maximum road thickness. The special feature of this approach is that the MDP transition matrix is obtained using a state-dependent deterioration process based on two deterioration parameters, namely the LCP and the DGR, as well as on the road characteristics: the total thickness and the composition in new and old underlying layers. The optimal decision consists in defining both the optimal thickness to remove and the resurfacing thickness to add. This changes the composition of the underlying layers with different mixes of degraded and non-degraded layers and, finally, different cracking patterns. Moreover, we introduce the possibility of actions that can which lead to a state better than new. This approach can be imple- mented directly in a decision tool for designing new roads. The model provides the optimal maintenance costs, given the initial road reliability performance (here defined as a function of the initial road thickness). We have already highlighted the trade-off in initial maximum thickness and maintenance costs.

Moreover, numerical examples show the importance of defining and estimating cost parameters and functions, especially the quality cost function. In fact the quality cost function represents a measure of deterioration risk, and is very important in determining the optimal maintenance to perform. Finally, we provide different numerical analyses of the optimal maintenance policies for different road sections which lead to the identification. All of them lead to the identification of some structural.

Table 3 Optimal policies with quadratic increasing quality cost in θ

(a) Gaussian form

ρ	θ 0.03	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(10, 20)	(10, 20)
0.8	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(10, 20)	(10, 20)
0.7	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(10, 20)
0.6	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(10, 20)
0.5	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(5, 15)
0.4	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(5, 15)
0.3	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(5, 15)
0.2	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(5, 15)
0.1	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)
0.03	(0, 0)	(0, 0)	(0, 0)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)
0.0	(0, 0)	(0, 0)	(0, 0)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)

(b) Linear form

ρ	θ 0.03	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(5, 15)	(10, 20)
0.8	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(5, 15)
0.7	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(5, 15)
0.6	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.5	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.4	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.3	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.2	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.1	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.03	(0, 0)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.0	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)

Table 4 Optimal policies with quadratic increasing quality cost in ρ for Sec2

(a) Gaussian form

ρ	θ = 0.03	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	(20, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(15, 15)	(5, 15)	(10, 20)	(10, 20)	(10, 20)
0.8	(20, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(15, 15)	(15, 15)	(10, 20)	(10, 20)	(10, 20)
0.7	(20, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(15, 15)	(15, 15)	(5, 15)	(10, 20)	(10, 20)
0.6	(20, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(15, 15)	(15, 15)	(5, 15)	(10, 20)	(10, 20)
0.5	(20, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(15, 15)	(15, 15)	(15, 15)	(5, 15)	(10, 20)
0.4	(20, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(15, 15)	(15, 15)	(15, 15)	(15, 15)	(5, 15)
0.3	(20, 5)	(20, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(15, 15)	(15, 15)	(0, 10)
0.2	(20, 5)	(20, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(15, 15)	(15, 15)	(0, 10)
0.1	(20, 5)	(20, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(0, 5)
0.03	(20, 5)	(20, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(0, 5)
0.0	(0, 0)	(20, 5)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(10, 10)	(0, 5)

(b) Linear form

ρ	θ = 0.03	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(5, 15)	(10, 20)	(10, 20)	(10, 20)
0.8	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(10, 20)	(10, 20)	(10, 20)
0.7	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(5, 15)
0.6	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.5	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.4	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.3	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.2	(20, 5)	(20, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.1	(20, 5)	(20, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.03	(0, 0)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.0	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 10)	(0, 10)	(0, 10)

Table 5 Optimal Gaussian-form policies for Sec2

(a) With removal of option for low risk

ρ	θ									
	0.03	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(20, 30)
0.8	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.7	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.6	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.5	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.4	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.3	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.2	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.1	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.03	(0, 0)	(0, 0)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.0	(0, 0)	(0, 0)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)

(b) Without removal of option for low risk

ρ	θ									
	0.03	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.8	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.7	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.6	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.5	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.4	(0, 5)	(0, 5)	(0, 5)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.3	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.2	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.1	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.03	(0, 0)	(0, 0)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
0.0	(0, 0)	(0, 0)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)

Table 6 Optimal Gaussian-form policies for Sec2 with (a) and without (b) removing option for high risk

(a) With removal of option for high risk

ρ	θ 0.03	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(10, 20)	(10, 20)	(10, 20)	(10, 20)	(20, 30)
0.8	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(10, 20)	(10, 20)	(10, 20)	(10, 20)	(10, 20)
0.7	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(10, 20)	(10, 20)	(10, 20)	(10, 20)	(10, 20)
0.6	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(10, 20)	(10, 20)	(10, 20)	(10, 20)
0.5	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(10, 20)	(10, 20)	(10, 20)	(10, 20)
0.4	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(10, 20)	(10, 20)	(10, 20)	(10, 20)
0.3	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(10, 20)	(10, 20)	(10, 20)
0.2	(0, 10)	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(10, 20)	(10, 20)	(10, 20)
0.1	(0, 10)	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(10, 20)	(10, 20)
0.03	(0, 0)	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(10, 20)
0.0	(0, 0)	(0, 10)	(0, 10)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(10, 20)

(b) Without removal of option for high risk

ρ	θ 0.03	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.8	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.7	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.6	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.5	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.4	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.3	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.2	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.1	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.03	(0, 0)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)
0.0	(0, 0)	(0, 0)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)	(0, 10)

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6 The Partially Observable Problem

Optimal Resurfacing Decisions for Road Maintenance: A POMDP Perspective

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Keywords: Road deterioration, maintenance optimization, Partially Observed Markov Decision process, grid-based approximation

Notations

a	maintenance action	i_θ	discrete value of θ
A_h	set of actions for a section thickness h	i_ρ	discrete value of ρ
$\mathbf{b}$	belief state	m	last maintenance type
$\mathbf{b^k}$	belief state immediately after action a_k	N	number of states
$\mathbf{b}'$	updated belief state after observation	N_h	number of available actions
$\mathbf{b}^\theta$	belief state on θ	N_o	number of possible observations
B	set of state beliefs	N_ρ	number of possible values of ρ
c_0	inspection cost	N_θ	number of possible values of θ
c_f	total fixed cost	o	observation
c_n	fixed cost of milling	O	set of observations
c_p	fixed cost of resurfacing	r_a	DGR reduction factor
c_v	total variable cost	s	section state
c_{un}	unit cost of milling	s_a	section state after action a
c_{up}	unit cost of resurfacing	S	set of states
$C_q(\cdot)$	quality cost	V	Total expected discounted cost-to-go
h	section thickness	$\Omega(o\|\mathbf{b})$	probability of o at the decision epoch end when $\mathbf{b}$ at its beginning
h_a	section thickness after action a	ρ	longitudinal cracking percentage
h_{max}	maximum road thickness	θ	deterioration growth rate
h_n	milling thickness	$\tilde{\theta}$	average deterioration speed
h_p	resurfacing thickness	τ	inter-inspection period
H_n	set of possible milling thicknesses	$\phi(\cdot)$	maintenance effect function on θ
H_p	set of possible resurfacing thicknesses		

1 Summary & Conclusions

We develop an optimal maintenance policy for a road section to minimize the total maintenance cost over the infinite horizon when some deterioration and decision parameters are not observable. Both perfect and imperfect maintenance actions are possible through the application of various thicknesses of resurfacing layers. We use a two-phase deterioration process based on two parameters: the longitudinal cracking percentage and the deterioration growth rate. Our deterioration model is a state-based model based on the state-dependent Gamma process for the longitudinal cracking percentage and the Bilateral Gamma process for the deterioration growth rate. Moreover the maintenance decision is constrained by a maximum road thickness that makes the maintenance decisions more complex as it becomes how much surface layer to add as well as to remove. Because only one of the two deterioration parameters is observable, we formulate the problem as a partially observed Markov decision process and solve it using a grid-based value iteration algorithm.

Numerical examples have shown that our model provides a preventive maintenance policy that slows down the initiation as well as the propagation of longitudinal cracks and that may ameliorate the road state to a better than as-good-as-new one by altering its composition through additive resurfacing layers.

2 Introduction & Literature review

During the last decades, the interest in maintenance optimization has increased considerably and several approaches have been developed for different fields of applications. For a survey of maintenance optimization models and application, refer to [4, 20]. Civil and transportation infrastructure networks is one of the fields where maintenance optimization is very important due to the importance to these networks to social and economical development of societies and to the large expenditure and construction time needed to construct new facilities [2].

Maintenance optimization approaches aim to determine optimal strategies, i.e. what maintenance action to perform and when in order to minimize variable costs as well as to ensure the proper functioning of infrastructure.

From the literature, different approaches have been developed and applied to many national networks such as in Netherlands [9, 19] and in the USA [6, 18]. Many approaches have been formulated as Markov decision processes (MDPs). This approach has the advantage to be dynamic and easy to solve. Nevertheless, MDPs assume that the deterioration parameters can be perfectly observed and measured. This assumption may not be realistic in some cases: even if advanced inspection technologies minimize measure errors, some deterioration parameters, such as the instantaneous speed of deterioration cannot be observed, but just estimated using available information on the system state. In order to take into account the non-observability of decision parameters, MDPs have been extended to partially observed Markov decision processes (POMDPs) [14, 8]. Instead of observed states, POMDPs work with belief states that are represented by probability distributions over all the states.

Several imperfect information problems have been treated in literature. Madanat *et al.* [11] present a dynamic programming method to determine optimal maintenance and inspection policy, in the presence of inspection error. In [7], perfect maintenance of an aircraft engine component subject to cracking is considered. The problem is formulated as a POMDP model since cracks are partially observable and the maintenance scheduling decision is based on other collected information. A similar problem is presented in [12] where a POMDP over the finite horizon is formulated for a maintenance problem using the failure rate to update the knowledge on the system state. In [13] an optimal stopping problem with partial information is formulated for a system with obvious failure and transformed to a problem with complete information. Ghasemi *et al.* [5] formulated a POMDP for a system subject to perfect condition-based maintenance. Their POMDP formulation is combined to a proportional hazards model for the system degradation.

Moreover, POMDPs offer an elegant way to take into account the uncertainty about the system state. However, their most important challenge is solution computation. Exact solutions are almost impossible except for problems of small sizes. In the literature, several approaches are combined with value iteration or policy iteration [17] algorithms to approximate POMDPs. Examples of such approaches are grid-based approximations ([10] and [22]) and point-based approximations ([15], [3], [21] and [16]). In this work, we extend the model presented in [23] by considering a non-observable deterioration growth rate. We present an optimization framework for a road section maintenance strategy using a POMDP formulation and show the interest of such formulation.

In a road maintenance context, our model has the advantage to take into account an underlying deterioration parameter that cannot be observed: the deterioration speed. This deterioration speed information is a key decision parameter in condition-based maintenance optimization. Indeed, the knowledge of the deterioration growth rate allows updating the maintenance decision based on the single deterioration level. Moreover, unlike most of developed approaches for partially observed problems, our model considers multiple imperfect maintenance actions comprised of two maintenance components.

The remainder of this paper is structured as follows. The model is formulated in Section 2. In Section 3, we discuss a numerical example. Finally a conclusion is presented in Section 4.

3 Model formulation

3.1 Problem statement

Consider a road section that is continuously and stochastically deteriorating. A road is a complex system with several deterioration modes. However, we focus in this work on the longitudinal fatigue cracking process because of its frequency and economical consequences. This deterioration mode is a two phase process: a non-observable and an observable phase. The deterioration begins at the bottom of the road layer and propagates until reaching the surface giving way to longitudinal cracks that propagate on the surface. The current metric available in the French IQRN database [1] to describe the deterioration level of the road section is the longitudinal cracking percentage (LCP) denoted ρ and represented in Figure 1 where it is given by $\rho = \sum_{i=1}^{3} L_i/200$.

The LCP metric only reflects the observable deterioration level, especially for small val-

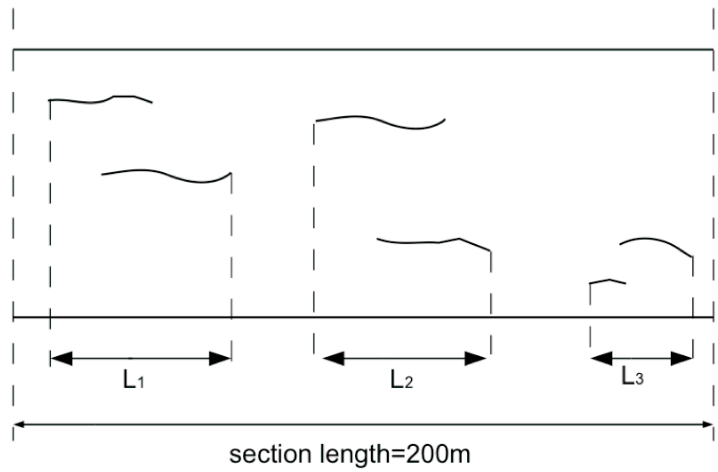

Figure 1: Construction of the longitudinal cracking percentage

ues of ρ. This indicator is not sufficient to predict the future behavior of the road cracking growth. Additional information has to be integrated in order to estimate the underlying deterioration. We introduce the deterioration growth rate (DGR), denoted θ, which is defined as the expected instantaneous deterioration speed and can be greater than zero when no cracks are observed, i.e., $\rho = 0$, to indicate cracking potential.

We consider, as in current practices, a constant maintenance decision interval equal to τ. At the beginning of each decision epoch, the road section is inspected at a cost c_0 and an observation of ρ is yielded. We assume that this observation is perfect in that the measure error is equal to zero. Unlike the LCP, θ can neither be observed nor measured. Only an average value can be estimated using successive observations of ρ. The complete deterioration is then partially observed.

After inspection, the decision-maker can opt to do nothing (DN) or perform one of several maintenance actions that have different effects on the road system. A maintenance action consists in resurfacing the section to conceal cracks. However, the road is constrained by a maximum thickness h_{max}. Therefore, the maintenance decision for a road section of thickness h consists in how much thickness, h_n, to remove and how much thickness, h_p, to add so that the thickness constraint is respected. The future behavior of the cracking process is a complex function of the state of the road before maintenance, the new thickness of the road and the underlying composition of the road characterized by h_n and h_p.

We assume that only R thicknesses to add are available. Let $H_p = \{h_{p_1}, \cdots, h_{p_R}\}$ be the set of available thicknesses to add. Therefore, given a current thickness h and a selected thickness to add h_p, the possible thicknesses to remove are given by $H_n(h, h_p) = [\min(0, h_p + h - h_{max}), h]$. We denote a maintenance action by $a = (h_p, h_n)$. Note that the DN action corresponds to $(h_p = 0, h_n = 0)$.

Each maintenance action incurs a fixed cost $c_f = c_n \; I_{\{h_n>0\}} + c_p \; I_{\{h_p>0\}}$ as well as a variable cost $c_v(h_n, h_p) = c_{un}\ h_n + c_{up}\ h_p$ where $I_X$is the binary indicator function (= 1 if X and = 0 otherwise), c_n and c_p are the fixed cost of removing and adding asphalt layers, respectively, c_{un} and c_{up} are the unit costs of removing and adding, respectively. Moreover, a quality cost, denoted $C_q(\cdot)$ is incurred in order to prevent the decision maker from always select the DN action and to penalize reaching high deterioration levels.

We define the state of the road section by $s = (\rho, \theta, h, m)$, where ρ and θ are the current LCP and the DGR, respectively, h is the current section thickness and m is the last maintenance type performed on the section (m is equivalent to the thickness of the last added layer).

Let A_h be the set of the feasible actions when the current section thickness is h and let N_h denote its cardinality.

$$A_h = \{a = (h_p, h_n)\ ;\ h_p \in H_p, h_n = h_n(h_p) \in H_n(h, h_p)\}.$$

We assume that an action a is instantaneously selected and performed. Moreover, if the DN-action is selected, the system state just after decision remains in s. Otherwise, according to the maintenance assumptions, the LCP is reset to zero and the new value of the DGR is reduced to a function of both its state before maintenance and the type of the last maintenance, $\phi(s, a)$, and the new characteristic of the road defined by $h_a = h - h_n + h_p$. h_p determines the future deterioration behavior. Finally, the new state of the road is $s_a = (0, \phi(s, a), h_a, h_p)$.

In [23], the perfect information problem is formulated as an MDP where the instantaneous DGR is approximated by the average using two successive observations of ρ. The model is solved for the optimal policy that minimizes the discounted cost-to-go over the infinite horizon subject to the maximum thickness constraint. However, numerical examples showed that our policy is very sensitive to the DGR θ. In addition, we introduced the DGR as a deterioration parameter to complete the information that the LCP gives about the deterioration level of the section. This approximation limits the significance of θ. These are the main motivations to consider a partially observed problem in this work.

3.2 Decision process formulation

The state transitions during next decision epoch are given by the deterioration model defined by Zouch *et al.* [24]. This model is a two-phase deterioration model based on a Poisson process modeling the first crack arrival and a gamma process for the deterioration growth rate in the initiation phase. In the propagation phase, ρ is modeled a state-dependent gamma (SDG) process, i.e; the increments of the process depend on both the current ρ and θ, whereas the GDR is modeled as a bilateral gamma (BG) process. For details of the deterioration model refer to [24].

For numerical convenience, a state space discretization is considered. Denote $S = s_1, \cdots, s_N$ the state set where $N = |S|$ and $s = (\rho_i, \theta_j, h, m)$ where $i = \{1, \cdots, N_\rho\}$ and $j = \{1, \cdots, N_\theta\}$. In practice, we can aggregate the possible levels of instantaneous speed of deterioration into qualitative states, e.g. *very slow, slow, fast, very fast*.

When the last performed maintenance is of type m, the transition probability from state $s = (\rho, \theta, h, m)$ at the beginning of the decision epoch t to $s' = (\rho', \theta', h, m)$ at its end $t + \tau$

is given by

$$\begin{aligned} p_{ss'}^{m} &= Pr\{s_{t+\tau} = s' | s_t = s\} \\ &= Pr\{\rho_{t+\tau} = \rho' | \rho_t, \theta_t, h, m\} \; Pr\{\theta_{t+\tau} = \theta' | \rho_t', \rho_t, h, m\} \\ &= Pr\{\rho' | \rho, \theta, h, m\} \; Pr\{\theta' | \rho', \rho, h, m\}, \end{aligned}$$

where $Pr\{\rho'|\rho, \theta, h, m\}$ is given by the density of the SDG process and $Pr\{\theta'|\rho', \rho, h, m\}$ by the density of the BG process [24].

As the section state is partially observed, the knowledge on the section state is represented by a probability distribution over the real states, referred to as the belief state and denoted $\mathbf{b} = (b_1, \cdots, b_N)$, where $b_j, j \in \{1, \cdots, N\}$ is the probability that the real state (deterioration level) is s_j. More specifically, if $s = (\rho, \theta)$ and the observed LCP level is different from ρ, then $b_j = 0$, otherwise b_j is the probability that true DGR level is θ given the LCP observation ρ. In this last case, $\mathbf{b}$ is equivalent to the $N_\theta-$ vector $\mathbf{b}^\theta$, the belief on the DGR θ. Finally, the belief state verifies $b_j \geq 0$ for $j \in \{1, \cdots, N\}$ and $\sum_{j=1}^{N} b_j = 1$. Let B be the set of all possible beliefs, i.e., the set of probability distributions over the section states.

Based on this knowledge, the decision-maker selects an action $a = (h_p, h_n)$. As the DN action does not alter the current state, it will also not alter the belief. Any other action changes the belief into a new belief $\mathbf{b^a}$ given by a deterministic function of the performed action and the belief just before action. At the end of the decision epoch, given the last maintenance type m the true state s of the road section evolves to a new state s' with probability $p_{ss'}^{m}$ defined in Equation (1) and the section is inspected to yield an observation of ρ. This LCP observation is used to estimate an average deterioration speed $\tilde{\theta}$ on the last section. Let o be the state observation $(\rho, \tilde{\theta})$. Let O be the set of observation and assume that there is a finite number N_o of possible observations of (ρ, θ).

For $s \in S$ and $o \in O$, let q_{so} be the probability observing o when the real section state is s. Let $Q = [q_{so}]$ be the $N \times N_o$ observation matrix.

Given the last maintenance type m, let $\Omega(o|\mathbf{b})$ be the probability of observing o at the end of the current decision epoch when the belief is $\mathbf{b}$ at its beginning.

$$\Omega(o|\mathbf{b}) = \sum_{s\in S} \sum_{s'\in S} b_s \, p_{ss'}^{m} \, q_{s'o},$$

$\Omega(o|\mathbf{b})$ is the sum of all possible transitions from s to s' that may yield the observation o weighted by the belief distribution. Based on the observation o, the belief state is updated to $\mathbf{b}'$ using the Bayes formula,

$$\mathbf{b}'_{s'} = \frac{\sum_{s\in S} b_s \, p_{ss'}^{m} \, q_{s'o}}{\sum_{s\in S} \sum_{s'\in S} b_s \, p_{ss'}^{m} \, q_{s'o}}, \quad \forall s'.$$

3.3 Cost decision criterion

The objective is to find the optimal policy that minimizes the discounted cost-to-go over the infinite horizon.

$$V^*(\mathbf{b}, h, m) = c_0 + \min\left\{DN(\mathbf{b}, h, m), MX(\mathbf{b}, h, m)\right\} \tag{1}$$

where

$$DN(\mathbf{b}, h, m) = \sum_{o \in O} \Omega(o|\mathbf{b}) \left\{ C_q(\mathbf{b}, \mathbf{b}') + \lambda V^*(\mathbf{b}', h, m) \right\} \tag{2}$$

$$MX(\mathbf{b}, h, m) = \min_a \left\{ MX_{a_1}(\mathbf{b}, h, m), \cdots, MX_{a_2}(\mathbf{b}, h, m) \right\} \tag{3}$$

and for $k \in 1, \cdots, N_h$ and $a_k = (h_p, h_n)$,

$$MX_{a_k}(\mathbf{b}, h, m) = c_v(h_n, h_p) + DN(\mathbf{b^k}, h - h_n + h_p, k).$$

where λ is the discount factor and $\mathbf{b^k}$ is the belief state just after a maintenance action is performed at a current belief $\mathbf{b}$.

Equation (2) states that following the DN-action when the current belief is $\mathbf{b}$ incurs the expected quality cost plus the cost-to-go to the updated belief $\mathbf{b}'$ by the end of the decision epoch. The action a_k in Equation (3) incurs a fixed and a variable maintenance costs plus the DN-cost of a system with a new belief after maintenance.

4 Solution procedure & Numerical example

4.1 Solution procedure

POMDPs offer the possibility to take into account the non observability of some decision variables. However, they are very hard to solve and computing the optimal policy is challenging because of the curse of dimensionality [15]. The main problem with solving exactly POMDPs is that it is impossible to update all the belief values. Approximating a POMDP solution consists of applying value updates in only some specific beliefs. Different approaches to select these specific beliefs to update have been developed such as grid-based approximation ([10] and [22]) and point-based approximation ([15, 3, 21] and [16]). Once the beliefs are selected, their value updating routine is standard [15]. In this work, we use the value iteration algorithm with the regular grid-based approximation proposed by Lovejoy (1991) [10].

4.2 Numerical example

In this section we discuss our model using a numerical example. For the transition probabilities, we consider the same example deterioration model as proposed in [23] and [24]. We assume that, given the real DGR, the observed one $\tilde{\theta}$ is normally distributed with mean equal to θ and a a given variance σ^2.

$$Pr\{\tilde{\theta} \mid \theta\} = N(\theta, \sigma^2).$$

The maintenance effect function on the DGR θ is given for $s = (\rho, \theta, h, m)$ by

$$\phi(s, a) = r_a(\theta + \frac{\rho}{\tau} + \frac{1}{h})$$

where r_a is a reduction factor depending on the performed action. Cost parameters are given by the following values (in Euros)

$$c_n = 200 \; ; \;\; c_p = 600 \; , \;\; c_{un} = 30 \; ; \;\; c_{up} = 70.$$

The quality cost function is given by

$$C_q(\tau; \mathbf{b}) = \tau(800 \, \mathbf{b} \, + \frac{1000}{h})$$

The discount factor is $\lambda = 0.95$. We also consider an action set presented by the following Table (h_p and h_n are given in centimeters)

Table 1: Definition of the action set by the associated removed and added layers thicknesses

$\mathbf{a}$	1	2	3	4	5	6	7	8	9	10
$\mathbf{h_p}$	0	5	5	5	5	10	10	10	10	15
$\mathbf{h_n}$	0	0	2.67	5.33	8	3	4.67	6.33	8	8

Finally, we obtain the policy presented in Figure 2. The optimal policy is given here for a (6x5) state set and evaluated with the Value Iteration algorithm based on a grid based approximation with a grid resolution parameter $M = 3$. The decision matrix gives the optimal maintenance action for each observed ρ and each belief on θ when the current thickness of the road section is $h = 8$ and the last added layer is $h_p = 5$.

For example (frames in red full lines in Figure 2), for an observed $\rho = 0.4$ and a belief on θ, $\mathbf{b}^\theta = (0, 0, 1, 0, 0)$, the optimal maintenance action is $(h_p = 10, h_n = 8)$, i.e., it is optimal to remove all the road and rebuild a thicker one. When $\mathbf{b}^\theta = (0.67, 0, 0.33, 0, 0)$, the optimal action is to remove nothing and add a layer of thickness 5.

Note that, due to the deterioration growth rate parameter, the provided policy is a preventive maintenance policy: even when the road is free of cracks, it is optimal to maintain it for high values of θ. The GDR is then a risk measure. The presence of action 10 in the optimal policy ensures that the road will be totally renewed at least once.

The five first rows of the decision matrix corresponds to the perfect information case, i.e, when θ is observable. Note that considering a probability distribution over θ may make some savings. For example (frames in green dashed lines in Figure 2), the optimal action when $\rho = 0.2$ and $\theta = 0.6$ with probability 1 is action 6. Whereas, for the same level of cracking but with θ equal to 0.6 with probability 0.67 and equal to 0.4 with probability 0.33, the action 2 is recommended.

This confirms the importance of the growth rate as a deterioration parameter and the sensitivity of our model to it. From numerical examples, we can notice that obtained policies present a structural properties with respect to the stochastic ordering of each belief parameter as well as the belief, i.e., the action type increases as the LCP and the belief on θ increase, as well as when the belief state increases. Moreover, the structure of optimal policies depends especially on the maintenance effect function. This function reflects the preference between a

	(0,0,0.33,0.33,0.33)	6	6	9	9	9	9
	(0,0.33,0,0.33,0.33)	2	2	2	9	9	9
	(0,0.33,0.33,0,0.33)	2	2	2	2	9	9
	(0,0.33,0.33,0.33,0)	2	2	2	2	9	9
	(0.33,0,0,0.33,0.33)	2	2	2	2	9	9
	(0.33,0,0.33,0,0.33)	2	2	2	2	2	2
	(0.33,0,0.33,0.33,0)	2	2	2	2	2	2
	(0.33,0.33,0,0,0.33)	2	2	2	2	2	2
	(0.33,0.33,0,0.33,0)	2	2	2	2	2	2
	(0.33,0.33,0.33,0,0)	2	2	2	2	2	2
	(0,0,0,0.67,0.33)	6	6	6	9	9	9
	(0,0,0.67,0,0.33)	2	2	2	9	9	9
	(0,0.67,0,0,0.33)	2	2	2	2	2	2
	(0.67,0,0,0,0.33)	2	2	2	2	2	2
	(0,0,0,0.33,0.67)	9	9	9	9	9	9
	(0,0,0.67,0.33,0)	2	2	2	9	9	9
	(0,0.67,0,0.33,0)	2	2	2	2	2	2
b_θ	(0.67,0,0,0.33,0)	2	2	2	2	2	2
	(0,0,0.33,0,0.67)	9	9	9	9	9	9
	(0,0,0.33,0.67,0)	6	6	2	9	9	9
	(0,0.67,0.33,0,0)	2	2	2	2	2	2
	(0.67,0,0.33,0,0)	2	2	2	2	2	2
	(0,0.33,0,0,0.67)	9	9	9	9	9	9
	(0,0.33,0,0.67,0)	2	2	2	2	2	2
	(0,0.33,0.67,0,0)	2	2	2	2	2	2
	(0.67,0.33,0,0,0)	2	2	2	2	2	2
	(0.33,0,0,0,0.67)	9	9	9	9	9	9
	(0.33,0,0,0.67,0)	2	2	2	2	2	8
	(0.33,0,0.67,0,0)	2	2	2	2	2	2
	(0.33,0.67,0,0,0)	1	2	2	2	2	2
	(0,0,0,0,1)	9	9	9	9	9	10
	(0,0,0,1,0)	6	6	6	9	9	9
	(0,0,1,0,0)	2	2	2	8	9	9
	(0,1,0,0,0)	1	1	1	2	2	2
	(1,0,0,0,0)	1	1	1	1	1	1
		0	0.03	0.2	0.4	0.6	0.8
					ρ		

Figure 2: Optimal policy for a (6x5) problem where $h_{max} = 15$, $h = 8$ and $h_p = 5$.

newer and a thicker road.

A direct comparison with the current French practices in road maintenance is not provided here. The current road maintenance strategies are defined in a complex process based on the evaluation of various scenarios taking into account many external parameters. It may be noticed that the decision is based, among other things, on the observation of the deterioration level, i.e. ρ. This policy would be equivalent to a policy that only one action would be identified in each column of the decision matrix. The proposed optimized policy differentiates maintenance actions according to $\mathbf{b}^\theta$ highlighting the non-optimality of the current practices.

5 Conclusions

We propose a condition-based maintenance optimization approach for road section to minimize the total maintenance cost over the infinite horizon with perfect and different imperfect maintenance actions are available. Our approach is based on a two-phase deterioration process with two deterioration parameters: the longitudinal cracking percentage and the deterioration growth rate. Our deterioration model has the advantage to be state-dependent. Moreover, the introduction of the speed of deterioration as a decision parameter allows the model to take into account the evolution of the underlying deterioration and so to the prediction of the deterioration evolution to be more precise.

However, the deterioration speed cannot be observable. We model the problem as a partially observed Markov decision process and solve it using a grid-based value iteration algorithm. This algorithm is highly time-consuming when the size of the state space is big because it generates a large belief grid. In our case, the size of the grid is reduced because only one of two decision parameters is not observable.

In practice, excepted the difficulties to estimate the parameters of the two-phase deterioration model, our approach can easily be used to provide optimal policies where the maintenance decision is not based on the pre-definition of deterioration level thresholds, which is not possible in the initiation phase. We propose optimal preventive policies that allow to maintain the road when it is free of cracks.

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7 Structural Properties For Solution Procedures

Structural Property-Based Algorithms for MDP and POMDP with Decision-Dependent Uncertainty

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Abstract

We investigate structural properties of Markov decision processes (MDP) with endogenous uncertainty. This endogenicity arises from decision-dependent transition matrices which differs from classical MDP that have a unique transition matrix. We establish analytical conditions under which the optimal policy of the MDP with decision-dependent transitions is of the control limit type. We also determine conditions that guarantee that all actions are selected at least once in the optimal policy. We use these properties to significantly accelerate the policy iteration algorithm (PIA). These properties are also used to develop a novel heuristic for the imperfect information problem modeled as a partially observed MDP (POMDP). The heuristic develops a belief grid by establishing cuts between action regions and reduces the action space via a structural property based rule. Numerical examples comparing our heuristic to the regular grid-based approximation show substantial savings in computation time (up to 99%) as well as its minimal deviation in objective value. Our heuristic also allows solving large-scale problems in only a few hours that cannot be solved with classical methods. We motivate our work through an application to road maintenance, however this work can be utilized for many general decision-making problems with an investment/risk structure.

Keywords: Maintenance policy, Markov decision process, Partially observed Markov decision process, Control-limit policy, Policy iteration, Heuristic, grid-based approximation.

Notations

A	set of actions	s_0	as-good-as-new state
$\vec{b}$	belief state	s_N	most deteriorated state
b_s	probability that the true state is s	s_m	section state given the last performed action m
$\tilde{\vec{b}}(o, \vec{b})$	updated belief from $\vec{b}$ after observing o	$s^k(s)$	section state after performing action k in state s
B_{jk}	belief cut between actions j and k	$s^k_m(s_m)$	state after performing action k in state s given the last performed action m
$c(k)$	cost of action k	V	Total expected discounted cost-to-go over the infinite horizon
$C_q(\cdot)$	quality cost function	ρ	longitudinal cracking percentage
$e_{jk}(\cdot)$	immediate efficiency function between actions j and k	ρ_k	longitudinal cracking percentage after action k
G	belief grid	θ	deterioration growth rate
g	belief of the grid	θ_0	minimum deterioration growth rate
h	road thickness	θ_{max}	maximum deterioration growth rate
h_{max}	maximum road thickness	θ_k	deterioration growth rate after action k
m	last maintenance type	$\Delta_{jk}C_q(\cdot)$	quality cost between difference between j and k
N	number of states	$\bar{\theta}$	average cracking speed
N_ρ	number of possible values of ρ	$\phi(\cdot)$	maintenance effect function on θ
N_θ	number of possible values of θ	$\gamma()$	state score function
o	observation	τ	inter-inspection period
O	set of observations	λ	discount factor
$p^k_{ss'}$	transition probability after performing action k	π	maintenance policy
q_{so}	probability of observing o when the real state is s	π^*	optimal maintenance policy
R	number of available actions	σ	standard variation of the normal distribution of DGR observations.
S	set of states	$\Omega(o\vert\vec{b})$	probability of o at the decision epoch end when $\vec{b}$ at its beginning
s	section state		

1 Introduction

In many fields of application such as energy, industrial engineering and civil and transportation infrastructure, the maintenance of operating systems is a very important issue of the management process as it offers an opportunity to ameliorate the reliability of operating systems, ensure acceptable levels of production or service quality and reduce total long-term repair and replacement expenses. Moreover, in many fields like civil and transportation infrastructure management, maintenance optimization involves user safety and comfort issues.

As most decision-making problems, maintenance optimization involves uncertainty that concerns the performance of operation systems, or equivalently their deterioration processes. Several techniques of dynamic decision-making under uncertainty exist to provide decision-makers with efficient policies for finite or infinite decision horizons. These technics are strongly based on the stochastic process modeling the uncertainty and its evolution. Markov decision processes (MDP) are a dynamic programming tool widely used to handle decision processes that are based on system state evaluation. They provide decision-makers with policies that determine the optimal maintenance action to perform in each possible system state, usually revealed by inspection.

In this paper, the maintenance optimization problem of a stochastically deteriorating road section with both perfect and imperfect maintenance options is studied. The deterioration level of the road section is described using two parameters: the longitudinal cracking percentage (LCP) and the deterioration growth rate (DGR). We use the action-dependent deterioration process proposed by Zouch *et al.* [24] to illustrate the decision-dependent uncertainty, but the results in this paper are not dependent upon the use of this specific process.

One of the special features of this deterioration model is that the evolution law changes in each decision epoch as it depends not only on the section state at its beginning but also on the last performed action type. This implies an endogenous, i.e., decision-dependent uncertainty. The deterioration model is incorporated into an MDP framework to derive state transition probability matrices and optimal maintenance policies that minimize the discounted expected maintenance cost-to-go over the infinite-horizon. Hence, unlike existing MDP-based models where the transition matrix is unique for all the actions (i.e., exogenous uncertainty), the MDP in this work considers a different transition matrix after each maintenance. To the best of our knowledge, this is the first proposed MDP model with endogenous uncertainty.

One of the most important results in MDP theory is structural properties. The most significant MDP structural property is the optimal policy being of the control limit type [7]. A control limit policy [6, 17] is a policy that dictates to replace the system whenever the system state exceeds some threshold. These types of policies are appealing because they are easy to understand and to implement. Structural properties help decision-makers to determine the effects of the system and cost parameters on the decision process. Moreover, they can be highly valuable to reduce the search for the optimal policy, especially for POMDP, by defining efficient problem-specific solution procedures or accelerating considerably the convergence of generic solution procedures such as the value iteration or policy iteration algorithms.

In this paper, conditions under which an MDP with decision-dependent uncertainty demonstrates appealing structural properties such as being of the limit control rule type are presented. These properties are then exploited to accelerate the convergence of the policy iteration algorithm (up to 70%) while maintaining an exact solution.

Along with the Markovian property of the state, which is a central assumption in MDP, the main restrictive assumption in MDP models is the complete observability of the state assumption. In many real-world case problems, all or some of the decision variables are not observable or cannot be directly observed, e.g., in road maintenance, the DGR is not a measurable metric. In [24], the authors approximate the DGR by the average speed of cracking obtained from successive observations of the LCP. They also extended the model to the partially observed Markov decision process (POMDP) formulation in [25] to take into account the non-observability of the DGR parameter by considering a probabilistic observation density on the DGR instead of the deterinistic approximation of the MDP formulation. In POMDP models the state of the system is replaced by a belief state which a vector of the state set size representing the probabilities associated with each possible state. Due to the continuity of the belief state space, POMDP are numerically intractable and exact solution are possible only for very small size problems.

We are not able to prove the structural properties derived for the MDP model in this work for the POMDP model because of the imperfect information that characterizes observations. However, we utilize these MDP properties to define a heuristic solution procedure for the POMDP. The solution procedure is based on the definition on action cuts that represent immediate action control limits (i.e., for the current decision epoch). These cuts are used to define belief regions where the set of possible actions is reduced using a structural property derived for the MDP model, and build a gird comprised of both beliefs from cuts and cut intersections. Numerical results given by the proposed heuristic are compared to the regular grid-based approximation (GBA) proposed by Lovejoy [9] and show substantial savings in computation time (up to 99%) with minimum objective value deviation (4%-13%). The heuristic also gives the capacity to solve large-size problems that are not possible using the grid-based approximation method.

Moreover, although the proposed solution procedures are applied to road maintenance, they can be also applied to many decision-making problem with an investment/risk structure.

The remainder of this paper is structured as follows. Section 2 presents a brief literature review on structural properties for MPD and POMDP. The road maintenance optimization problem is described in Section 3. Section 4 is dedicated to the MDP problem where structural properties are derived and compared to the existing properties in the literature. A solution procedure is also presented in Section 4 and its numerical results compared to the policy iteration algorithm results. In Section 5, the maintenance optimization problem is formulated as a POMDP with imperfect information. Derived structural properties for the perfect information case (MDP) are used to derive a heuristic solution procedure that is described and compared to the regular GBA. Finally, we conclude the paper in Section 6.

2 Literature review

The control limit rule is a property that originally characterized optimal stopping problems such as asset selling and purchasing with a deadline [3]. Puterman [15] presents general conditions under which an MDP model, equivalent to a stopping problem, demonstrates the control limit property. Note that an MDP can be equivalent to a stopping problem in the case of a set of actions comprised of "Do" and "Do Not" actions. In addition to these general conditions, application-specific conditions have been demonstrated in the literature. For example, in a health care context, Alagoz et *et al.* [1] provide clinically realistic conditions under which it is optimal to perform a living donor liver transplant. In a maintenance optimization context, results for optimization problems with perfect actions (equivalent to optimal stopping problems) follow directly from the results given by [15].

Douer and Yachiali [7] generalize these result to a problem with imperfect actions. The authors define a generalized control limit policy as a rule by which the system should be perfectly or imperfectly maintained whenever current state exceeds some limit state.

Only few works attempt to extend structural properties to POMDP problems. In the maintenance optimization context, most of these works consider a maintenance problem where partial observability is a product of aperiodic inspections. A general POMDP formulation of this problem is given in [8]. Smallwood and Sondik [18] show that for any finite-horizon POMDP, the optimal value function of the remaining decision periods is piecewise linear and convex. Sondik [20] extend these results to the infinite-horizon discounted case. Ross [17] present conditions for the optimal policy to be a monotonic "At-Most-Four-Region" (AM4R) policy for the problem with two possible states: failed and not failed. This property was then extended by Rosenfield in [16] to problems with more than two possibles states. White [21] prove similar results for systems with silent failures under less stringent conditions by formulating the problem as a POMDP and using stochastic ordering to derive monotonicity results on subsets of the action space. More specifically, he proves that the optimal policy for a silent failure system along straight lines of knowledge states in increasing stochastic order is a monotonic AM4R policy if the state transition matrix has an increasing failure rate. Maillart [10] extend the AM4R property to systems with obvious failure and perfect information. They then use this result to define a heuristic to solve the problem with imperfect information.

Different numerical methods proposed in the literature to solve POMDP are based on a reduction of the number of possible belief states to a finite set of points and the use of interpolation or intermediate beliefs to approximate the value function. Lovejoy [9] proposes a grid-based approximation method that applies a bounding procedure to the algorithm proposed by [18]. More recent approaches are those with point-based approximations such as [22] and [13]. These approaches are generally time-consuming as the number of vectors that express the value function can increase exponentially as the size of the state space increases. Other algorithms use dominance criteria and pruning mechanisms to reduce the number of vectors to describe the value function. Such algorithms, as the linear support algorithm and the witness algorithm, can be found in [4]. An alternative opportunity to solve POMDP can be offered by the structural properties of the POMDP problem to define efficient heuristics [10].

Different from literature, this paper considers a decision problem where the state transition process, i.e., the deterioration process is action dependent. Under this assumption, additional conditions should be considered (i.e., conditions to compare different transition processes) in order to optimal MDP policies to be control limit rules. Moreover, whereas the generalized control limit policy [7] only states the existence of a control threshold between maintaining (perfectly or imperfectly) or not maintaining, we add conditions under which optimal policies of the MDP model divide the state space into exactly as many regions as the number of actions. From a practical point of view, such a property helps decision-makers, especially in large action set cases, define a set of significant maintenance actions by eliminating "useless" actions according to the conditions that should verify cost and transition structures.

Moreover, the imperfect information in this works stems from the non-observability of the second deterioration parameter rather than aperiodic inspections. Although structural properties are not proven for the POMDP model, they are used to define a heuristic solution procedure.

3 The road section maintenance problem

Consider a road section that is subject to stochastic and continuous deterioration. The deterioration mode of interest is longitudinal cracking problem or fatigue cracking that is usually caused by sustained traffic loads and harsh environment conditions.

The section is characterized by an initial thickness h and its deterioration level is represented by the longitudinal cracking percentage (LCP) and the deterioration growth rate (DGR) [24]. The LCP, denoted ρ, is the ratio of the cracked section length over its total length as illustrated by Figure 3.1. Whereas the DGR, denoted θ, represents an instantaneous speed of deterioration and equivalently a potential of deterioration when no cracks are observed during the initiation phase. The section state is defined by its deterioration level (ρ, θ) and its thickness h.

The infinite horizon is divided into equal decision epochs of length τ at the beginning of each the

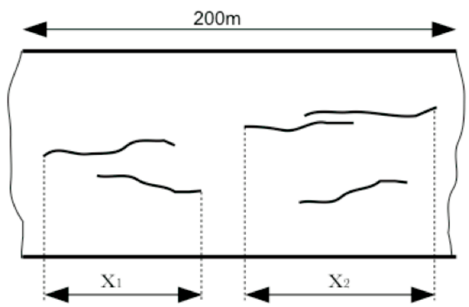

Figure 3.1: The LCP metric: $\rho = (x_1 + x_2)/200$

road section is inspected to measure its deterioration level (ρ, θ). In practice, only ρ can be observed. However, we assume that θ is also fully observable by approximating the DGR by the average cracking speed using successive observations of ρ.

After inspection, the decision-maker chooses whether to do nothing (DN) and wait until the next decision epoch or to perform one of several available maintenance actions (MX). Given an initial thickness h, a maintenance action consists in milling some or all the existing road pavement and resurfacing with a new concrete layer so that the section thickness does not exceed a maximum thickness h_{max}. In this work, we only consider actions that mill and add the same thickness so that the total road thickness does not change. An action corresponds then to the thickness of the renewed layer. Note that the DN action corresponds to renewing zero thickness.

The set of actions is discretized into R possible actions where 1 indexes the DN action and $k \in \{2, \cdots R\}$ indexes $R - 1$ different possible thicknesses to renew.

The state space $S = \{(\rho, \theta) \in [0, 1] \times [\theta_0, \theta_{max}]\}$ is also discretized so that ρ and θ represent the first value in N_ρ and N_θ equal size discrete intervals, respectively, such that N_ρ and N_θ divide evenly into 100.

Maintaining the road section conceals surface cracks, i.e., resets ρ to zero, and reduces the DGR of the road section according to a deterministic function $\phi(\rho, \theta; k)$ where ρ, θ are the section LCP and DGR just before maintenance, respectively, and k represents the type of the performed action. If the DN action is selected, then the section state just after decision is assumed to be the state revealed by inspection. It is assumed that the deterioration process is changed after each maintenance. Therefore, in order to ensure the Markovian property to the state definition, the type of the last maintenance, denoted $m = 2, \cdots, R$, is added to the state definition.

Let $s_m = (\rho, \theta, m)$ denote the section state. In order to lighten notation in the remainder of the paper, the following state notation is used $s_m = (s, m)$, where $s = (\rho, \theta) \in S$. The separation between s and m in the state definition is motivated by the fact that only s (representing the section deterioration level) will be used for cost evaluation and that m is only used to determine the deterioration process in case the DN action is selected ($m = k$ when action k is selected).

A maintenance action incurs a setup cost as well as a variable cost given unit milling and resurfacing costs. Let $c(k)$ be the total cost of action k such that $c(1) = 0$ and $c(2) < c(3) < \cdots < c(R-1) << c(R)$. Moreover, in order to prevent the decision-maker from always selecting the DN action and penalize leaving the road section during the decision epoch without maintenance, a penalty cost called quality cost $C_q(\tau; \rho_k, \theta_k)$ is considered, where ρ_k and θ_k are the section LCP and DGR immediately after performing an action $k \in \{1, \cdots, R\}$.

The following notations will be used in the remainder of the paper. For a given state $s_m = (s = (\rho, \theta), m)$, let

$$s_m^k(s_m) = \begin{cases} (s, m) & \text{if } k = 1 \\ (s^k(s), k) & \text{otherwise} \end{cases}$$

denote the section state immediately after performing action k in state s_m. Moreover $s^k(s) = (\rho_k, \theta_k)$, where $\rho_1 = \rho$, $\rho_k = 0, k > 1$, and $\theta_k = \phi(\rho, \theta, k)$ are the section LCP and DGR immediately after performing action k. Finally, with the new notations, the quality cost function and the maintennace effect function on the DGR are denoted $C_q(\tau, s)$ and $\phi(s, k)$, respectively.

4 The fully observable problem

4.1 The MDP model

In this section, the DGR is assumed to be observable and is approximated by the average cracking speed using successive observations of the LCP, i.e., $\theta \sim (\rho_{t+\tau} - \rho_t)/\tau$. Thus, the problem can be formulated as an MDP. Note that, in existing MDP models, transitions are said to be action-dependent since the state from which the system transits, i.e., the state immediately after maintenance, depends on the selected action. However, in theses models the transition process (i.e., transition probability matrix) remains unique, identical to the new system one. In this work, we use the state-dependent deterioration model developed in [24] where the state evolution law does not only depend on the system state immediately after maintenance, but also changes with the last maintenance type.

Let $p_{ss'}^k$ denote the transition probability from state s to s' after performing an action k. The MDP model is given by the following. For $s_m = (s, m)$,

$$V^*(s) = c_o + \min\{DN(s), \min_k\{MX_k(s)\}\} \tag{1}$$

where

$$DN(s) = C_q(\tau; s) + \lambda \sum_{s' \in S} p^m_{ss'} V^*(s') \quad (2)$$

$$MX_k(s) = c(k) + C_q(\tau; s^k(s)) + \lambda \sum_{s' \in S} p^k_{s^k(s)s'} V^*(s') \quad (3)$$

where $\lambda \in [0, 1]$ is the discount factor. Equation (2) states that following the DN action when the current state is s incurs the immediate quality cost plus the cost-to-go of the system evolving from the state s to any state s'. The MX_k action (3) incurs a maintenance cost $c(k)$ plus an immediate quality cost of the new state $s^k(s)$ resulting in performing action k to state s, plus the cost-to-go of the system beginning with $s^k(s)$, or equivalently, to the DN cost of the same system beginning in state $s^k(s)$.

4.2 Structural properties

Structural properties are defined for stochastically ordered state sets (since the state evolves according to a stochastic process). More specifically, given a current state s and two maintenance actions j and k, the state may evolve at the end of the decision epoch to the random states $s'(s, j)$ and $s'(s, k)$ if action j and k were performed at the beginning of the decision epoch, respectively. The state $s'(s, j)$ is said to dominate $s'(s, k)$, denoted $s'(s, j) \succeq s'(s, k)$, on a set of real-valued functions U if [19, 2, 12]:

$$E[u(s'(s, j))] \geq E[u(s'(s, k))], \quad \forall u \in U$$

When U is the set of real-valued-non decreasing functions on S, the stochastic ordering corresponds to the usual first-order stochastic dominance (FSD), and when U is the set of real-valued-non decreasing and concave functions on S, the stochastic ordering corresponds to second-order stochastic dominance (SSD) which we need in this paper. In a reliability context, the SSD corresponds to the increasing failure rate property which is defined for a transition matrix $P = (p_{ss'})_{s,s'=1,..,N}$ by:

Definition (IFR property): For each $i = 1, \cdots, N$, the function $D_i(s) = \sum_{s'=i}^{N} p_{ss'}$ is non-decreasing in s.

From [6], the IFR property is equivalent to the following.

Proposition For every nondecreasing function u in s, the function $K(s) = \sum_{s'=1}^{N} p_{ss'} u(s')$ is also nondecreasing in s.

Note that the stochastic order is also defined for ordered states. When the state is defined by a single deterioration parameter, the states can be ordered according to an increasing deterioration level. Here the deterioration level of the road section is represented using two variables ρ and θ. The state order can be defined using a one-to-one score function $\gamma(s) = \gamma(\rho, \theta) : S \rightarrow \mathbb{R}^+$. A state $s_1 = (\rho_1, \theta_1)$ is said to be less deteriorated than a state $s_2 = (\rho_2, \theta_2)$, denoted $s_1 < s_2$, if

$$\gamma(\rho_1, \theta_1) \leq \gamma(\rho_2, \theta_2).$$

Note that propositions that will be presented in this section do not depend on the score function γ.

Assumption 1 *The state transition matrix P^k after a maintenance action $k = 2, \cdots, R$ is IFR.*

The IFR property states that more deteriorated states are more likely to deteriorate more in the next periods which is an acceptable assumption for a cumulative deteriorating system.

The next assumption states that the more deteriorated is the state, the higher is the DGR after maintenance. Similarly, the weaker is the performed action, the higher is the DGR after maintenance.

Assumption 2 *The maintenance effect on the DGR function $\phi(s, k)$ is:*

(i) non-decreasing in the state s.

(ii) non-increasing in the action $k \in A$.

Assumption 2 (i) implies that the order of the states does not change after maintenance. This is implied by the deterministic effect of maintenance actions.

Lemma 1 *Under Assumption 2, if $s_1 < s_2$, then $s^k(s_1) < s^k(s_2), \quad k = 1, \cdots, R$.*

Proof

- $k = 1$, obvious since $s^k(s_1) = 1$ and $s^k(s_2) = s_2$

- $k > 1$, $s^k(s_1) = (0, \phi(s_1, k))$ and $s^k(s_2) = (0, \phi(s_2, k))$, and (ii) of Assumption 2 completes the proof.

□

Assumption 3 *For $j, k \in \{1, \cdots, R\}$ such that $j < k$, then $c(j) \leq c(k)$.*

Assumption 4 *The quality cost function $C_q(\tau; s)$ is non-decreasing in s.*

Assumption 3 states that the more expensive is the action, the stronger (i.e., immediately more efficient) it is. Assumption 4 states that the more deteriorated is the system, the higher is the penalty cost paid for deterioration risk. Note that Assumptions 3 and 4 are classical and not restrictive.

Assumption 5 *The quality cost function $C_q(\tau; s)$ is concave.*

Assumption 5 implies that the quality cost function is super-additive, i.e.,

$$C_q(\tau;\ s_1 + s_2) \geq C_q(\tau; s_1) + C_q(\tau; s_2), \quad \forall s_1, s_2.$$

The super-additive property is an intuitive assumption in road maintenance. It reflects a risk-averse policy in order to prevent leaving the road in highly deteriorated states, i.e., it is better to maintain the road twice in two intermediate states s_1 and s_2 than to wait a total deteriorated state $s = s_1 + s_2$.

Assumption 6 *The function $\Delta_{jk} C_q(s) = C_q(\tau; s^j(s)) - C_q(\tau; s^k(s))$ is non-decreasing in s, for $j < k$.*

Assumption 6 is an intuitive assumption stating that the gain in quality cost between two maintenance actions increases with more deteriorated states. It indicates that stronger actions become more beneficial in more deteriorated states.

We introduce the following function representing the difference of immediate efficiency (i.e., of the single, current decision period) of two maintenance actions $j < k$ in a given state s.

$$e_{jk}(s) \quad = \quad [C_q(\tau; s^j(s)) - C_q(\tau; s^k(s))] - [c(k) - c(j)]$$

Note that $e_{jk}(s)$ is the difference of two terms. The first one, $(C_q(\tau; s^j(s)) - C_q(\tau; s^k(s)))$, is the difference between the quality costs after performing actions j and k and represents the potential gain in risk in the current decision epoch or the immediate reward incurred by performing a stronger action. The second term is the difference in action costs and represents the additional investment that should be made to perform this stronger action. Three cases can be distinguished:

- $e_{jk}(s) > 0$, action k is immediately more efficient than action j since it makes risk saving greater than its additional investment.

- $e_{jk}(s) = 0$, actions j and k have the same immediate efficiency, however the cheaper action should be preferred.

- $e_{jk}(s) < 0$, action j is immediately more efficient than action k and it is not worth investing more in action k.

Assumption 6 implies that $e_{jk}(s)$ is non-decreasing in s.

All the properties presented next hold under Assumptions 1-6.

Proposition 1 *The value function of the MDP model (1) is non-decreasing concave in the state $s = (\rho, \theta)$, for any given $m = 2, \cdots, R$.*

Proof (by induction on the iterations i of value iteration)

- $i = 1$. $V_1(s) = \min\{Cq(\tau; s), \min_k\{c(k) + Cq(s^k(s))\}\}$.
  Assumption 4 ensures the property for $i = 1$.

- Assume that the property holds for iteration $i-1$, we shall prove it for iteration i. Let $s_1 < s_2$ in S
  If the selected action at stage i is the DN action, then

$$\begin{aligned}
V_i^*(s_2) &= C_q(\tau; s_2) + \lambda \sum_{s'} p_{s_2 s'}^m V_{i-1}^*(s') \\
&\geq C_q(\tau; s_1) + \lambda \sum_{s'} p_{s_1 s'}^m V_{i-1}^*(s') \\
&\geq V_t^*(s_1)
\end{aligned}$$

  The first inequality follows from Assumption 4, and Derman's property applied to Assumption 1 and Proposition 1 at $i-1$. The second inequality follows from Equation (1) of the MDP model.
  Similarly, if the selected action at stage i is a maintenance action k, then

$$\begin{aligned}
V_t^*(s_2) &= c(k) + C_q(\tau; s^k(s_2)) + \lambda \sum_{s'} p_{s^k(s_2)s'}^k V_{i-1}^*(s') \\
&\geq c(k) + C_q(\tau; s^k(s_1)) + \lambda \sum_{s'} p_{s^k(s_1)s'}^k V_{i-1}^*(s') \\
&\geq V_i^*(s_1)
\end{aligned}$$

  The first inequality follows from Lemma 1, in addition to Assumptions 4, 1 and Proposition 1 at $i-1$.

The same proof can be applied to prove the concavity of the value function under Assumption 5. □

In order to extend the classical MDP structural results such as the control limit rule property to the case of action-dependent transitions, the following assumption is necessary.

Assumption 7 *(Action-dependent IFR)*
For $j \leq k$ and $s_1 < s_2$, $\sum_{s'=1}^{N}(p_{s_1 s'}^j - p_{s_2 s'}^k) \geq 0$.

Assumption 7 compares the system transitions after different actions. It states that the system is more likely to deteriorate after performing a weaker action in a given state s since weaker actions reset the system to a more deteriorated state than a stronger one. Note that Assumption 7 becomes equivalent to the IFR property in the classical context, i.e., action-independent transition process.

Therefore, we denote the assumption as *action-dependent IFR*.

The next proposition states that if among two actions the stronger one is immediately more efficient, then it will also be more efficient for the infinite horizon, i.e., optimal over the weaker one. Whereas if the weaker action is immediately more efficient then we cannot draw any conclusion. However, if it is known that the weaker action is optimal over the stronger one (i.e., more efficient in the infinite horizon), then it will also be immediately more efficient.

Proposition 2 *For $j < k$:*

(i) If $e_{jk}(s) > 0$ then action k is optimal over action j.

(ii) If $e_{jk}(s) = 0$ then action j is optimal over action k.

(iii) If action j is optimal over k then $e_{jk}(s) \leq 0$.

Proof

(i) $MX_j(s) - MX_k(s) = e_{jk}(s) + \lambda \sum_{s' \in S}(p^j_{s^j(s)s'} - p^k_{s^k(s)s'})V(s')$
If $j < k$, then $s^k(s) < s^j(s)$, and under Assumption 7, $\sum_{s' \in S}(p^j_{s^j(s)s'} - p^k_{s^k(s)s'})V(s') \geq 0$.
Therefore, if $e_{jk}(s) \geq 0$, $MX_j(s) - MX_k(s) \geq 0$, and action k is prefered to j.

(ii) j is optimal over k in s, then $MX_j(s) - MX_k(s) \geq 0$. Since $\sum_{s' \in S}(p^j_{s^j(s)s'} - p^k_{s^k(s)s'})V(s') \geq 0$, then $e_{jk}(s)$ must verify $e_{jk}(s) \leq 0$.

□ The next two propositions determine the optimal policies in the least and most deteriorated states.

Proposition 3 states that, given the last maintenance type m, the optimal action is DN or an action that is stronger or equal to the last performed one m. This property ensures that it is possible to derive preventive policies.

Proposition 3 *If the section state is $s_m = (0, \theta_0, m)$, then $\pi^*(s) \in \{1\} \cup \{ k \geq m\}$.*

Proof If $s = (0, \theta_0, m)$, then

$$DN(s) - MX_k(s) \quad = \quad e_{1k}(s) + \lambda \sum_{s' \in S}(p^m_{ss'} - p^k_{s^k(s)s'})V(s').$$

$s^k(s) = (0, \theta_0, k)$. Therefore, If $k < m$, then using Assumption 7 and Assumption 4, we have:

$$\begin{aligned} \sum_{s' \in S}(p^m_{ss'} - p^k_{s^k(s)s'})V(s') &\leq 0 \\ C_q(\tau; s) - C_q(\tau; s^k(s)) &\leq 0 \end{aligned}$$

Thus, $DN(s) \leq MX_k(s)$. $\square$ Let $s_0 = (0, \theta_0, R)$ be the as-good-as-new (AGAN) and assume that $C_q(\tau; s_0) = 0$, an immediate result from Proposition 3 is $\pi^*(s_0) = DN$.

Proposition 4 *For $m = 2, \cdots, R$, let $s_m = (1, \theta_{max}, m)$. If*

$$\begin{cases} C_q(\tau; s) \geq c(R) \\ C_q(\tau; s^k(s)) \geq c(R) - c(k), \quad k = 2, \cdots, R-1 \end{cases}$$

then, $\pi^*(s) = R$.

Proof Recall that $c_q(\tau; s_0) = 0$. $s_m = (1, \theta_{max}, m)$, then

$$DN(s) - MX_R(s) \quad = \quad e_{1R}(s) + \lambda \sum_{s' \in S} (p^m_{ss'} - p^R_{s^R(s)s'}) V(s')$$

Under Assumption 7, $\sum_{s' \in S} (p^m_{ss'} - p^R_{s^R(s)s'}) V(s') \geq 0$. From assumption of proposition 4, $e_{1R}(s) \geq 0$. Thus,

$$DN(s) \geq MX_R(s) \tag{4}$$

Similarly,

$$MX_k(s) - MX_R(s) \quad = \quad e_{kR}(s) + \lambda \sum_{s' \in S} (p^k_{s^k(s)s'} - p^R_{s^R(s)s'}) V(s')$$

Under Assumption 7, $\sum_{s' \in S} (p^k_{s^k(s)s'} - p^R_{s^R(s)s'}) V(s') \geq 0$. From assumption of proposition 4, $e_{kR}(s) \geq 0$. Thus,

$$MX_k(s) \geq MX_R(s) \tag{5}$$

From (4) and (5), $\pi^*(s) = R$. $\square$ Let $\pi = (\pi(s_1), \cdots, \pi(s_N))$ be the A^N vector denoting a maintenance policy where A is the set of R available actions and N is the number of possible states s. π^* is the optimal policy and $\pi^*(s)$ denotes the optimal action in state s.

Proposition 5

$$\textit{If} \quad \begin{cases} s_1 < s_2 \\ \pi^*(s_1) = k \end{cases} \qquad \textit{then} \quad \pi^*(s_2) \geq k.$$

Proof (by induction on the iterations i of the Value iteration)

- For the first iteration, $i = 1$

$$\pi^*(s_1) = k \quad \Leftrightarrow \quad \begin{cases} e_{jk}(s_1) \geq 0, \forall j < k \\ e_{kl}(s_1) \leq 0, \forall l > k \end{cases}$$

From Assumption 6, $e_{jk}(s_2) \geq 0, \forall j < k$. Proposition 2 completes the proof for the $i = 1$.

- Assume that the proposition holds for the iteration $i - 1$, we shall prove it for iteration i. Let $\pi^*(s, t)$ denote the optimal action in state s in iteration i.

$$\pi^*(s_1, i) = k \quad \Rightarrow \quad \begin{cases} MX_k(s_1, i) < DN(s_1, i) \\ MX_k(s_1, i) < MX_j(s_1, i) \quad \forall j < k \\ MX_k(s_1, i) < MX_l(s_1, i) \quad \forall l > k \end{cases}$$

$$\Rightarrow \quad \begin{cases} \lambda \sum_{s' \in S} (p^k_{s^k(s_1)s'} - p^m_{s_1 s'}) V(s', i-1) & < e_{1k}(s_1) \\ \lambda \sum_{s' \in S} (p^k_{s^k(s_1)s'} - p^j_{s^j(s_1)s'}) V(s', t-1) & < e_{jk}(s_1), \quad \forall j < k \end{cases} \tag{6}$$

Let $l^* \in \{1, \cdots, R\}$ such that $s^{l^*}(s_2) \leq s^k(s_1) < s_{l^*-1}(s_2)$, then $l^* \geq k$.
From Assumption 6, the system of inequalities (6) is equivalent to

$$\begin{cases} \lambda \sum_{s' \in S} (p^k_{s^{l^*}(s_2)s'} - p^m_{s_1 s'}) V(s', t-1) & < e_{1k}(s_2) \\ \lambda \sum_{s' \in S} (p^k_{s^{l^*}(s_2)s'} - p^j_{s_j(s_1)s'}) V(s', t-1) & < e_{jk}(s_2), \quad \forall j < k \end{cases}$$

$$\Rightarrow \quad \begin{cases} \lambda \sum_{s' \in S} (p^k_{s^{l^*}(s_2)s'} - p^{l^*}_{s^{l^*}(s_2)s'}) + (p^{l^*}_{s^{l^*}(s_2)s'} - p^m_{s_1 s'}) V(s', t-1) & < e_{1k}(s_2) \\ \lambda \sum_{s' \in S} (p^k_{s^{l^*}(s_2)s'} - p^{l^*}_{s^{l^*}(s_2)s'}) + (p^{l^*}_{s^{l^*}(s_2)s'} - p^j_{s_j(s_1)s'}) V(s', t-1) & < e_{jk}(s_2), \quad \forall j < k \end{cases}$$

$l^* \geq k$ and Assumption 7 give $\sum_{s' \in S} (p^k_{s^{l^*}(s_2)s'} - p^{l^*}_{s^{l^*}(s_2)s'}) V(s', t-1) \geq 0$. Thus,

$$\begin{cases} \lambda \sum_{s' \in S} (p^{l^*}_{s^{l^*}(s_2)s'} - p^m_{s_1 s'}) V(s', t-1) & < e_{1k}(s_2) \\ \lambda \sum_{s' \in S} (p^{l^*}_{s^{l^*}(s_2)s'} - p^j_{s_j(s_1)s'}) V(s', t-1) & < e_{jk}(s_2), \quad \forall j < k \end{cases}$$

From Assumption 1 and Proposition 1,

$$\begin{cases} \lambda \sum_{s' \in S}(p^{l^*}_{s^{l^*}(s_2)s'} - p^m_{s_2 s'})V(s', t-1) & < e_{1k}(s_2) \\ \lambda \sum_{s' \in S}(p^{l^*}_{s^{l^*}(s_2)s'} - p^j_{s_j(s_2)s'})V(s', t-1) & < e_{jk}(s_2), \ \ \forall j < k \end{cases}$$

If l^* verifies also $e_{kl^*}(s_2) \leq 0$, then ($e_{jk}(s) = e_{jl^*}(s) + e_{l^*k}$)

$$\begin{cases} \lambda \sum_{s' \in S}(p^{l^*}_{s^{l^*}(s_2)s'} - p^m_{s_2 s'})V(s', t-1) & < e_{1l^*}(s_2) \\ \lambda \sum_{s' \in S}(p^{l^*}_{s^{l^*}(s_2)s'} - p^j_{s_j(s_2)s'})V(s', t-1) & < e_{jl^*}(s_2), \ \ \forall j < k \end{cases}$$

Thus, for any action $j < k$, such action l^* verifies $l^* \geq k$ and l^* optimal over j.

□ Douer

and Yachiali [7] define a generalized control limit rule as a maintenance rule of the form: repair or replace the system if and only if the state of the system exceeds a certain threshold being the control limit. The authors state that the term "generalized" stands for the generalization of the well known control limit rule ([6, 17]) by adding imperfect repairs in addition to the renewal action. In this work, the result of [7] is extended to the case of action-dependent transition processes. The next theorem states that under Assumptions 1-4 and the additional Assumption 7, the optimal policy of the MDP problem (1)-(3) is a generalized control limit rule with action-dependent transitions.

Theorem 1 *With decision-dependent uncertainty, the optimal policy π^* is a generalized control limit rule.*

Proof The proof follows directly from Proposition 5. □ So far, we have proved that

the optimal policy divides the state space into *at most* R regions, and that the policy is monotonic in increasingly-ordered states. From a solution procedure perspective, the control limit rule property is useful to ameliorate the performance of the policy iteration algorithm by reducing the set of actions for increasingly-ordered states.

We propose to develop further the control limit rule property by giving conditions under which the optimal policy divides the state space into exactly R regions. In fact, Theorem 1 does not guarantee that all the actions are chosen for at least one system state. More, specifically, if a region k is missed, this means that, in practice, action k can never be the best action to perform and should be removed from the action set. Whereas, if the state space is divided into exactly R regions, it is possible to ameliorate the performance of the policy iteration by reducing the set of possible actions to two successive actions which we demonstrate in the following propositions.

Assumption 8 *(Non-linear action costs and risk-averse loss function)*
For $j = 1, \cdots, R-1$,

(i) $c(j+1) - c(j)$ is increasing in j.

(ii) $\Delta_{j,k} C_q(\tau; s) = C_q(\tau; s^j(s)) - C_q(\tau; s^k(s))$, where $k = j+1$, is decreasing in j, for all s.

Assumption 8 states that (i) the difference in action cost between successive actions, or the additional investment necessary to perform a stronger action is not constant but increasing in actions. This relaxation of the linear cost assumption that is usually considered in maintenance optimization problems. In the case of the road maintenance problem, this assumption can be motivated by the fact that renewing a thickness $2x$ does not need the same labor, the same kind of engines and the same working time as renewing a thickness x two times. Whereas, (ii) the difference in quality cost or the risk gain between successive actions is decreasing in actions (otherwise it is always optimal to perform stronger actions). This assumption can be explained by the fact that the first maintenance investments are the most valuable ones since cracks of the road bottom are less "dangerous" than those of the surface and the risk gain decreases as the renewed layer becomes thicker.

Proposition 6 *For $j \in A$, let $k = j+1$ and $l = j+2$ and s_{jk}, when it exists, be the solution of the equation $e_{jk}(s) = 0$. Under Assumption 8,*

$$s_{jk} \leq s_{kl}.$$

Proof

$$\begin{aligned} c(k) - c(j) &= C_q(\tau; s^j(s_{jk})) - C_q(\tau; s^k(s_{jk})) \\ c(l) - c(k) &= C_q(\tau; s^k(s_{kl})) - C_q(\tau; s^l(s_{kl})) \end{aligned}$$

(i) of Assumption 8 gives,

$$\begin{aligned} C_q(\tau; s^j(s_{jk})) - C_q(\tau; s^k(s_{jk})) &\leq C_q(\tau; s^k(s_{kl})) - C_q(\tau; s^l(s_{kl})) \\ \Delta_{j,k} C_q(s_{jk}) &\leq \Delta_{k,l} C_q(s_{k,l}) \end{aligned}$$

From (ii) of Assumption 8,

$$\Delta_{k,l} C_q(s_{k,l}) \leq \Delta_{j,k} C_q(s_{k,l})$$

which gives

$$\Delta_{j,k} C_q(s_{j,k}) \leq \Delta_{j,k} C_q(s_{k,l})$$

Finally, Assumption 6 stating that $\Delta_{j,k}C_q(s)$ is non decreasing in s completes the proof. □

The next proposition follows directly from Proposition 6 and presents a sufficient condition for the optimal policy to be preventive. Recall that a maintenance policy is preventive when it advises to maintain the system when no deterioration is observed.

Proposition 7 *If equation*

$$e_{12}(0, \theta, m) > 0, \quad m = 2, \cdots, R \tag{7}$$

yields a solution $\theta_p \in [\theta_0, \theta_{max}]$, then the optimal policy π^* is a preventive policy.

Proof The result follows directly from proposition 2 (i) that implies that action 2 is optimal over the DN action in state $s = (0, \theta_p, m)$. □

Proposition 8 *(Action utilization guarantee)*
For $j = 1, \cdots, R-1$ and $k = j+1$, if the transition matrices P^j and P^k verify :

$$\sum_{s' \in S} (p^j_{s^j(s)s'} - p^k_{s^k(s)s'}), \quad \forall s$$

is decreasing in j, then the optimal policy π^* divides the state space into exactly $R-1$ or R regions in the order $2 \to \cdots \to R$ or $1 \to 2 \to \cdots \to R$ in increasingly-ordered states.

Proposition 8 presents conditions that ensure that all the maintenance actions ($R-1$ actions) are significant to the the decision problem, i.e., all the maintenance actions are an optimal action for at least one system state. However, it does not ensure that the DN action is selected for at least one state. In fact, the DN action is certain to be in the optimal policy (i.e., the state space is divided into R regions) only when equation (7) in Proposition 7 is met.

Proof Let $j \in \{1, 2, \cdots, R-2\}$, $k = j+1$ and $l = j+2$. Let s^*_{kl} denote the control limit state between actions k and l, i.e., s^*_{kl} is the solution of equation

$$MX_k(s) - MX_l(s) = 0.$$

Because of the state space discretization described in Section 3, if x^*_{kl} corresponds to the solution of the continuous equation

$$MX_k(x) - MX_l(x) = 0,$$

then s^*_{kl} corresponds to the largest discrete state that verifies $s^*_{kl} \leq x^*_{kl}$.
we shall prove that

$$s^*_{jk} \leq s^*_{kl}.$$

We have

$$\begin{aligned} e_{jk}(s^*_{jk}) &= -\lambda \sum_{s'} (p^j_{s^j(s^*_{jk})s'} - p^k_{s^k(s^*_{jk})s'}) V(s') \\ e_{kl}(s^*_{kl}) &= -\lambda \sum_{s'} (p^k_{s^k(s^*_{kl})s'} - p^l_{s^l(s^*_{kl})s'}) V(s') \end{aligned}$$

From Assumption of the proposition, $e_{jk}(s^*_{jk}) \leq e_{kl}(s^*_{kl})$ and from Assumptions 6, $e_{kl}(s^*_{kl}) \leq e_{jk}(s^*_{kl})$. Since e_{jk} is non-decreasing in the state s, then $s^*_{jk} \leq s^*_{kl}$. Finally, conditions of proposition 4 that ensure that the optimal action in the most deteriorated state is R, and under conditions of proposition 7, the state space is divided into at least $(R-1)$ regions. □

4.3 Solution procedure

In this section, we use the structural properties presented in the previous subsection to accelerate the policy iteration algorithm by reducing the set of possible actions in the policy improvement step using Theorem 1 and Proposition 8. The solution procedure is described in Algorithm 1 below and works as follows: (1) Any given policy is first evaluated in the policy evaluation step. (2) The search for a better policy in the policy improvement step is accelerated by (i) reducing the set of candidate actions using Proposition 8 and (ii) using the immediate efficiency rule to possibly avoid computing the transitions and future costs. More specifically, for two states s_a and s_b such that $s_a < s_b$, if the optimal action in s_b is known, e.g., $\pi^*(s_b) = j$, then $\pi^*(s_a)$ can be determined from the reduced action set $\{j, j-1\}$ in the policy improvement policy step of the policy iteration algorithm.

Note that our algorithm works backwards in states (i.e., begins from the most deteriorated state) since only the fact that R is performed in the most deteriorated state can be ensured based on the problem parameters. Let $s_1 < s_2 < \cdots < s_N$ denote the N states ordered by the γ order.

4.4 Numerical performance

When all assumptions are met, the solution procedure gives an exact solution. However, in practice, it may be difficult to ensure that all the assumptions met. In this case, the obtained solutions are heuristic. In Table 1, some numerical examples are presented for different problem sizes and parameters to compare the performances of three different algorithms: the policy iteration algorithm (PIA), the algorithm based on the control limit rule property in Proposition 1 (CL1) and the algorithm based on the control limit rule property and Proposition 8.

Note that even if the policy iteration algorithm converges in average in a few iterations, an important computation effort is necessary for the transition matrices. Recall that there are as many transition probability matrices as actions. The results presented in Table 1 report the most efficient method for

Algorithm 1 Accelerated Policy Iteration Algorithm

INITIALIZATION

$\pi^*(s_N) \leftarrow R$

$k^* \leftarrow \pi^*(s_N)$

EVALUATION

Evaluate $V(\pi)$ by solving the system of linear equations

$$V(s) = c_0 + c(\pi(s)) + C_q(\tau; s_{\pi(s)}(s)) + \lambda \sum_{s'} p^{\pi(s)}_{s^{\pi(s)}(s)s'} V(s')$$

IMPROVEMENT

for $s = s_{N-1} \cdots s_1$ **do**
  if $k^* > 2$ **then**
    $j \leftarrow k^* - 1$
    Evaluate $e_{j,j+1}(s)$
    if $e_{j,j+1}(s) > 0$ **then**
      $\pi(s) \leftarrow k^*$
    else
      if $e_{j,j+1}(s) = 0$ **then**
        $\pi(s) \leftarrow k^* - 1$
      else
        $A_s \leftarrow \{k^* - 1, k^*\}$ and solve
          $\pi(s) = arg \max_{k \in A_s} c_0 + c(k) + C_q(\tau; s^k(s)) + \lambda \sum_{s'} p^a_{s^k(s)s'} V(s')$
      end if
    end if
  else
    Evaluate $e_{12}(s)$
    if $e_{12}(s) > 0$ **then**
      $\pi(s) \leftarrow k^*$
    else
      if $e_{12}(s) = 0$ **then**
        $\pi(s'') \leftarrow 1, \forall s'' \leq s$
      else
        $A_s \leftarrow \{1, 2\}$ and solve
          $\pi(s) = arg \max_{k \in A_s} c_0 + c(k) + C_q(\tau; s^k(s)) + \lambda \sum_{s'} p^a_{s^k(s)s'} V(s')$
      end if
    end if
  end if
end for

computing the matrices for each case. For the PIA, the transition matrices are computed apriori and then called in the algorithm. Whereas for CL1 and CL2, the transitions are computed dynamically as they are not all needed.
CL2 uses more structural properties then CL1 andthus performs better. However, since CL2 is based on more restrictive assumptions than CL1, it should be used unless the assumptions of Proposition 8 are not met.
The discount factor $\lambda = 0.95$.

Table 1: Numerical results for different problem sizes comparing the PIA to two other algorithms using the control limit property.

N	R	CPU (PIA) *min*	CPU (CL1) *min*	CPU (CL2) *min*	CPU reduction $(CL1)\%$	CPU reduction $(CL2)\%$	Value increase $(CL1)\%$	Value increase $(CL2)\%$
30	4	1.456	1.007	0.868	30.83	40.38	0	0
	7	3.158	1.628	1.106	48.44	64.97	0	0
	10	5.131	1.610	1.255	68.62	75.54	0	1.2
110	4	24.69	18.65	14.16	24.46	42.65	0	0
	7	50.43	29.33	21.20	41.84	57.96	0	0
	10	77.78	21.98	10.96	71.74	85.90	0	3

The performance of the three algorithms is compared in terms of objective value and computation time. From Table 1, CL1 as well as CL2 reduce considerably the computation time (up to 71.74% and 85.90%, respectively). However, note that the PIA in the examples shown in Table 1 is not itself highly time consuming. This is due to the fact that, in the deterioration problem, the transition matrices are upper triangular which makes its computation as well as the policy evaluation step faster. However, the presented results in Table 1 are for a single road section. The gain in computation time will be much more significant when considering a network of road sections.

Note that the performance of the solution procedure in terms of optimum value depends on the discretization of the state space. In fact, the solution procedure ensures that the control limits s^*_{jk} and s^*_{kl} between successive actions $j < k < l$ are such that $s^*_{jk} < s^*_{kl}$, and therefore that every action is selected at least in one state. However, since the state space is discretized, it may happens that $s^*_{jk} - s^*_{kl} < \delta$ where δ is the discretization granularity, especially when the size of the action set increases. Therefore, the solution is not exact such as in case of $(N = 30, R = 10)$ represented in Figure 4.2 where the decision matrices (a) and (b) are the exact and non-exact solutions given by the PIA and CL2, respectively. The 1.2% deviation from the exact objective value given in Table

1 is caused by the fact that actions 7 and 8 are never selected in the optimal policy. Note that the deviation is minimal. This could be explained by the fact that when the optimal action in a given state is selected in the policy improvement step, this state is not excluded from the policy evaluation step to optimize the policy objective value. Therefore, the optimal policy is changed in order to minimize the value function, given the error that was made by selecting action 8 in the state $(\rho = 0.8; \theta = 0.6)$ instead of action 6.

The two policies in Figure 4.2 may seem very different, but they are not in terms of objective value. This can be explained by the fact that when there are many actions available to the decision maker, the difference between the actions is generally not very significant (compared with the quality cost value ensuring the strongest action to be selected in the most deteriorated state). However, when the number of actions is not large and the discretization is greater than the difference between two control limits, the policies are not very different as shown in an example in Figure 4.1 where the decision matrices (a) and (b) are the exact and non-exact solutions given by the PIA and CL2.

	0.8	4	4	4	4	4
	0.6	2	2	2	2	4
ρ	**0.4**	2	2	2	2	2
	0.2	2	2	2	2	2
	0.03	2	2	2	2	2
	0.0	1	1	2	2	2
		0.03	**0.2**	**0.4**	**0.6**	**0.8**
			θ			

(a)

	0.8	4	4	4	4	4
	0.6	2	2	3	4	4
ρ	**0.4**	2	2	2	2	2
	0.2	2	2	2	2	2
	0.03	2	2	2	2	2
	0.0	1	1	2	2	2
		0.03	**0.2**	**0.4**	**0.6**	**0.8**
			θ			

(b)

Figure 4.1: Differences in optimal policies given by PIA (a) and CL2 (b) for $R = 4$ when CL2 solution is not exact.

	0.8	6	6	6	9	10
	0.6	6	6	6	6	6
ρ	**0.4**	4	4	4	6	6
	0.2	2	2	2	3	3
	0.03	2	2	2	2	2
	0.0	1	1	2	2	2
		0.03	**0.2**	**0.4**	**0.6**	**0.8**
			θ			

(a)

	0.8	8	8	8	9	10
	0.6	8	8	8	8	8
ρ	**0.4**	8	8	8	8	8
	0.2	4	5	6	7	8
	0.03	3	3	3	3	3
	0.0	1	1	2	2	3
		0.03	**0.2**	**0.4**	**0.6**	**0.8**
			θ			

(b)

Figure 4.2: Differences in optimal policies given by PIA (a) and CL2 (b) for $R = 10$ when CL2 solution is not exact.

5 The partially observable problem

5.1 The POMDP model

In the previous sections, the non-observability of the DGR is not taken into account and θ is approximated by the average cracking speed which is observable (given last observation of the LCP). However, this approximation does not represent the main motivation behind introducing the DGR as an additional deterioration and decision parameter, i.e., taking into account the underlying, non-observable deterioration process.

In this section, the deterministic approximation of the DGR is relaxed and probabilistic knowledge on θ is built using the same available information, i.e., LCP observations. Observations of the DGR are approximated by the average cracking speed $\bar{\theta} = (\rho_{t+\tau} - \rho_t)/\tau$ that are supposed, with respect to the true DGR θ, to be normally distributed with mean θ and a given standard deviation σ. Hence, the maintenance optimization problem is formulated as a POMDP.

In the partially observed problem, the knowledge upon the section state is represented by a belief state denoted $\overrightarrow{b} = (b_1, \cdots, b_s, \cdots, b_N)$ where $b_s, s = 1, \cdots, N$ is the probability that the true state is $s = (\rho, \theta)$. Recall that $\rho \in \{\rho_a, a = 1, \cdots, N_\rho\}$ and $\theta \in \{\theta_b, b = 1, \cdots, N_\theta\}$. In the remainder of the paper, the states are ordered so that the state index s representing the state (ρ_a, θ_b) is given by $s = (N_\rho - 1)a + b$. Since the LCP is perfectly observed, then for any state s with ρ different from the observed LCP, $b_s = 0$.

At the beginning of each decision epoch, the belief state is updated using the partial information yielded by inspection. Let o denote a state observation given by the couple of LCP and DGR observations. Since the available observation are the (perfect) LCP observations, we build (imperfect) observations of θ represented by the average cracking speed, denoted $\bar{\theta}$, using two observations of ρ. An observation o is therefore given by $(\rho, \bar{\theta})$ where ρ is the observed LCP value and $\bar{\theta}$ is a possible DGR observation. Observations are related to the true section state through an observation probability q_{so} which is the probability of observing o when the true state is $s = (\rho, \theta, m)$. Given the true section state s, we assume that the probability distribution of observations the DGR is given by the normal distribution with mean θ and standard deviation σ. This implies that the probability q_{so} of observation $o = (\rho, \bar{\theta})$ is given by the Normal distribution with mean θ and standard deviation σ if the observed ρ is equal to the true one, and 0 otherwise.

$$\begin{cases} q \sim N(\theta, \sigma) & \text{if the observed } \rho \text{ is in the belief} \\ q = 0 & \text{otherwise} \end{cases}$$

The relationship between belief states and observations are given by $\Omega(o|\overrightarrow{b})$ which is the probability of observing o at the end of a decision epoch if the belief state at the beginning is $\overrightarrow{b}$. Using the true state transitions, $\Omega(o|\overrightarrow{b})$ can be expressed as follows:

$$\Omega(o|\overrightarrow{b}) = \sum_{s}\sum_{s'} b_s p_{ss'} q_{s'o}.$$

where $p_{ss'}$ is the transition probability from state s at the beginning of the decision epoch to s' at the end. $\Omega(o|\overrightarrow{b})$ is the sum over all the possible transitions that may lead to the observation o, weighted by the belief distribution.

After each observation, the belief state is updated using the last belief $\overrightarrow{b}$ and the observation o to $\overrightarrow{\tilde{b}}(o, \overrightarrow{b})$ as follows:

$$\overrightarrow{\tilde{b}}_{s'}(o, \overrightarrow{b}) = \frac{\sum_s b_s p_{ss'} q_{s'o}}{\sum_s \sum_{s'} b_s p_{ss'} q_{s'o}}$$

The maintenance optimization problem can be formulated as a POMDP where the objective is to find the optimal maintenance policy that minimizes the discounted cost-to-go over the infinite horizon.

$$V^*(\overrightarrow{b}) \quad = \quad c_0 + \min\{DN(\overrightarrow{b}), \min_{k \in A}\{MX_k(\overrightarrow{b})\}\} \tag{8}$$

where,

$$DN(\overrightarrow{b}) \quad = \quad C_q(\tau; \overrightarrow{b}) + \lambda \sum_{o \in O} \Omega(o|\overrightarrow{b}) V^*(\overrightarrow{\tilde{b}}(o, \overrightarrow{b})) \tag{9}$$

$$MX_k(\overrightarrow{b}) \quad = \quad C(k) + C_q(\tau; \overrightarrow{\phi}(\overrightarrow{b}, k)) + \lambda \sum_{o \in O} \Omega(o|\overrightarrow{b}) V^*(\overrightarrow{\tilde{b}}(o, \phi(\overrightarrow{b}, k))) \tag{10}$$

where $C_q(\tau; \overrightarrow{b}) = \sum_{s=1}^{N} b_s C_q(\tau; s)$, $C_q(\tau; s)$ is the quality cost function defined in the previous section, and $\overrightarrow{\phi}(\overrightarrow{b}, k)$ is the belief state immediately after performing action k in belief $\overrightarrow{b}$.

It is well known that the main obstacle of POMDP models is the “curse of dimensionality” and exact solutions are almost impossible except for problems of very small sizes. Several approximation approaches have been proposed and combined with the value iteration or policy iteration algorithms to approximate POMDP solutions. Examples of such approaches are grid-based approximations ([9] and [23]) and point-based approximations ([13, 5] and [14]).

One of the ways to solve POMDP is to derive structural properties that can be used to define a solution procedure that is much less time-consuming. However, structural property based solution procedures are generally problem specific and may not generalize to solve generic POMDP models. Moreover, they depend on the existence of such properties. Structural properties in a fully observed problem are usually relatively easy to verify and reflect realistic deterioration or economic assumptions. They are harder to verify in the partially observed case except for finite-horizon, perfect information (i.e., when the partial observability results from non periodic inspection, but the inspection yields perfect observation) or total absence of information [11]. For example, Maillart [10] investigate a finite-horizon maintenance optimization problem where partial observability comes from the aperiodicity of inspection. They study first the case of perfect information, i.e., observations, when they happen, are perfect, and prove that the optimal policy is a monotonic, AM4R policy. Based on these properties, they derived analytical expressions of decision region bounds. They state that these properties do not necessarily hold for imperfect observations. However, they use the perfect information problem properties to define an efficient heuristic that they compare to the regular grid-based procedure in [9].

In our road maintenance problem, partial observability comes from the fact that only ρ can be observed but not θ; this is a case of imperfect information. Therefore, the POMDP model (8)-(10) does not necessarily present structural properties. Like in [10], we use results derived for the perfect information case to define a solution procedure for the imperfect information case. The perfect information case corresponds here to the MDP problem from the previous section. Different from [10], we consider the infinite-horizon case, and analytical expressions of decision region bounds cannot be derived. Hence, properties derived for the MDP model can only be used to propose a heuristic solution procedure.

We propose in this section, to adapt the solution procedure of our MDP in order to derive a heuristic solution for the POMDP model (8)-(10) that is less time consuming than the exact grid-based algorithm [9] used by Zouch *et. al* in [25] to solve the same model.

5.2 Heuristic solution of the POMDP

The result presented in Proposition 2 for the MDP model states that if, among two actions, the strongest one is more efficient immediately, i.e., for the current decision epoch, then it remains more efficient when considering future discounted rewards. Whereas, when the weakest action is more efficient immediately, comparison of future expected rewards is necessary to decide of the optimal action to perform. This result cannot be proven for the imperfect information POMDP case because of the imperfect information yielded by observations. However, we propose to use it to define a heuristic solution procedure since it is intuitive for cumulative deteriorating systems (i.e., without maintenance, the system state can not be ameliorated). Nevertheless, since the optimal policy is not certain to be monotone and that control limit property as well as Proposition 8 cannot be proven, the result of Proposition 2 is changed to the following rule.

Rule 1: for any actions $j < k < l \in A$ and belief $\overrightarrow{b}$

$$\text{If } \begin{cases} e_{jk}(\overrightarrow{b}) < 0 \\ e_{jl}(\overrightarrow{b}) > 0 \end{cases} \quad , \quad \text{then } j \notin A_b$$

$$\text{Otherwise} \quad , \quad j \in A_b$$

where A_b is the reduced set of candidate actions for belief $\overrightarrow{b}$.
Rule 1 states that if, according to the immediate efficiency property, action j should be eliminated by a stronger action k, then it is eliminated only when all actions stronger than k also eliminate it.

For each couple of actions (j, k), such that k is stronger than action j, let B_{jk} be the set of beliefs solution of equation (11).

$$e_{jk}(\overrightarrow{b}) = 0 \tag{11}$$

Set B_{jk} defines a cut in the belief space between a region where action k is preferred to action j and another region where both j and k are candidates. These cuts may intersect and divide the belief space into different regions characterized by a set of candidate actions as defined by Rule 1.

The idea is to build a grid G by selecting beliefs from the cuts $B_{jk}, j, k \in A$ and their intersections. More specifically, if $(j, k, l, m) \in A$ such that $j < k$ and $l < m$, then beliefs from the intersection of B_{jk} and B_{lm}, if not empty, are added to the grid. If the intersection is empty, two beliefs from B_{jk} and B_{lm} each are added to the grid. Values of the grid beliefs denoted $\overrightarrow{g}$ are determined using the value iteration algorithm. The grid beliefs are then used to approximate the value of any belief $\overrightarrow{b}$.

The solution procedure is described in more detail in Algorithm 2.

Algorithm 2 Heuristic for the POMDP

1. BUILD GRID G
for (j,k) and $(l,m) \in A$ **do**
  if $B_{jk} \cap B_{lm} \neq \phi$ **then**
    Add randomly a belief $\overrightarrow{g}$ from the intersection to G
  else
    Add randomly $\overrightarrow{g}_{jk}$ and $\overrightarrow{g}_{lm}$ from B_{jk} and B_{lm}, respectively.
  end if
end for

2. FIND VALUES OF GRID BELIEFS
Use VIA to find optimal values of $\overrightarrow{g} \in G$ so that in each iteration, the value of any updated $\overrightarrow{b}$ is found as follows.

- $\overrightarrow{c} \leftarrow (c_1, \cdots, c_N)$ such that $c_s = \max\{g_s \mid g_s \leq b_s\}, \forall s$
- $\overrightarrow{d} \leftarrow \overrightarrow{b} - \overrightarrow{c}$
- $\overrightarrow{p}$ the vector containing the permutation of the indexes $1, 2, \cdots, N$ that orders elements of $\overrightarrow{d}$ decreasingly
- Find the weights $\beta_i, i = 1, \cdots, N$ as follows:

$$\begin{aligned} \beta_i &\leftarrow d_{p_{i-1}} - d_{p_i}, \quad i = 2, \cdots, N \\ \beta_1 &\leftarrow 1 - \sum_{i=2}^{N} \beta_i \end{aligned}$$

- The value of $\overrightarrow{b}$ is given by

$$V(\overrightarrow{b}) \leftarrow \sum_{i=1}^{N} \beta_i V(g_i)$$

3. FIND POLICY OF ANY BELIEF
Find the optimal policy in $\overrightarrow{b}$ using optimal values of the grid beliefs from Step 2 to linearly interpolate values of $\overrightarrow{\tilde{b}}$,

- Find A_b with *Rule 1*, $\forall \overrightarrow{b}$,
- $\pi^*(\overrightarrow{b}) \leftarrow \operatorname{argmin}_{k \in A_b}\{C(k) + C_q(\tau; \overrightarrow{\phi}(\overrightarrow{b}, k)) + \lambda \sum_{o \in O} \Omega(o|\overrightarrow{b}) V^*(\overrightarrow{\tilde{b}}(o, \phi(\overrightarrow{b}, k)))\}$
  where $V^*(\overrightarrow{\tilde{b}}) \leftarrow \sum_{i=1}^{|G|} \lambda_i V^*(\overrightarrow{g})$

5.3 Analytical expressions of cuts and cut intersections

In this section, analytical expressions of the cuts and cut intersections of the belief space are given in order to determine the beliefs of the grid as described in Algorithm 2 above. For any state $s \in S$ and action $j \in A$, let $\alpha^j_{ss'}$ the binary variable indicating if the resulting state from state s after performing action j is s',

$$\alpha^j_{ss'} = \begin{cases} 1 & \text{if } s^j(s) = s' \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$\begin{aligned} e_{jk}(\overrightarrow{b}) &= 0 \\ \sum_s b_s \sum_{s'} (\alpha^j_{ss'} - \alpha^k_{ss'}) Cq(\tau; s') &= c(k) - c(j) \\ \sum_s b_s Q^{jk}_s &= c(k) - c(j) \end{aligned} \tag{12}$$

where $Q^s_{jk} = \sum_{s'} (\alpha^j_{ss'} - \alpha^k_{ss'})\ Cq(\tau; s')$.

Let i_ρ denote the index of the observed value of ρ in N_ρ possible discrete values as defined in Section 3. Recall that

$$b_s = 0,\ \forall s \notin \{r_1 = (i_\rho - 1)\ N_\theta + 1, \cdots, r_{N_\theta} = i_\rho\ N_\theta\}$$

Let

$$r^* = \min_{s = r_1, .., r_{N_\theta} - 1} \{\alpha^j_{sr_{N_\theta}} - \alpha^k_{sr_{N_\theta}} \neq 0\}, \tag{13}$$

Equation (12) implies the following

$$b_{r^*} = \frac{c(k) - c(j) - Q^{jk}_{r_{N_\theta}}}{Q^{jk}_{r^*} - Q^{jk}_{r_{N_\theta}}} - \frac{\sum_{r^*+1}^{r_{N_\theta}-1} b_s (Q^{jk}_s - Q^{jk}_{r_{N_\theta}})}{Q^{jk}_{r^*} - Q^{jk}_{r_{N_\theta}}} \tag{14}$$

Therefore, a grid belief $\overrightarrow{g}_{jk}$ in B_{jk} is given by the following procedure.

1. Set $(g_{jk})_s = b_{r^*}$, where $s = r^*$.
2. Generate $(g_{jk})_{r^*+1}$ to $(g_{jk})_{N_\theta - 1}$ randomly such that $\sum_{s=r^*+1}^{N_\theta - 1} (g_{jk})_s < 1 - (g_{jk})_{r^*}$.
3. Set $(g_{jk})_{r_{N_\theta} = 1 - \sum_{s = r^*, \cdots, r_{N_\theta} - 1}}\ (g_{jk})_s$
4. $(g_{jk})_s = 0, \forall s \notin \{r^*, \cdots, r_{N_\theta}\}$

Hence, a grid belief $\overrightarrow{g} \in B_{jk} \cap B_{lm}$ verifies

$$g_{r^{**}} = \frac{(E^{lm} - E^{jk}) - (F^{lm} - F^{jk})}{D(r^{**}; j, k, l, m)} \tag{15}$$

where,

$$\begin{aligned}
r^{**} &= \min_{s=r^*+1..r_{N_\theta}-1} \{D(s; j, k, l, m) \neq 0\} \\
E^{jk} &= \frac{c(k) - c(j) - Q^{jk}_{r_{N_\theta}}}{Q^{jk}_{r^*} - Q^{jk}_{r_{N_\theta}}} \\
F^{jk} &= \frac{\sum_{s=r^{**}+1}^{N_\theta-1} b_s(Q^{jk}_s - Q^{jk}_{N_\theta})}{Q^{jk}_{r^*} - Q^{jk}_{N_\theta}} \\
D(r^{**}; j, k, l, m) &= \frac{Q^{lm}_{r^{**}} - Q^{lm}_{r_{N_\theta}}}{Q^{lm}_{r^*} - Q^{lm}_{r_{N_\theta}}} - \frac{Q^{jk}_{r^{**}} - Q^{jk}_{r_{N_\theta}}}{Q^{jk}_{r^*} - Q^{jk}_{r_{N_\theta}}}
\end{aligned}$$

and r^* in $Q^{jk}_{r^*}$ is given by (13).
Finally, the grid belief $\overrightarrow{g} \in B_{jk} \cap B_{lm}$ is given by the following procedure.

1. Set $g_s = g_{r^{**}}$, where $s = r^{**}$.
2. Generate $g_{r^{**}+1}$ to $g_{N_\theta-1}$ randomly such that $\sum_{s=r^{**}+1}^{N_\theta-1} g_s < 1 - g_{r^{**}}$.
3. Set $g_{r^*} = b_{r^*}$ from (14))
4. Set $g_{r_{N_\theta}=1-\sum_{s=r_1,\cdots,r_{N_\theta}-1}} g_s$
5. $g_s = 0, \forall s \notin \{r^*, r^{**}, \cdots, r_{N_\theta}\}$

5.4 Numerical Example

In order to illustrate the construction of the heuristic solution grid, we consider an example where $N_\rho = N_\theta = 3$, e.g., $(\rho, \theta) \in \{0, 0.4, 0.8\} \times \{0.1, 0.4, 0.8\}$. We also consider three maintenance actions denoted $1, 2$ and 3.
For $N_\theta = 3$, the set of solutions of equation $e_{jk} = 0, \quad j, k \in \{1, 2, 3\}$ is represented by straight lines defined by equation (14). Figure 5.1(a-c) illustrate, separately, the three cuts defined by the three actions for the three different levels of ρ. Solid lines represent the cut between actions 1 and 2, dashed-dotted lines represent the cut between actions 1 and 3, and dashed lines are the cut between actions 2 and 3.

In Figure 5.1-b, the belief space is divided into 6 different regions denoted A-F, each characterized by its reduced set of actions. For example, any belief state in region B (i.e., with $\rho = 0.4$) has a reduced set of actions equal to $\{1, 3\}$. More specifically, any belief state in region B, is above the 1-2 cut (solid) which implies that actions 1 and 2 should be in A_b, it is also above the 1-3 cut (dashed-dotted) which means that actions 1 and 3 should also be in A_b. Finally, it is below the 2-3 cut (dashed), which implies that action 3 is preferred to action 2, and that action 2 can therefore be eliminated from A_b. Thus, the reduced set of actions for any belief state in region B with $\rho = 0.4$ is $A_b = \{1, 3\}$.

Region C of the same figure is below the 1-2 cut and above the 1-3 and 2-3 cuts, but action 1 is not eliminated though, since it was dominated by action 2 and not by action 3 which is stronger than action 2. Whereas, in region D, action 1 is dominated by both actions 2 and 3, and is therefore eliminated.

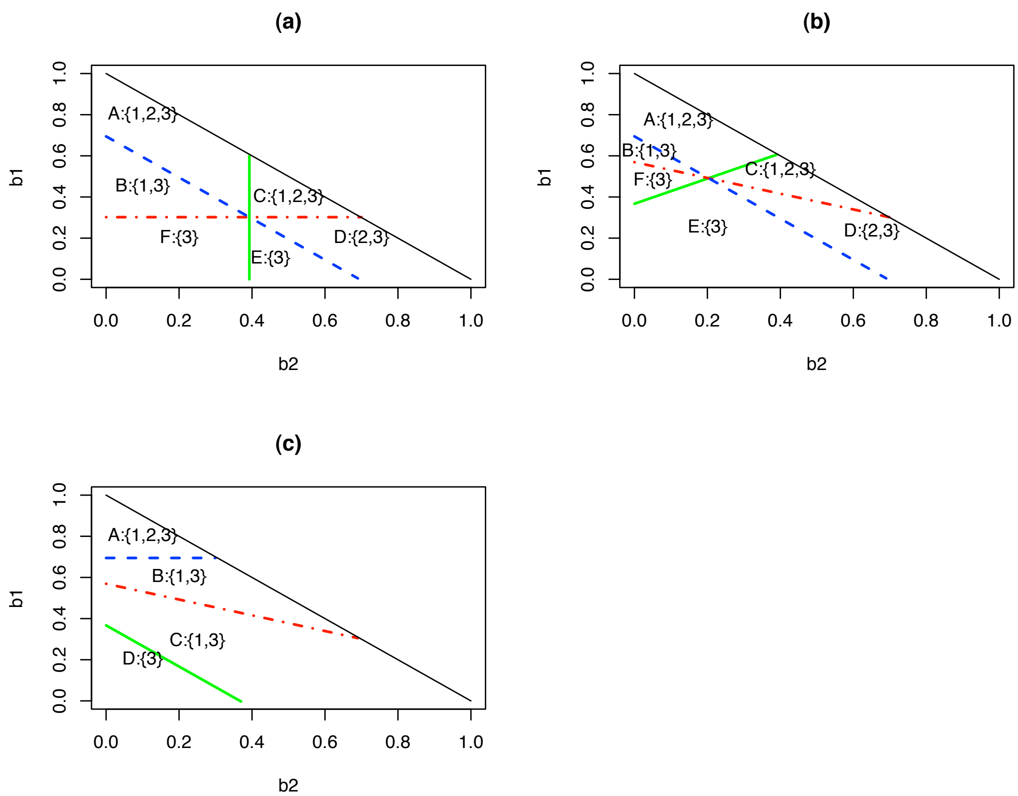

Figure 5.1: Example of separated cuts for a problem with $N = 3$ and $R = 3$

In Figure 5.2 all cuts for different levels of ρ are shown. G is comprised of the beliefs of cut intersection (different couples of actions and same value of ρ) (beliefs g_1 and g_2), as well as beliefs from cuts that do not intersect (beliefs g_3, g_4 and g_5). The size of G depends on the number

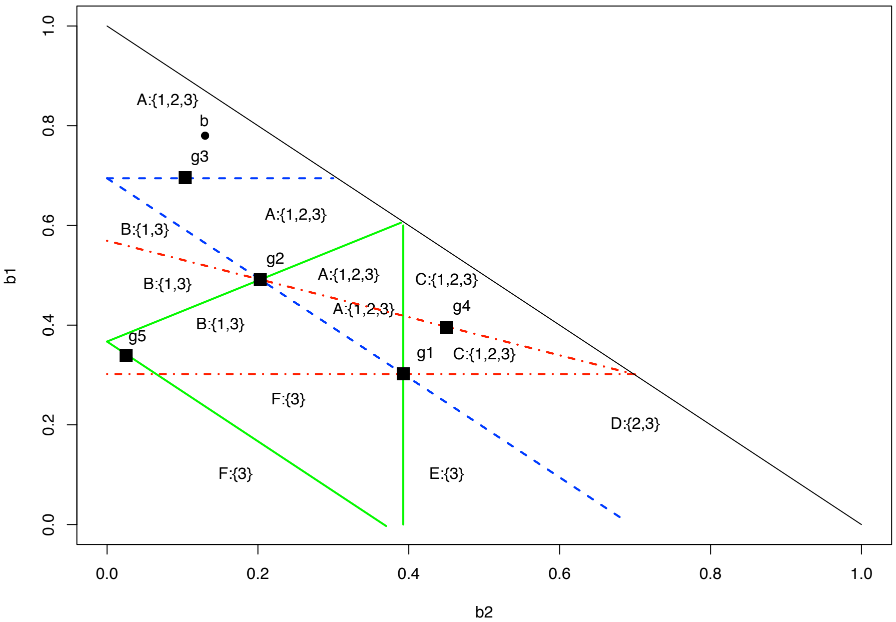

Figure 5.2: Gathered cuts for a problem with $N = 3$ and $R = 3$

of the available actions. If R is the number of available actions, then the size of G is at most $\frac{1}{8}R(R-1)[R(R-1)-2]$. Although in practice the number of available actions is not very large (compared to the size of the state space), the size of G can grow very quickly with R. In order to reduce the size of the grid G, we use the following property.

$$e_{jl} = e_{jk} + e_{kl}, \quad j, k, l \in A \tag{16}$$

Proof The proof follows directly from the definition of the immediate efficiency function. $\square$

A consequence of Equation (16), it is given by the following

Proposition 9 *For $j, k, l \in A$,*

$$B_{jk} \cap B_{jl} = B_{jk} \cap B_{kl} = B_{jl} \cap B_{kl}$$

When the number of actions is large, the size of G can be considerably reduced using Proposition 9.

Proof The proof follows from Equation (15) that gives expressions of beliefs in $B_{jk} \cap B_{jl}$, and Property (16) that implies that:

$$Q_s^{jl} = Q_s^{jk} + Q_s^{kl}\ , \quad \forall s,\ j < k < l \in A$$

□

5.5 Numerical results and comparison to the regular Grid-based approximation

In this section, some numerical results of the proposed heuristics are presented and compared to the grid-based approximation [9] in Tables 2 and 3 for grid resolution parameters equal to $M = 3$ and $M = 4$, respectively. The resolution parameter determines the accuracy of the discretization of the belief space. For both tables, we consider a state space of size $N = 30$ $(N_\rho = 6, N_\theta = 5)$. A sample of 72 belief states is used for each example of action set size R.

Results in Tables 2 and 3 are given by the heuristic described by Algorithm 2 and denoted (HL) as well as the heuristic, denoted (HS). Recall that (HS) uses the additional Proposition 9 to reduce the size of the grid. Results given by heuristics (HL) and (HS) are the average of 3 different grids for (HL) and (HS) (since the selection of grid beliefs is random). Note that for $R = 3$, only (HL) is used since (HS) requires more than 3 actions to build the grid.

The comparison results in both Tables 2 and 3 are given in terms of policies (% Pol), objective values (% V), and computation time (CPU and %CPU). Column (% Pol) gives the percentage of belief states where maintenance decisions given by the heuristics and the GBA are different. Column (%V) gives the relative difference between objective values of the GBA and the heuristics. A negative percentages mean that the GBA gives lower objectives, whereas positive percentages mean that our heuristics are able to find a better solution than the GBA.

Note that with both heuristics, we are able to reduce considerably the computation time. However, heuristic (HL) can be very time-consuming as the problem size is larger, e.g., in Table 3 where $N = 110$, it becomes more difficult to use (HL).

From numerical examples with different cost structures, we note that the quality of the solution does not only depend on the problem size but also on the problem parameters. More specifically, since grids of both heuristics (HL) and (HS) are built from cut intersections, their sizes depend on the existence of such intersections in the belief space. However, in the case of cost structures with comparable maintenance costs (investment) and quality cost (risk) (which makes the decision problem non-trivial), the effect of problem input parameters on the solution quality is not significant.

Note that the difference between heuristic and GBA results in terms of policies and objective

Table 2: Numerical results for different problem sizes comparing proposed heuristics to the GBA with $M = 3$ for $N = 30$.

R		**Value decrease** (/**GBA**)(%)	**ΔPolicy** (/**GBA**)(%)	**CPU** (min)	**CPU reduction** (%)
3	**HL**	1.52	9.1	1.73 (GBA: 38.39)	95.49
4	**HL**	-2.07	6.94	14.29	70.01
	HS	-7.06	5.55	1.53 (GBA: 47.67)	98.80
5	**HL**	4.25	9.8	81.30	40.30
	HS	2.46	11.18	3.76 (GBA: 136.19)	97.23
7	**HL**	-4.68	11.49	80.12	75.38
	HS	-6.07	9.55	4.55 (GBA: 325.51)	98.60
10	**HL**	-7.75	9.72	162.19	61.47
	HS	-9.37	9.72	9.26 (GBA: 421.03)	97.80

Table 3: Numerical results for different problem sizes comparing proposed heuristics to the GBA with $M = 4$ for $N = 30$.

R		**Value decrease** (/**GBA**)(%)	**ΔPolicy** (/**GBA**)(%)	**CPU** (min)	**CPU reduction** (%)
3	**HL**	0.63	11.5	1.73 (GBA: 124.76)	98.61
4	**HL**	-9.20	6.94	14.29	90.56
	HS	-10.50	5.55	1.53 (GBA: 151.38)	98.85
5	**HL**	1.52	9.8	81.30	59.79
	HS	-0.31	11.18	3.76 (GBA: 202.22)	98.14
7	**HL**	-5.39	11.49	80.12	82.65
	HS	-10.35	9.55	4.55 (GBA: 461.92)	99.01

values do not necessarily vary in the same way. For example, in Table 2 for $R = 5$, the difference in terms of policy between the heuristic (HL) and the GBA is 11.49% whereas in terms of objective value, it is only -4.68%. This can be explained by the fact that as the number of actions is larger, it becomes harder and harder for Rule 1 to hold (i.e., the difference in action effects is harder to distinguish), which explain the (relatively) large difference in policies. However, when the number of actions is large, the difference between the actions in terms of cost becomes slight, which explains the smaller differences in terms of objective values.

Note that in some cases such as in Table 2 for $R = 5$, both heuristics give a value function that is smaller than the one given by the GBA. We believe that this due to an underestimation of the value function rather than a better result than the GBA. In fact, examples show that in some cases, the objective value decreases with the heuristic grid size.

Table 4 presents computation time given by the proposed heuristics for a state space of size $N = 110$ that we do not solve using the grid-based approximation because of computation time limits, i.e., the GBA takes longer than 24 hours to solve.

Table 4: computation time for varying problem sizes using the proposed heuristic for $N = 110$.

R	**CPU** (min)	
3	40.94	(HL)
4	64.91	(HS)
5	168.09	(HS)
7	271.63	(HS)
10	323.82	(HS)

6 Conclusion

This paper proposes a condition-based maintenance optimization approach for a road section under stochastic deterioration. The section level of degradation is given by an observable and a non-observable parameter. The optimization problem is first formulated as a Markov decision process with endogenous uncertainty, unique to literature.

We show how the control limit property extends to the case of MDP with decision-dependent uncertainty and use it to accelerate the policy iteration algorithm. We are able to reduce the CPU time computation by up to 70% while maintaing optimal solutions. Moreover, some structural properties, such as the *action utilization guarantee* property have an important managerial implication. More specifically, conditions ensuring that all the available actions are significant, i.e., are the optimal decision for at least one state, helps managers to define the set of actions by eliminating superfluous ones.

In order to take into account the partial observability of the state, we extend the MDP to a partially observed Markov decision process. Although MPD structural properties do not necessarily hold for the POMDP model, we use these properties to define a heuristic solution procedure for the POMDP model. Numerical experiments show the performance of our solution in terms of substantial savings in computation time (up to 99%) with minimum objective value deviation (4%-13%). The heuristic also gives the capacity to solve large-size problems that are not possible using the grid-based approximation.

Finally, although the proposed heuristic solution procedure is applied to the road maintenance problem, it can be also applied to many POMDP problems with an investment/risk or an investment/penalty structure such as in industrial (e.g., production), business (e.g., marketing) and social (e.g., education, health care) applications.

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8 Structural Properties For Evolving Systems

MDP Structural Properties with Action-Dependent Transitions for Evolving Systems: Application to Road Maintenance

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Abstract

We investigate the maintenance optimization problem of stochastically deteriorating systems. After maintenance, the system transition, i.e., deteriration process is changed according to the last performed action type. Moreover, available maintenance actions have perfect or imperfect impacts on the road state, but also offer the possibility to change the system by changing its performance characteristic. This option implies a state "better-than-new". One of the consequences of such actions is that the strength action order becomes state-dependent and proofs of a control limit policy rule may not hold. We extend the control limit rule definition to the state-dependent control limit rule by defining a state-dependent action order.

Keywords: Markov decision process, Partially observed Markov decision process, Structural properties, Control-limit policy, evolving systems.

Notations

a_k	action of index k	s	section state
A_h	set of feasible actions for thickness h	s_{0h}	as-good-as-new state for thickness h
c_n	fixed milling cost	s_{Nh}	most deteriorated state for thickness h
c_p	fixed resurfacing cost	$s^k(s)$	section state after performing action k in state s
c_{un}	unit milling cost	s_m	section state given the last performed action type m
c_{up}	unit resurfacing cost	$s^k_m(s_m)$	state after performing action k in state s given the last performed action m
$c(\cdot)$	action cost	V	Total expected discounted cost-to-go over the infinite horizon
$C_q(\cdot)$	quality cost	ρ	longitudinal cracking percentage
$e_{jk}(\cdot)$	immediate efficiency function between actions j and k	θ	deterioration growth rate
h	section road thickness	$\theta_0(h)$	minimun DGR for thickness h
h_{max}	maximum road thickness	θ_{max}	
h_n	milling thickness	$\phi(\cdot)$	maintenance effect function on θ
h_p	resurfacing thickness	τ	inter-inspection period
k	action index in A_h	λ	discount factor
m	index in A_h of the last performed action	$\gamma(\cdot)$	state score function
N	number of states	$\Delta_{jk}C_q(\cdot)$	quality cost between difference between j and k
$p^k_{ss'}$	transition probability after performing action k	π	maintenance policy
R_h	number of actions in A_h	π^*	optimal maintenance policy

1 Introduction & Literature review

Many real-world problems include management decisions over multiple time periods and under uncertainty. For example, in financial planning, production or maintenance management, periodical decisions for cash and stock movements, production quantities and plans or maintenance actions under uncertain interest rates, demands or deterioration processes. In order to actual decisions to be efficient and optimal in the long horizon time, they need to take into account immediate as well as stochastic future behaviors and outcomes.

The maintenance management problem has gained an increasing interest during the last decades as decision-makers are becoming more aware of the potential gains in renewal costs as well as in failure risks that can be made by defining optimal and efficient dynamic maintenance strategies. Many approaches have been proposed for different fields such as industrial and civil engineering. These approaches aim to define optimal maintenance policies that help decision-makers to ensure acceptable performance levels and minimize total maintenance costs over a finite or the infinite horizon. For a recent survey on maintenance optimization approaches, the reader can refer to [13]. These approaches are usually classified into time-based and condition-based models. In the first-class approaches ([6, 12]), the maintenance decision is based on the system age and the uncertainty comes from a failure risk that is modeled using a lifetime distribution. Whereas, in condition-based maintenance (CBM) approaches, the decision is based on the dynamic evaluation of the system state, i.e., the deterioration level, as well as its evolution. CBM approaches are more recent and generally more efficient as they allow dynamic updating of available information about the system condition. However, they require more efforts in terms of modeling and deterioration data collection.

Most of the proposed maintenance optimization approaches model the impact of maintenance on the system by an immediate reduction of its deterioration level, but assume that the system keeps on deteriorating under the same process. In this work, as proposed by Zouch *et al.* [15] a deterioration model where the deterioration process changes after each maintenance is considered.

Moreover, existing approaches consider only perfect or imperfect actions, which implies that the best option to the decision-maker, despite multiple time period decisions, is to replace the system by a new but identical one. In this work, the option of changing the system by different one (i.e., different performance characteristics) is available to the decision-maker. More specifically, we consider the maintenance optimization problem of a road section that is deteriorating under the state-dependent model proposed by Zouch et al. [14]. This model is said to be state-dependent since its evolution law depends on its current level. Since the last performed action type is part of the state definition in this model, the evolution law also action-dependent which makes the deterioration process change after each maintenance. Maintaing the road section consists in renewing partially or completely the road section thickness, but allows also for changing the initial road thickness which implies the possibility of changing the road section by a new but different one.

The dynamic maintenance optimization problem is formulated as a Markov Decision Processe (MDP) where the system state evolution is given by the action-dependent model [14].

An appealing advantage from using MDPs is that they present structural properties that give an insight into the decision process and the investment/risk structure. Moreover, structural properties such as the control limit property ([4, 10]) may characterize optimal policies. For example, when the maintenance options available to the decision-maker are doing nothing and replacing the system, a control limit policy has the following form: do nothing until the system state exceeds a control limit state. These type of policies are appealing because they are easy to understand and helpful for decision-makers to determine conditions that input parameters should verify and help understanding the effects of these parameter variations. Moreover, structural properties can be highly valuable to reduce the search of optimal policies and accelerate generic solution MDP procedures.

The control limit property characterizes optimal stopping problems such as asset selling and purchasing with deadline problems [3]. Puterman [9] presents general conditions for the MDP transitions and rewards to ensure the existence of a control limit policy for MDPs. Note that an MDP can be equivalent to a stopping problem in the case of a set of actions comprised of a "Do" and "Do Not" actions. In addition to these general conditions, application-specific conditions have been demonstrated in the literature. For example, in a health care context, Alagoz et al. [1] provide clinically realistic conditions under which it is optimal to perform a living donor liver transplant. In a maintenance optimization context, results for optimization problems with perfect actions (equivalent to optimal stopping problems) follow directly from the results presented by [9].

Douer et al. [5] generalize these result to a problem with imperfect actions. They define a generalized control limit policy as a rule by which the system is maintained whenever current state exceeds some limit control. Zouch et *al.* [15] consider the road maintenance optimization problem with both perfect and imperfect actions. Different from [5], imperfect actions do not only change the resulting system state after maintenance but also the deterioration process. Therefore, the MDP model has action-dependent transition probability matrices. While the results presented by [5] give conditions that ensure the existence of control limit states between repairing and not repairing the system, Zouch et *al.* [15] present additional conditions for the existence of control limits between different actions and that ensure that every available maintenance action is selected at least for one state.

In this work, we give conditions under which optimal policies of the MDP model with action-dependent transition and actions that involve the possibility of changing the system are control limit ruels. The paper is structured as follows. In Section 2, the road maintenance optimization problem is described and formulated as an MDP. Structural properties are then discussed in Section 3.

2 Problem statement

Consider a road section that is subject to stochastic and continuous deterioration. The deterioration mode of intrest here is the longitudinal cracking problem or fatigue cracking that is usually caused by increasing traffic loads and harsh environment conditions.

The section is characterized by an initial thickness h and its deterioration level is represented by the longitudinal cracking percentage (LCP) and the deterioration growth rate (DGR) [14]. The LCP, denoted ρ, is the ratio of the cracked section length over its total length as illustrated by Figure 2.1. Whereas the DGR, denoted θ, represents an instantaneous speed of deterioration, and equivalently a potential of deterioration when no cracks are observed during the initiation phase. The section state is defined by its deterioration level (ρ, θ) and its characteristic thickness h.

The infinite horizon is divided into equal decision epochs of length τ in the beginning of

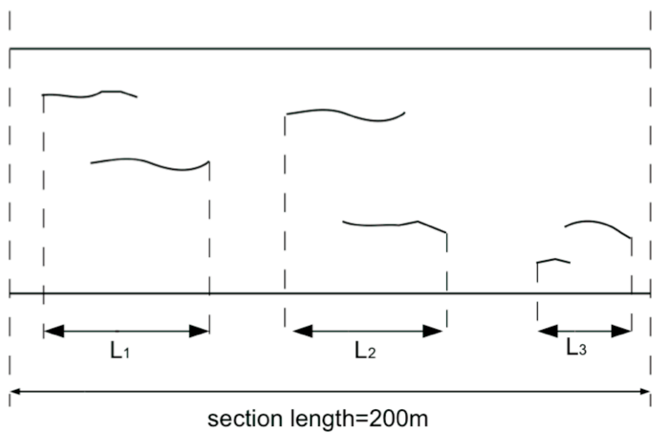

Figure 2.1: Pourcentage de fissuration longitudinale d'une section de longueur 200m : $\frac{L_1+L_2+L_3}{200}$

each the road section is inspected to measure its deterioration level (ρ, θ). In practice, only ρ can be observed. However, we assume that θ is also fully observable by approximating the DGR by the cracking average speed using successive observations of ρ.

After inspection, the decision-maker choses whether to do nothing (DN) and wait till the next decision epoch or to maintain (MX) the section. A maintenance consists in renewing partially or completely the road section by milling a thickness h_n and resurfacing it with a thickness h_p. Maintaining the road section conceals surface cracks, i.e., resets ρ to zero, and reduces the DGR of the road section according to a deterministic function $\phi(\rho, \theta, h; a)$ where

ρ, θ and h are the section LCP, DGR and thickness just before maintenance, respectively and $a = (h_n, h_p)$ is the selected action. If the DN action, also denoted $a_1 = (h_n = 0, h_p = 0)$, is selected then the section state just after maintenance decision is assumed to be the state revealed by inspection. Unlike [15] where Zouch *et al.* consider only the case of actions that do not change the road thickness, i.e., actions with of the same milling and resurfacing thicknesses, this paper allows for the possibility of milling and resurfacing with different thicknesses. Thus, the total road thickness may decrease, remain unchanged or increase after maintenance. In all cases, it is assumed that the road thickness is constrained by a maximum thickness h_{max}. It is also assumed that it is not possible to mill the road without resurfacing it.

The deterioration process of the road section is modeled by the bivariate, state-dependent and two-phase deterioration process developed in [14]. This state-dependent process assumes that the deterioration evolution during a decision epoch depends on the deterioration level in its beginning. Moreover, it is assumed that the deterioration process is changed after each maintenance. Therefore, in order to ensure the Markovian property to the state definition, the type of the last maintennace, denoted m, is added to the state definition.

Let $s_m = (\rho, \theta, h, m)$ denote the section state. In order to lighten the notation for the remainder of the paper, the following state notation is used $s_m = (s, m)$, where $s = (\rho, \theta, h)$. The seperation between s and m in the state definition is motivated by the fact that only s (representing the section deterioration level and characteristic) will be used for the costs evaluation and that m is only used to determine the deterioration process in case the DN action is selected.

Removing a thickness h_n incurs a fixed cost c_n and a total milling cost $c_{un}h_n$ where c_{un} is the unit milling cost. Similarly, adding a resurfacing thickness h_p incurs a fixed cost c_p and a total resurfacing cost $c_{up}h_p$ where c_{up} is the unit resurfacing cost. Let $c(a) = c_n 1_{hn>0} + c_{un}h_n + c_p 1_{hp>0} + c_{up}h_p$ denote the cost of the maintenance action $a = (h_n, h_p)$. Moreover, in order to prevent from always choosing the DN action and penalize leaving the road section without maintenance during the decision period of length τ, a quality cost denoted $C_q(\tau; \rho_a, \theta_a, h_a)$ is incurred, where ρ_a, θ_a and h_a are the section LCP, DGR and thickness immediately after performing an action a.

For a given section thickness h, let A_h be the set of available actions given by

$$A_h = \{(h_n, h_p) | h_n \in [0, h], h_p \in [0, h_{max}], h - h_n + h_p \leq h_{max}\}$$

Let R_h be the cardinality of A_h and $a_k, k = 1, \cdots, R_h$ denote an action in A_h such that actions a_k are ordered in an increasing cost order, i.e., if $j < k \in \{1, \cdots, R_h\}$, then $c(a_j) < c(a_k)$.

Immediate consequences of considering different milling and resurfacing actions are:

- The set of available actions depends on the actual road thickness, given a maximum road thickness.

- The definition of the as-good-as-new (AGAN) state becomes function of the road thickness unlike the case of unchanged thickness [15] where the AGAN state is constant. The definition of a relative AGAN state implies the possibility to reset the system to state better than new after maintenance. Let $s_{0h} = (0, \theta_0(h), h, R_h)$ be the relative AGAN state where $\theta_0(h)$ is the DGR of a new section of thickness h. $\theta_0(h)$ is a decreasing function in h. We assume that $\theta_0(h) \geq \theta_0$ where $\theta_0 = \theta_0(h_{max})$ and h_{max} is the maximum road thickness allowed.

- The strength order of actions may change from one state to another since the definition of the state includes the road section thickness in addition to the deterioration level.

- The cost order of actions does not correspond necessarily to the action strength order as in the unchanged thickness case. More specifically, it is possible to cheaper actions in some states to have better immediate effect on the road performance.

The following notations will be used in the remainder of the paper. For a given state $s_m = (s, m)$, let

$$s_m^k(s_m) = \begin{cases} (s, m) & \text{if } k = 1 \\ (s^k(s), k) & \text{if } k > 1 \end{cases}$$

$s_m^k(s) = (s^k, k)$ denote the section state after performing action a_k in state s_m, where $s^k = (\rho_k, \theta_k, h_k)$ and ρ_k, θ_k and h_k are the section LCP, DGR and thickness immediately after performing an action a_k.
In the remainder of the paper, the quality cost function and the maintennace effect function on the DGR are denoted $C_q(\tau, s)$ and $\phi(s; a)$, respectively.

Let $p^k_{s_k s'_k}$ denote the transition probability from a state s_k in the beginning of the decision epoch to another state s'_k at its end when the last performed action is a_k.

The maintenance optimization problem is formulated as a Markov decision process (MDP) to minimize the total expected maintenance cost-to-go over the infinite horizon. The MDP model is given for each state $s_m = (s, m)$, by the following:

$$V^*(s) = c_o + \min\{DN(s), \min_k\{MX_k(s)\}\} \tag{1}$$

where

$$\begin{aligned} DN(s) &= C_q(\tau, s) + \lambda \sum_{s' \in S} p^m_{ss'}\, V^*(s') &(2)\\ MX_k(s) &= c(a_k) + C_q(\tau, s^k(s)) + \lambda \sum_{s' \in S} p^k_{s^k(s)s'}\, V^*(s') &(3)\end{aligned}$$

where $\lambda \in [0, 1]$ is the discount factor.

Equation (2) states that following the DN action when the current state is s incurs the immediate quality cost plus the cost-to-go of the system evolving from the state s to any state s'. The MX_k action (3) incurs a maintenance cost $c(a_k)$ comprised of the fixed and variable maintenance costs plus the quality cost of beginning the decision epoch in state $s^k(s)$ plus the cost-to-go of the system beginning in state $s^k(s)$, i.e., with ρ restored to zero and θ to $\phi(\rho, \theta, h; a_k)$, or equivalently, to the DN cost of the same system beginning in state $s^k(s)$.

3 Structural properties

Structural properties are properties that optimal policies verifiy under specific conditions of cost structures and state transitions such as monotonicity or the existence of control limits. Because the state realizations and transitions are uncertain, structural properties require a stochastically ordered state spaces. For example, given two different state s_1 and s_2 in the beginning of the decision epoch that may evolve to $s'(s_1)$ and $s'(s_2)$, respectively, at its end. State $s'(s_1)$ is said to stochastically dominate $s'(s_2)$ on a set of real-valued function U, denoted $s'(s_2) \preceq_U s'(s_1)$, if [11, 2, 8]:

$$E[u(s'(s_2))] \leq E[u(s'(s_1))], \quad \forall u \in U$$

The stochastic order corresponds to the usual first-order stochastic dominance (FSD) for the set of real-valued and non-decreasing functions on states, and to the second second-order stochastic dominance (SSD) for the set of real-valued, non-decreasing and concave functions on states. In a reliability context, the SSD corresponds to the increasing failure rate (IFR) property which is defined for a transition matrix $P = (p_{ss'})_{s,s'=1,..,N}$, where $p_{ss'}$ is the transition probability from state s to s', by:

Definition (IFR property): For each $i = 1, \cdots, N$, the function $D_i(s) = \sum_{s'=i}^{N} p_{ss'}$ is non-decreasing in s.
From Derman [4], the IFR property is equivalent to the following.
Proposition For every non-decreasing function u in s, the function $K(s) = \sum_{s'=1}^{N} p_{ss'} u(s')$ is also non-decreasing in s.

From the IFR definition and Derman's proposition, note that the stochastic order definition requires an usual state order, i.e., an order that allows to say which states are better than others. Since the definition of the state contains more than one parameter, we use a score function as in [15]. However, since the state definition contains the section characteristic h in addition to the deterioration parameters (ρ, θ), we propose to order the states according to the following separable score function:

$$\gamma(s) = \gamma_1(\rho, \theta) + \gamma_2(h), \quad s = (\rho, \theta, h). \tag{4}$$

where $\gamma_1(\rho, \theta)$ is a one-to-one function that is non-decreasing in both ρ and θ and $\gamma_2(h)$ is a decreasing function in h. We assume that the parameter of the last performed action only accounts for the stochastic order, i.e., for state transitions. Note that the order of states with the score function γ is a tradeoff between the level of deterioration and the type of road section in thickness (equivalently the level of used technology).

Assumption 1 *The state transition matrix P^k after any action $a_k, k = 2, \cdots, R_h$ is IFR, $\forall h$.*

The IFR property states that, after a given action a_k, more deteriorated states are more likely to deteriorate more in the next periods.

Assumption 2 *The minimum DGR $\theta_0(h)$ is decreasing in the road thickness h.*

With unchanged section thickness [15], the effect of a maintenance depends only on the thickness of the new layer. Thus, the strength order of actions is obvious: the thicker the new layer is, the stronger the action is. With the possibility of changing the road thickness by milling and resurfacing with different thicknesses, the maintenance effect on the road state becomes function of both the new layer thickness and the total road thickness. The dependence of maintenance action effect on the state at which it is performed implies that the action strength order is state-dependent. More specifically, for a given state $s_m = (s, m)$ where $s = (\rho, \theta, h)$, let a_j and a_k be two different actions in A_h. Using the score function

(4), a_j is said to be weaker than a_k in state s, and denoted $a_j <_s a_k$, if

$$\gamma_1(\rho_{a_j}, \phi(s_m; a_j)) - \gamma_1(\rho_{a_k}, \phi(s_m; a_k)) \quad > \quad \gamma_2(h_{a_k}) - \gamma_2(h_{a_j}). \tag{5}$$

Assumption 3 *For* $s_m = (s = (\rho, \theta, h), m)$ *and* $s'_m = (s' = (\rho', \theta', h'), m)$ *such that* $s < s'$*, and any action* $a_k \in A_h \cap A_{h'}$

- $\phi(s; a_k) \leq \phi(s'; a_k)$
- *If* $h_1 = h_2$*, then* $\gamma(s^k(s)) \leq \gamma(s'^k(s'))$

Assumption 3 states that the maintenance effect function on the DGR conserves the state order, whereas, maintenance does not necessarily conserve the state order, only for states with same characteristic h.

Note that with actions that do not change the section thickness, more expensive actions correspond to stronger actions, whereas, with different added and removed thicknesses, the cost and strength orders of maintenance actions do not meet any more as the strength order becomes state-dependent.

Assumption 4 *The quality cost function is non-decreasing in the state.*

We introduce the immediate efficiency function that compares present investment (maintenance cost) and risk (quality cost) differences of two different actions in a given state. Let $s_m = (s = (\rho, \theta, h), m)$ and $a_j, a_k \in A_h$ such that $a_j <_s a_k$,

$$e_{jk}(s) \quad = \quad \Big[C_q(\tau, s^j(s)) - C_q(\tau, s^k(s))\Big] - \Big[c(a_k) - c(a_j)\Big] \tag{6}$$

The immediate efficiency function $e_{jk}(s)$ is the difference of two terms. The first one, $(C_q(\tau, s^j(s)) - C_q(\tau, s^k(s)))$, represents the gain in risk that can be realized by performing the strongest action, whereas, the second term represents the additional investment necessary to perform this strongest action. Thus, $e_{jk}(s)$ represents the tradeoff between risk and investment for a single decision epoch.

The next Assumption states that the gain in quality cost between two given maintenance actions becomes more important for more deteriorated states. It implies the same properties for the immediate efficiency function.

Assumption 5 *For* $s_m = (s, m)$ *and* $s'_m = (s', m)$ *such that* $s < s'$ *and* $a_j, a_k \in A_h \cap A_{h'}$

such that $a_j <_s a_k$*,*

$$\begin{cases} \Delta_{jk}C_q(s) \leq \Delta_{jk}C_q(s'), & j < k \\ \Delta_{jk}C_q(s) \geq \Delta_{jk}C_q(s'), & j > k \end{cases}$$

where $\Delta_{jk}C_q(s) = C_q(\tau, s^j(s)) - C_q(\tau, s^k(s))$*.*

Proposition 1 *The value function of the MDP model (1) is non-decreasing concave in the state* s*.*

Proof (by induction on the iterations of the Value Iteration Algorithm)
Let $m \in \{2, \cdots, R_h\}$ and $s_m = (s, m)$,

- t=1: $V_1(s) = \min\{Cq(\tau, s), \min_{a_k}\{C(k) + Cq(\tau, s_k(s))\}\}$.
  Assumptions 3 and 4 ensure the property for $t = 1$.

- Assume that the property holds for iteration $t-1$, we shall prove it for iteration t.
  Let $s_1 < s_2$, if the selected action at stage t is the DN action, then

$$\begin{aligned} V_t^*(s_2) &= C_q(\tau, s_2) + \lambda \sum_{s'} p_{s_2 s'}^m V_{t-1}^*(s') \\ &\geq C_q(\tau, s_1) + \lambda \sum_{s'} p_{s_1 s'}^m V_{t-1}^*(s') \\ &\geq V_t^*(s_1) \end{aligned}$$

  The first inequality follows from Assumption 4 for the first term, and Derman's property for IFR transitions with the property at iteration $t-1$ for the second term. The second inequality follows from Equation (1) of the MDP model.
  If the selected action at stage t is $a_k \in A_h$, then

$$\begin{aligned} V_t^*(s_2) &= c(a_k) + C_q(\tau, s^k(s_2)) + \lambda \sum_{s'} p_{s^k(s_2)s'}^k V_{t-1}^*(s') \\ &\geq c(a_k) + C_q(C_q(\tau, s^k(s_1)) + \lambda \sum_{s'} p_{s^k(s_1)s'}^k V_{t-1}^*(s') \\ &\geq V_t^*(s_1) \end{aligned}$$

  The first inequality follows from Assumptions 3 and 4 for the second term, and the Derman's proposition for IFR transitions with the property at iteration $t-1$ for the third term. The second inequality follows from Equation (1) of the MDP model.

The concavity is a well known result from [7]. □

Considering different state transition processes after different maintenance actions as in [15] implies that a stochastic order that compares future random states resulting from different deterioration processes is also needed as stated by the following Assumption.

Assumption 6

$$\sum_{s'=i}^{N}\left(p^{j}_{s^j(s)s'} - p^{k}_{s^k(s)s'}\right) \geq 0,\ \forall s,\ \ a_j <_s a_k.$$

Using the efficiency function defined in (6), action a_j is said to be immediately more efficient than action $a_k >_s a_j$ in a state s if the gain in risk is greater than the investment, i.e., if $e_{jk}(s) > 0$. The next proposition states that if among two different actions, the strongest one is immediately more efficient in a given state, then it is also more efficient for the infinite decision horizon.

Proposition 2 *For any section state $s_m = (s = (\rho, \theta, h), m)$ and actions $a_j, a_k \in A_h$ such that $a_j < a_k$:*

(i) If $e_{jk}(s) > 0$ then action a_k is preferred to action a_j.

(ii) If $e_{jk}(s) = 0$ then action a_j is preferred to action a_k.

(iii) If action a_j is preferred to a_k then $e_{jk}(s) \leq 0$.

Proof

(i) $MX_j(s) - MX_k(s) = e_{jk}(s) + \lambda \sum_{s' \in S}(p^{j}_{s^j(s)s'} - p^{k}_{s^k(s)s'})V(s')$
If $a_j <_s a_k$, then $s_k(s) < s_j(s)$.
Under Assumption 6, $\sum_{s' \in S}(p^{j}_{s_j(s)s'} - p^{k}_{s_k(s)s'})V(s') \geq 0$.
Therefore, if $e_{jk}(s) \geq 0$, $MX_j(s) - MX_k(s) \geq 0$, and action a_k is preferred to a_j.

Proofs of (ii) and (iii) are similar to (i). □

The next two propositions give the optimal actions in both the AGAN and most deteriorated states. Proposition 3 states that it is never optimal to reduce the road section thickness or to renew it to the same thickness when the section is in the AGAN state $(0, \theta_0(h), h)$. Whereas, Proposition 4 gives conditions under which the section has to be renewed to the maximum thickness.

Proposition 3 *For a given thickness h,*

$$\pi^*(s_{0h}) \in \{DN\} \cup \bar{A}_h$$

Proof Consider a new road section with thickness h and $a_k = (h_n, h_p) \in A_h$. The section state immediately after performing a_k is given by $s^k(s_{0h}) = (0, \phi(s_{0h}; a_k), h - h_n + h_p)$.

$$\begin{aligned} DN(s_{0h}) - MX_k(s_{0h}) &= C_q(\tau, s_{0h}) - C_q(\tau, s^k(s_{0h})) - c(a_k) + \\ &\quad \lambda \sum_{s'} (p^{R_h}_{s_{0h}s'} - p^k_{s^k(s_{0h})s'}) V(s') \end{aligned}$$

• If $a_k \in \underline{\mathrm{A}}_h$, then from Assumption 2, as $h_k = h - h_n + h_p < h$, then $\theta_0(h_k) > \theta_0(h)$. Moreover $h_p \leq h_k$, therefore, $\phi(s_{0h}; a_k) \geq \theta_0(h_k)$ and $s^k(s_{0h}) > s_{0h}$ according to the score function γ.
From Assumption 4,

$$C_q(\tau, s_{0h}) - C_q(\tau, s^k(s_{0h})) \leq 0$$

From Assumption 1 and Proposition 1,

$$\sum_{s'} (p^{R_h}_{s_{0h}s'} - p^k_{s^k(s_{0h})s'}) V(s') \leq 0$$

Thus,

$$DN(s_{0h}) \leq MX_k(s_{0h}) \;\; \forall a_k \in \underline{\mathrm{A}}_h$$

• Similary, if $a_k \in \hat{A}_h$, then $\phi(s_{0h}; a_k) \geq \theta_0(h)$ and $s^k(s_{0h}) > s_{0h}$ according to the score function γ.
Thus,

$$DN(s_{0h}) \leq MX_k(s_{0h}), \;\; \forall a_k \in \hat{A}_h$$

□

Let $s_{Nh} = (1, \theta_{max}, h)$ and $a_{R_h} = (h, h_{max})$. We assume that $C_q(\tau, s_0) = 0$, where $s_0 = (0, \theta_0, h_{max})$.

Proposition 4 *For a given road thickness h, if*

$$C_q(\tau, s^j(s_{Nh})) > c(a_{R_h}) - c(a_k), \;\; k = 1, \cdots, R_h - 1$$

then

$$\pi^*(s_{Nh}) = a_{R_h}$$

Proof

$$DN(s_{Nh}) - MX_{R_h}(s_{Nh}) \quad = \quad C_q(\tau, s_{Nh}) - C_q(\tau, s_0) - C(a_{R_h}) + \lambda \sum_{s'} (p^m_{s_{Nh}s'} - p^{R_h}_{s_0 s'}) V(s')$$

The assumption of the proposition applied for $k = 1$ implies $C_q(\tau, s_{Nh}) - C_q(\tau, s_0) - C(a_{R_h}) > 0$.
Assumption 1 and Proposition 1 imply $\sum_{s'} (p^m_{s_{Nh}s'} - p^{R_h}_{s_0 s'}) V(s') \geq 0$.
For $k < R_h$

$$\begin{aligned} MX_k(s_{Nh}) - MX_{R_h}(s_{Nh}) \; = \; & \Big[C_q(\tau, s_k(s_{Nh})) - C_q(\tau, s_{0h})\Big] - \Big[C(a_{R_h}) - c(a_k)\Big] \\ & + \lambda \sum_{s'} (p^k_{s_k(s_{Nh})s'} - p^{R_h}_{s_0 s'}) V(s') \end{aligned}$$

From the assumption of the proposition

$$\Big[C_q(\tau, s_k(s_{Nh})) - C_q(\tau, s_0)\Big] - \Big[C(a_{R_h}) - c(a_k)\Big] > 0$$

.
From Assumption 1 and Proposition 1, $\sum_{s'} (p^k_{s_k(s_{Nh})s'} - p^{R_h}_{s_0 s'}) V(s') \geq 0$. □

Note that, when there are many actions available to the decision-maker, Proposition 4 can help to decide, given a maximum quality cost (i.e., penalty) of the strongest action that is sufficient to ensure an optimal policy.

Proposition 5

$$\textit{If} \quad \begin{cases} s_1 < s_2 \\ \pi^*(s_1) = a_k \end{cases} , \quad \textit{then} \;\; \pi^*(s_2) \geq_{s_1} a_k$$

Proof (by induction on the iterations t of the value iteration algorithm)

$$\pi^*(s_1) = a_k \Rightarrow MX_k(s_1) \leq MX_j(s_1), \forall a_j \neq a_k$$

We shall prove that :

$$\forall a_j <_{s_1} a_k, \quad \exists\, a_l >_{s_1} a_k \text{ such that } MX_l(s_2) \leq MX_j(s_2).$$

- t=1
  Let $a_j <_{s_1} a_k$, we need to prove that:

$$\Big[C_q(s^j(s_2)) - C_q(s^k(s_2))\Big] - \Big[c(a_k) - c(a_j)\Big] \geq 0 \tag{7}$$

  - If $j < k$, then $0 \leq e_{jk}(s_1) \leq e_{jk}(s_1)$ (from Assumption 5).
  - If $j > k$, then $c(a_k) - c(a_j) < 0$.
  Moreover, if $a_j <_{s_2} a_k$, then $C_q(s^j(s_2)) - C_q(s^k(s_2)) \geq 0$ and Equation (7) holds.
  Thus any action $a_l \in \Big\{a_l; a_l >_{s_1} a_j \mid (j < l) \cup \Big((j > l) \cap (a_l >_{s_2} a_j)\Big)\Big\}$ verifies Equation (7).

- Assume that the property holds for iteration $t-1$, we shall prove it for iteration t

$$\pi^*(s_1, t) = a_k \Rightarrow MX_k(s_1, t) < MX_j(s_1, t), \;\; \forall a_j \neq a_k.$$

  $\Rightarrow$

$$\lambda \sum_{s' \in S} (p^k_{s^k(s_1)s'} - p^j_{s^j(s_1)s'}) V(s', t-1) < \Big[C_q(s^j(s_1)) - C_q(s^k(s_1))\Big] - \Big[c(a_k) - c(a_j)\Big]$$

  Let l^* such that $s^{l^*}(s_2) \leq s^k(s_1)$, then $a_{l^*} \geq_{s_1} a_k$

  - If $j < k$, then from Assumption 5, $e_{jk}(s_1) < e_{jk}(s_2)$, and

$$\lambda \sum_{s' \in S} (p^k_{s^{l^*}(s_2)s'} - p^j_{s^j(s_1)s'}) V(s', t-1) \quad < \quad e_{jk}(s_2)$$

    Assumption 6 implies

$$\lambda \sum_{s' \in S} (p^k_{s^{l^*}(s_2)s'} - p^j_{s^j(s_2)s'}) V(s', t-1) \quad < \quad e_{jl^*}(s_2) - e_{kl^*}(s_2)$$

    Thus if l^* is solution of the following

$$\begin{cases} s^{l^*}(s_2) \leq s^k(s_1) \\ C_q(\tau; s^k(s_2)) - C_q(\tau; s^{l^*}(s_2)) \geq c(a_{l*}) - c(a_k) \end{cases}$$

then, $\pi^*(s_2) \geq_{s_1} a_k$.

– If $j < k$, if l^* is solution of the following

$$\begin{cases} s^{l^*}(s_2) \leq s^k(s_1) \\ \Big[C_q(\tau; s^j(s_1)) - C_q(\tau; s^j(s_2))\Big] - \Big[C_q(\tau; s^k(s_1)) - C_q(\tau; s^{l^*}(s_2))\Big] \leq c(a_k) - c(a_{l*}) \end{cases}$$

then $\pi^*(s_2) \geq_{s_1} a_k$.

□

Douer and Yechiali [5] define a generalized control limit rule as a maintenance rule of the form: repair or replace the system if and only if the state of the system exceeds a certain threshold being the control limit. The authors say that the term "generalized" stands for the generalization of the well known control limit rule ([4, 10]) by adding imperfect repairs in addition to the renewal action. Zouch *et al.* extend In [15] the result of Douer and Yechiali [5] to the case of action-dependent transition processes. They also give conditions that ensure that every action is selected at least for one state. In this work, we extend further the control limit property to evolving systems by defining a state-dependent control limit policy. Unlike the usual control limit policies for which the action order is constant, the action order in a state-dependent control limit policy is state dependent, which mean that an action may return in a along straight lines states in increasing γ order.

The next theorem states that under Assumptions 1-4 and the additional Assumption 6, the optimal policy of the MDP problem (1)-(3) is a state-dependent control limit rule with action-dependent transitions.

Theorem 1 *Under Assumptions 1-5, optimal policy of the MDP (1)-(3) is a state-dependent control limit rule.*

Proof The proof follows directly from Proposition 5. □

In [15], the authors present an additional condition on the transition probabilities of different maintenance actions ensure that the monotone, optimal policy divides the state space into exactly a number of regions equal to the number of available actions. This property is not verified in the case of changing road thickness since the action order is state-dependent. Solution Procedure

4 Solution Procedure

The solution procedure proposed in [15] for the case of unchanged road thickness can be adapted to the case of evolving thickness by using the control limit property for reordered actions in strength in each state. The following algorithm describes policy iteration modified by the control limit property in the policy improvement step (Step 3).

Step 1- Initialize π' such that $\pi'(s_{Nh}) = a_{R_h}$

set $a.sucess = \pi'(s_{Nh})$ and $stop = false$

Step 2- Evaluate $V(\pi')$ by solving the linear equations

$$V(s) = c_0 + C(\pi(s)) + C_q(\tau, s^{\pi(s)}(s)) + \lambda \sum_{s'} p^{\pi(s)}_{s^{\pi(s)}(s)s'} V(s')$$

Step 3- for each s

- Reorder actions in $<_s$ order
- Find $A_s^* = \{a_j \in A_h \mid a_j <_s a.sucess\}$
- while (a.sucess>1), repeat

  $j = a.sucess - 1$ and evaluate $e_{jj+1}(s)$

  If $e_{jj+1}(s) > 0$, set $\pi(s) = a.sucess$

  If $e_{jj+1}(s) = 0$, $\pi(s) = a.sucess - 1$

  If $e_{jj+1}(s) < 0$, set $A_s = \{a.sucess - 1, a.sucess\}$ and improve by solving ’

$$\pi(s) = arg \max_{a \in A_s^*} c_0 + c(a) + C_q(\tau, s^a(s)) + \lambda \sum_{s'} p^a_{s^a(s)s'} V(s')$$

- if $(a.sucess = 1)$, then $\pi(s) = a.sucess$

Step 4- If $\pi \neq \pi'$ go to Step 3, otherwise $stop = true$

5 Conclusion

This paper presents the structural properties of a Markov decision process for a road section maintenance optimization. This MDP model presents two special features: (i) the state transition process is action-dependent, i.e., it changes after each maintenance action. (ii) Available actions to the decision-maker include actions that change the operating system by a different one allowing for the operating system to evolve. In a more general context, this corresponds to the possibility to take into account the possibility of adopting new or different systems in maintenance planning.

Relaxing the assumption of a unique state transition process (i.e., deterioration process) implies additional conditions on the transition probabilities that relate different actions. Whereas, considering evolving systems, i.e., actions that change the system, incurs that cost structure becomes different from one state to another (e.g., the action strength order). In order to prove that the optimal policy for evolving system can be a control limit rule, we define a state-dependet action strength order and prove that according to this order and under transition and cost structure conditions, the optimal policy is a state-dependent control limit rule.

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CONCLUSION GÉNÉRALE

Les travaux de thèse présentés dans ce rapport s'inscrivent dans le cadre du projet régional SBaDFoRM qui a pour objectif l'application des processus gamma pour l'optimisation de la gestion des réseaux routiers. Nous nous sommes intéressés à définir des approches conditionnelles pour la gestion des activités de maintenance des chaussées au niveau de la section aussi bien que celui du réseau, bien que la résolution numérique du modèle développé pour ce dernier reste en perspective.

Une revue des approches et pratiques existantes d'optimisation de la maintenance nous a permis d'en identifier certaines limites dont les plus importantes concernent la modélisation stochastique du processus de dégradation ainsi que la modélisation de l'impact des maintenances imparfaites dans un contexte conditionnel.

Alors que le seul paramètre de dégradation fourni par l'inspection des sections de route est le pourcentage de fissuration longitudinale, paramètre de dégradation observable, nous avons introduit un deuxième paramètre, appelé le taux de croissance de dégradation, pour décrire la dégradation sous-jacente et non-observable de la section. L'introduction du taux de croissance de dégradation nous a permis de modéliser la dégradation non-observable pendant les deux phases de dégradation et d'intégrer des comportements individualisés dans ce modèle.

Nous avons alors développé un modèle de dégradation bivarié (i.e., modélise la dépendance entre les deux paramètres de dégradation) et à deux phases. De plus, il a la particularité d'être un processus conditionnel dans le sens où sa loi d'évolution dépend du niveau de dégradation courant. Nous avons alors proposé un processus gamma conditionnel où la fonction de forme s'exprime à l'aide du niveau de dégradation donné par les deux paramètres de dégradation. Le développement de ce processus est motivé par le fait qu'un processus gamma stationnaire imposant une évolution de dégradation constante en moyenne est loin d'être un modèle réaliste. De plus, sa généralisation à un processus gamma non-stationnaire en faisant dépendre son évolution de l'âge du système et son introduction dans un contexte d'optimisation de la maintenance entraîne une inextricabilité du problème. Le processus gamma conditionnel peut être vu comme un processus gamma stationnaire pour lequel la fonction de forme est actualisée sur chaque intervalle par le niveau de dégradation courant.

Quant à la modélisation de l'impact de la maintenance imparfaite, d'une part, l'introduction du taux de croissance de dégradation comme indicateur de dégradation non observable a permis de modéliser l'impact de différentes actions de maintenance imparfaites qui ont le même effet sur le pourcentage de fissuration longitudinale. D'autre part, le processus de dégradation conditionnel a permis de relâcher l'hypothèse classique de même processus de dégradation après toutes les maintenances. Ainsi, la loi d'évolution de dégradation du modèle proposé est fonction du niveau de dégradation juste avant la maintenance ainsi que de l'action entreprise.

Bien que le taux de croissance de dégradation ne soit pas observable, nous l'avons approché par la vitesse moyenne de fissuration et avons formulé le problème d'optimisation de la maintenance d'une section comme un processus de décision markovien dont la matrice des probabilités de transition, classiquement déterminée par l'analyse statistique de données ou un avis d'expert, est donnée par le modèle stochastique de dégradation développé.

Nous avons ensuite relâcher l'hypothèse d'observabilité du taux de croissance de dégradation et étendu la formulation du problème d'optimisation de la maintenance d'une section à un processus de décision markovien partiellement observable. Cependant, les POMDPs sont numériquement insolubles sauf pour des problèmes de petites tailles. Bien que plusieurs heuristiques génériques existent pour fournir des solutions presque-optimales des POMDPs, nous avons choisi de prendre avantage de la structure du problème de maintenance pour définir une heuristique efficace de résolution. En effet, nous avons démontré que sous certaines conditions de transition stochastique et de structures de coût, le problème observable présente certaines propriétés structurelles comme la monotonie et l'existence de limites de contrôle. Ces propriétés ont permis d'accélérer l'algorithme classique policy iteration utilisé pour résoudre le MDP. Cette accélération atteint dans certains exemples 60% de gain de temps avec une moyenne de 40%.

Ces propriétés ne sont pas nécessairement vérifiées pour le problème partiellement observable, cependant nous les avons exploitées pour définir une heuristique efficace de résolution du modèle POMDP. Bien que cette heuristique soit proposée pour le problème de maintenance d'une section de route, nous pensons qu'elle peut être généralisable à d'autres problèmes présentant une structure investissement/risque.

Au niveau réseau, nous avons considéré le problème d'optimisation de la double décision d'inspection et de maintenance d'un réseau routier divisé en sections égales que nous avons formulé comme un *multi-stage stochastic program* (MSSP) avec incertitude endogène. Le modèle du réseau devrait permettre d'assurer une solution optimale non seulement du point de vue gestionnaire mais aussi du point de vue usagers et ce en considérant des coûts usagers et environnementaux. Nous avons aussi considéré, en plus de la contrainte budgétaire, une contrainte de qualité de trafic et ce en considérant des demandes et des capacités de trafic et un seuil acceptable de qualité de trafic.

Bien que, pour des contraintes de temps, nous n'ayons pas encore résolu le modèle MSSP, une analyse de la structure du problème et de l'arbre de décision nous engage à penser qu'il est possible de tirer avantage de la structure du problème ainsi que l'effet de chacune des décisions d'inspection et de maintenance sur la résolution de l'incertitude pour déterminer l'application des contraintes de non-anticipativité et définir une procédure de résolution du problème.

Nous pensons aussi que le problème d'optimisation de l'inspection et de la maintenance du réseau routier ouvre des perspectives de recherche très intéressantes aussi bien en programmation stochastique avec incertitude endogène qu'en pratiques d'optimisation de la maintenance.

D'autres perspectives à ces travaux de thèse peuvent être facilement proposées. La première est sûrement la confrontation des modèles développés avec des données réelles et des pratiques actuelles au travers d'études statistiques pour le choix et l'estimation des paramètres du modèle de dégradation. Toujours dans le contexte des routes et afin de rejoindre les pratiques actuelles, il serait intéressant de pouvoir étendre nos résultats pour la prise en compte de différents modes de dégradation. La troisième perspective pourrait être de chercher à généraliser l'approche développée à d'autres secteurs d'activités principalement pour bénéficier de la modélisation de la maintenance imparfaite, de l'extension des paramètres de décision pour ne pas se limiter à la seule observation courante et enfin la possibilité d'intégrer des actions d'amélioration des systèmes vis-à-vis du mode de dégradation étudié.

ANNEXES

A A Multistage Stochastic Programming Approach for the Road Network Inspection and Maintenance Problem

A Multistage Stochastic Programming Approach for the Road Network Inspection and Maintenance Problem

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Abstract

We consider the problem of optimizing inspection and maintenance decisions for a road network under uncertainty. In the beginning of each stage, the decision-maker should decide the network sections to be inspected, and the maintenance actions to be performed on each section, even those uninspected. The uncertainty concerns the road state, revealed only through inspection, and its future degradation, which is dependent upon the current state and previous maintenance actions. We maximize both the road and network quality considering the demand and capacity of routes subject to a budget constraint. We formulate this problem as a multi-state stochastic program (MSSP) with endogenous uncertainty that makes the scenario tree decision-dependent. Differing from the literature, we have two sources of endogenous uncertainty and the uncertainty returns in every stage rather than being only a single event. We also present the decision-dependent scenario trees for this MSSP model.

Keywords: Markov decision process, Partially observed Markov decision process, Structural properties, Control-limit policy, Heuristic.

Notations

B_t	budget for stagte t	n_{klrst}	number of alternative routes for route r in interval k of stage t under scenario s
b_1	unit cost of driving an additional distance	$\overrightarrow{R}_l$	set of routes that connect destination points of link$l$
b_2	user cost	s	scenario
c_0	fixed route inspection cost	S	number of scenarios
c_1	nit route inspection cost	t	stage
c_2	setup cost for single-lane sections	T	length of the finite decision horizon
c_3	setup cost for double-lane sections	u_{klst}	total available capacity of link l in interval k of stage t under scenario s
c_4	unit maintenance cost	v	renewed thickness
$C(\cdot)$	quality cost function	α_i	binary variable indicating if section i is single or double-lane
d_{lr}	traffic demand on route r of link l	β_l	traffic quality threshold
D_l	total traffic demand of link l	δ_{lr}	length of route r of link l
j	arc	Δ	maximum number of sections in a maintenance area
J	number of arcs	η_{lr}	initial capacity of route r of link l
l	link= couple of destination points	γ_{lrj}	capacity of arc j reserved for route r of link l
L	number of links	ρ_{it}	longitudinal cracking percentage of section i in stage t
r	route of a link	$\tilde{\rho}$	estimated longitudinal cracking percentage when no inspection is made
i	road section	θ_{it}	deterioration growth rate of section i in stage t
I	number of road sections	$\phi(\cdot)$	maintenance effect function on θ
$\overrightarrow{I}_j$	set of sections on arc j	τ	maximum time needed for a maintenance area
$\overrightarrow{J}_{lr}$	set of arcs on route r of link l	ϵ	maximum number of simultaneous maintenance areas
k	maintenance intervals within a stage	ω	section length
K	maximum number maintenance intervals within a stage	ξ_l	initial capacity of link l

1 Introduction and literature review

During the last decades, the maintenance of civil engineering structures has been identified as “hot” topic in many countries such as the USA, France, The Netherlands, Hungry, Spain,etc [3]. Several pavement management systems (PMS) have been developed in order to assist experts to define efficient and optimal maintenance policies. For a survey on existing PMSs, refer to [8]. A PMS may operate at a section and/or a network level and is generally based on a deterioration model that predicts the future evolution of the deterioration level of the network. The first PMS was proposed by Golbai *et al.* [7] for the state of Arizona, and is based on a network optimization system comprised of short-term and long-term Markov decision processes solved by linear programming. It is reported that this PMS resulted in a savings of 14 million dollars in the first year, and next year savings were estimated to 101 million dollars. It was also adopted by several other highway agencies.

In this paper, we consider the problem of inspection and maintenance of a road network that is continuously and stochastically deteriorating. Different from the approaches proposed in the literature, this paper proposes a stochastic model that optimizes inspection decisions in addition to maintenance decisions.

In order to highlight the contributions of this work, we first present a brief review of the literature on the road network management problem. Bakó *et al.* [1] propose to classify the network roads into different categories and to optimize the distribution of the funds to different categories given a total budget and road quality constraints. Worm *et al.* [10] propose a hierarchic planning approach for a network with four levels. The two first levels concern the single-sector maintenance optimization problem. In the third level, clusters of sections are defined and optimal actions are selected on the basis of the possible maintenance projects. In the fourth level, the budget constraint is considered for project assignment. This approach is modeled as a binary programming problem that minimizes the present value of the total maintenance costs under budget and quality constraints and is solved by the branch and bound method. Dahl *et al.* [2] present a pavement resurfacing plan for a road network under budget constraints. The problem is modeled as an integer program with underlying dynamic programming structure and solved by a Lagrangian relaxation-based method. The element of the network considered in [2] is one road and not just a section. The single-road problem is first considered and modeled as a shortest path problem and then generalized to the network. The main limits of this model is that the proposed policies force the maintenance of the entire road, and do not take into consideration the differences as well as the dependencies between the sections of the same road that may lead to more efficient maintenance strategies.

Our model considers both the section and network level with road and traffic quality constraints in addition to the budget constraint. This notion of traffic quality is introduced by considering the demand and the capacity of the network and gathering maintenance actions in a way that ensures acceptable traffic conditions. Hence, the decision optimization has two levels: the maintenance and the action scheduling levels. The road network maintenance and inspection problem (RNIMP) is formulated as an MSSP. Unlike the cited models, we consider the inspection optimization problem in addition to the maintenance one. Moreover, the model allows optimizing the inspection in a non-periodic fashion. However, it renders the stochastic structure of the optimization problem more complicated since maintenance decisions will depend on inspection decisions and outcomes. In fact, the uncertainty in the RNIMP concerns the road section states and their evolution. The uncertainty of the true states can be resolved before a maintenance decision by inspection. This is known in the literature as endogenous uncertainty. Another source of endogenous uncertainty in our model is the fact that the deterioration evolution law is action-dependent. Unlike exogenous uncertainty where random parameters are realized independently from decisions, endogenous uncertainty happens when (i) the time point of random variable realizations depends on decisions (e.g., observation decisions) or (ii) when decisions affect probability distributions by making some transitions more probable than others (e.g., maintenance decisions in our model). Classical stochastic programming (SP) models only handle exogenous uncertainty. There are very few studies on SP problems with endogenous parameters [9]. Moreover, in most of such existing SP models ([4, 5, 6] and [9]), the uncertainty is endogenous because of inspection decisions. In this work, the uncertainty in the RNIMP is endogenous because of both types of causes.

In the RNIMP the uncertainty returns in every stage after performing maintenance decision rather than being only a single event. In fact, in classical MSSP models where the uncertainty is resolved after few stages and the problem becomes deterministic after then. However the uncertainty upon the section states in the RNIMP returns in every decision epoch even if it was resolved in its beginning by inspection. Moreover, the objective of our optimization model is not classical optimal resources allocation from a manager point of view. The proposed objective function tends to minimize user costs incurred by maintenance activities and ensure acceptable traffic conditions.

This paper is structured as follows. Sec. 2 presents the road network inspection and maintenance problem. The MSSP model for the RNIMP is formulated in Sec. 3 and illustrate through example the scenario tree with the non-anticipativity constraints in Sec. 4. Finally, conclusions and perspectives are presented in Sec. 5.

2 Problem definition and assumptions

Consider a road network comprised of road segments we denote as arcs $j = 1, \cdots, J$ that connect a set of nodes defined by intersections of roads. A node can be a destination point, e.g., a city. Let $l = 1, \cdots, L$ denote a link defined as a couple of destination points. A route r for link l is a path connecting the destination points of l. The roads are divided into equal sections $i \in I$, where I is the set of all road network sections. A road section can be either single-lane or double-lane. Let α_i be the binary variable defined as follows.

$$\alpha_i = \begin{cases} 1 & \text{if section } i \text{ is single-lane} \\ 0 & \text{if section } i \text{ is double-lane} \end{cases}$$

The network represented in Fig. 2.1 is comprised of 4 nodes and 3 destination points $\{A, B, C\}$, that is 5 arcs and 3 links $\{(A, B), (A, C), (B, C)\}$. Link (A, B) has 4 possible routes.

The road sections are continuously and stochastically deteriorating due to cumulative fatigue. We consider

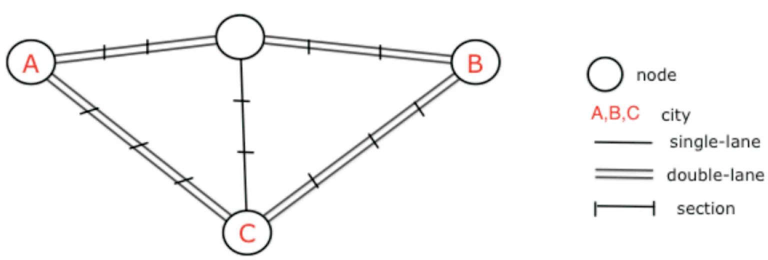

Figure 2.1: A 3-city road network

the longitudinal cracking deterioration model and represent the deterioration level of each section using two parameters: the longitudinal cracking percentage (LCP) and the deterioration growth rate (DGR) denoted ρ and θ, respectively.

A finite decision horizon of length T is considered and discretized into equal length decision epochs called stages, denoted by $t = 1, \cdots, T$ at which inspection and maintenance decisions are made. In our road maintenance problem, a stage refers to one year. A budget B_t is fixed for each stage and has to be allocated to inspect and maintain the road network so as to maximize its quality.

At the beginning of each stage, some routes of the network may be inspected to yield perfect observations of both the LCP and the DGR of each section of inspected routes. Inspecting a route r incurs a fixed cost c_0 plus a variable cost proportional to its length given a length unit inspection cost c_1. Inspections are assumed to be instantaneous and resolve the uncertainty by revealing the true state of inspected road sections. Let (ρ_{it}, θ_{it}) denote the true state of section i at stage t.

After inspection, the decision-maker must decide for each section, whether it was inspected or not, to do nothing (DN) and wait till the next epoch or to maintain it (MX). A maintenance action consists of renewing totally or partially the road section by milling and adding the same thickness v [11]. Note that the DN action corresponds to removing and adding a thickness equal to zero.

A maintenance action conceals surface cracks and makes the section more rigid by reducing its DGR according to a maintenance effect function $\phi(\rho, \theta, x)$ that is a function of the section state (ρ, θ) just before maintenance and the maintenance decision v. Renewing the total road thickness corresponds to a perfect maintenance (resets the state to the as-good-as-new), whereas renewing a partial thickness is an imperfect action. Maintaining a road section may incur a setup cost denoted c_2 and c_3 for single and double-lane sections, respectively. More specifically, we assume that to perform maintenance actions, construction areas are opened by installing barriers and road signs and mobilizing workers and equipment. We define a maintenance area as Δ consecutive sections among which at least one section is under maintenance. Single-lane sections are closed for maintenance, whereas double-lanes are transformed into single-lane sections. Maintenance incurs a variable cost per section that is function of the added layer thickness given a resurfacing cost per unit c_4.

In addition to the maintenance cost, a quality cost is considered for leaving the section in an undesirable state. In practice, the quality cost is the penality that the maintenance contractor must pay for unmet road quality standards prescribed in the maintenance contract. We assume that the road network quality standards are defined based on a threshold of the observable cracking level (LCP). More specifically, a penalty is paid when the value of ρ exceeds a fixed threshold and is proportional to the threshold exceedance. Let $C(\rho)$ denote the quality cost function where ρ is the current LCP before maintenance decision. When no inspection is made, the true LCP value cannot be known. We use an estimated value $\tilde{\rho}$ using the known deterioration law from the last known deterioration state.

When construction areas are installed, some routes are closed and some other have reduced capacity. Therefore, maintenance activities incur user costs (e.g., costs of taking longer routes) that should be taken into account when defining maintenance strategies. We detail further the definition of user costs in the next section.

We assume that the time needed to close a maintenance area does not exceed a certain interval length τ. Maintenance decisions for stage t should be implemented during stage t but must respect the connectivity of all links, i.e., ensure that all destination points of the network are continuously connected. Moreover, a road section is characterized by a traffic capacity that indicates the number of cars that can circulate in normal conditions per time unit. Bad traffic conditions happen when the traffic demand flow on an arc, i.e., the number of users per time unit that need to take this arc, exceeds the capacity. Given the network conditions, the traffic demand flows are assumed to be fixed and known for each link on the road network.

A traffic quality constraint can be defined for each link of the network to ensure that the difference between the available capacity on the link and its traffic demand flow does not fall below a certain threshold. Note that verifying the traffic quality constraint for each link of the network, ensures the connectivity of the network. In order to uphold the traffic quality, maintenance decisions for each stage should be implemented in different time intervals. Thus, each decision epoch is divided into equal intervals that correspond to the maximum duration of a maintenance area τ to schedule maintenance actions that have to be performed during the current decision epoch. Let $k = 1, \cdots, K$ indicate the maintenance intervals within a stage, where K represented the maximum number of intervals that can considered in a single decision epoch. We also consider that a maintenance of a maximum of ϵ maintenance areas that may be opened in an interval. This corresponds to a constraint on the number of available maintenance crews.

The optimal inspection and maintenance strategy is the strategy that minimizes quality and user costs under budget, connectivity and traffic quality constraints.

3 Model formulation

We formulate the RNIMP as a multistage stochastic programming (MSSP) problem with endogenous uncertainty. In the beginning of each stage of the decision horizon, the goal of the road network manager is to find the optimal budget allocation to inspections and maintenance actions that minimizes both quality and user costs. However the future realization of uncertain parameters as well as the possibility of recourse actions must be taken into account to derive the current decision.

In the remainder of the paper a vector A will be denoted $\overrightarrow{A}$, and its cardinality $|A|$. For each link l, let $\overrightarrow{R_l}$ be the set of routes that connect the destination points of l so that the indexes $r = 1, \cdots, |R_l|$ order the routes by increasing length, i.e., if $\overrightarrow{\delta}_l = (\delta_{l1}, \cdots, \delta_{l|R_l|})$ is the vector of elements δ_{lr} equal to the length of route $r \in \overrightarrow{R_l}$, then $\delta_{l1} \leq \delta_{l2} \leq \cdots \leq \delta_{l|R_l|}$. For each link l and route $r \in \overrightarrow{R_l}$, let $\overrightarrow{J}_{lr}$ be the set of arcs that are on route r. Define $\overrightarrow{I}_j$ as the set of road sections that are on arc $j \in \{1, \cdots, J\}$. Therefore,the length of a route r on a link l is given by

$$\delta_{lr} = \sum_{j=1}^{|J_{lr}|} \sum_{i=1}^{|I_j|} \omega, \quad r = 1, \cdots, |R_l|.$$

where ω is the section length.
Let $s = 1, \cdots, S$ denote a scenario and define the following decision variables at each stage t of the decision time horizon.

v_{ikst} = maintenance decision, i.e., resurfacing thickness, for section i implemented in interval k

$$w_{ikst} = \begin{cases} 1 & \text{if section } i \text{ opens a maintenance area in interval } k \\ 0 & \text{otherwise} \end{cases}$$

$$x_{lrst} = \begin{cases} 1 & \text{if route } r \text{ of link } l \text{ is inspected} \\ 0 & \text{otherwise} \end{cases}$$

$$y_{klrst} = \begin{cases} 1 & \text{if route } r \text{ of link } l \text{ is closed in the interval } k \\ 0 & \text{if route } r \text{ is open without maintenance in the interval } k \end{cases}$$

$$z_{klrst} = \begin{cases} 1 & \text{if route } r \text{ of link } l \text{ is open but under maintenance in the interval } k \\ 0 & \text{otherwise} \end{cases}$$

3.1 The user cost

We consider two types of user costs: costs incurred by closing single-lane routes, and costs incurred by transforming double-lane routes into single-lanes.

(i) When a single-lane route r is closed for maintenance, users are obliged to take another route to get to their destination. We assume that the alternative routes for r are longer than route r otherwise users would have already taken them. We also assume that the demand d_{lr} of the closed route r is evenly distributed on the available longer routes. Recall that indexes in routes are given by the order of the elements of $\overrightarrow{\delta}_l$ that are ordered by increasing length. Since the alternative routes for a closed route r are the open longer routes, then the number of alternative routes for route r in interval k of stage t under scenario s, denoted n_{klrst}, is given by the sum of open routes of the link as follows.

$$n_{klrst} = \sum_{r'=r+1}^{|R_l|} \Big((1 - y_{klr'st}) + z_{klr'st} \Big), \quad \forall l, r \in \overrightarrow{R_l}$$

The user cost incurred by closing route r of link l in interval k of stage t under scenario s corresponds to the sum of extra distance costs weighted by the evenly distributed demands on open routes.

$$\sum_{r'=r+1}^{|R_l|} \frac{d_{lr}}{n_{klrst}} (\delta_{lk} - \delta_{lr})\, b_1\, y_{klrst}, \quad \forall l, r \in \overrightarrow{R_l}$$

where b_1 is the unit cost of driving an additional distance unit.

(ii) When a double-lane is transformed into a single-lane, we assume that users do not change their route. However, each user incurs a cost b_2 due to the deterioration of traffic quality and additional fuel consumption due to overloading the route. The user cost incurred by transforming a double-lane route r of link l to a single-lane route is given by the cost incurred by all the users of this route.

$$d_{lr}\, b_2\, z_{klrst}$$

3.2 The traffic quality constraint

The traffic quality represents the capacity of the network to absorb the traffic demand. Hence, the quality of each link of the network at each stage t and interval k can be defined as the difference between its traffic demand D_l and the total available capacity of its open routes, denoted u_{klst}. The link traffic quality constraint is therefore defined using an acceptable quality threshold β_l as follows:

$$u_{klst} - D_l \quad \geq \quad \beta_l, \quad \forall k, l, s, t \tag{1}$$

If ξ_l is the initial capacity of link l, i.e., when all routes are open, then the available capacity of l is the difference between ξ_l and the capacity lost because of maintenance. Recall that maintenance reduces the capacity of doube-lane routes to the half, whereas single-lanes are closed. It is therefore necessary to redefine the traffic redistribution in each stage and each interval.

We propose to divide the total capacity of each arc on the different routes of different links that contain it so that links with bigger capacities have more capacity reserved on each arc. Let γ_{lrj} be the capacity of arc j reserved for route r of link l given by the following.

$$\gamma_{lrj} = \frac{\eta_{lr}\ q_{lrj}}{\sum_{l=1}^{L} \sum_{r=1}^{|R_l|} \eta_{lr}\ q_{lrj}}$$

where q_{lrj} is defined for each route r of link l by

$$q_{lrj} = \begin{cases} 1 & \text{if arc } j \in \overrightarrow{J}_{lr} \\ 0 & \text{otherwise} \end{cases}$$

η_{lr} is the initial capacity of route r of link l defined by the minimum arc in $\overrightarrow{J}_{lr}$ capacity. Note that the denominator is a normalization constant so that the sum over all links and routes is equal to the initial arc capacity when no maintenance areas are opened. Therefore, the available capacity of link l, u_{klst}, is given by the difference between the initial link capacity and the lost capacity of arcs of closed routes as follows

$$u_{klst} \quad = \quad \xi_l - \sum_{r=1}^{|R_l|} \sum_{j=1}^{|J_{lr}|} \gamma_{lrj} \ \Big((1 - y_{klrst}) + 0.5\ z_{klrst} \Big) \, \xi_l$$

The complete MSSP model for the RNIMP is given by the following.

$Minimize$

$$\sum_{s} p_s \sum_{t=1}^{T} \lambda^{t-1} \sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{r=1}^{|R_l|} \begin{bmatrix} \sum_{r'=r+1}^{|R_l|} \left(\frac{d_{lr}}{n_{krlst}} (\delta_{lk} - \delta_{lr}) b_1 y_{klrst} + d_{lr}\ b_2\ z_{klrst} \right) \\ + \\ \sum_{j=1}^{|J_{lr}|} \sum_{i=1}^{|I_j|} \left(C(\rho_{ist}) x_{lrst} + C(\tilde{\rho}_{ist})(1 - x_{lrst}) \right) \\ -M' w_{ikst} \end{bmatrix} \tag{2}$$

$s.t.$

$$\sum_{l=1}^{L}\sum_{r=1}^{|R_l|} \left[\begin{matrix} (c_0 + \delta_{lr}\ c_1) x_{lrst} + \\ \sum_{k=1}^{K}\ \sum_{j=1}^{|J_{lr}|} \sum_{i=1}^{|I_j|} \begin{pmatrix} (w_{ikst}(c_2\ \alpha_i + c_3\ (1-\alpha_i))) \\ +(\alpha_i\ v_{ikst}\ c_3 + (1-\alpha_i)\ v_{ikst}\ c_4) \end{pmatrix} \end{matrix} \right] \le B_t, \qquad \forall s,t \tag{3}$$

$$u_{klst} - D_l \ge \beta_l, \qquad \forall k,l,s,t \tag{4}$$

$$\sum_{i \in I} w_{ikst} \le \epsilon, \qquad \forall k,s,t \tag{5}$$

$$M\ y_{klrst} \ge \sum_{j=1}^{|J_{lr}|}\sum_{i=1}^{|I_j|} v_{ijst}\alpha_i, \qquad \forall k,l,r \in \overrightarrow{R_l}, t, s \tag{6}$$

$$M\ z_{klrst} \ge \sum_{j=1}^{|J_{lr}|}\sum_{i=1}^{|I_j|} v_{ijst}(1-\alpha_i), \qquad \forall k,l,r \in \overrightarrow{R_l}, t, s \tag{7}$$

$$\begin{cases} M'\ w_{ikst} \le M'\ x_{ikst} & \forall i,k,t,s \\ (i-1)\ w_{ikst} \le \sum_{i'=1}^{i-1}(1 - w_{i'kst}) & i = 2,..,\Delta-1, \forall k,t,s \\ (\Delta-1)\ w_{ikst} \le \sum_{i'=i-\Delta+1}^{i-1}(1 - w_{i'kst}) & \forall j, i = \Delta,..,|I_j|,k,t,s \end{cases} \tag{8}$$

$$x_{lrs1} = x_{lrs'1} \qquad \forall l, r \in \overrightarrow{R_l}, s < s' \tag{9}$$

$$\begin{bmatrix} \sigma_{ss't} \\ v_{ikst} = v_{iks't} \\ y_{klrst} = y_{klrs't} \\ z_{klrst} = z_{krls't} \\ w_{ikst} = w_{iks't} \\ x_{lrs(t+1)} = x_{lrs'(t+1)} \end{bmatrix} \cup \begin{bmatrix} \neg\sigma_{ss't} \\ v_{ikst}, v_{iks't} \ge 0 \\ y_{klrst}, y_{krls't} \in \{0,1\} \\ z_{klrst}, z_{klrs't} \in \{0,1\} \\ w_{ikst}, w_{iks't} \in \{0,1\} \\ x_{lrs(t+1)}, x_{lrs'(t+1)} \in \{0,1\} \end{bmatrix} \qquad \forall i,k,l,r \in \overrightarrow{R_l}, s < s' \tag{10}$$

$$\sigma_{ss't} \Leftrightarrow \bigcap_{\substack{i \in E_{lr}(s,s',t) \\ i \notin F_{lr}(s,s',t)}} \left(\neg x_{lrst} \right) \qquad \forall i,t,s < s' \tag{11}$$

$$v_{ikst} \ge 0 \qquad \forall i,k,s,t \tag{12}$$

$$w_{ikst} \in \{0,1\} \qquad \forall i,k,s,t \tag{13}$$

$$x_{lrst} \in \{0,1\} \qquad \forall l, r \in \overrightarrow{R_l}, s, t \tag{14}$$

$$y_{klrst}, z_{klrst} \in \{0,1\} \qquad \forall k,l,r \in \overrightarrow{R_l}, s, t \tag{15}$$

where, $\lambda \in [0,1]$ is the discount factor, M and M' are large real numbers and $\tilde{\rho}_{ist}$ is the estimated value of LCP when section i is not inspected.

Note that after maintenance is selected, the LCP is reset to zero and therfore known even no inspection was performed. However, the new DGR after maintenance cannot be known with certainty if no inspection was performed. If the DN-action is selected, both the LCP and DGR just after the maintenance decision cannot be known with certainty.

The first term in (2) is the user cost whereas the second represents the quality cost. The budget constraint (3) states that the sum of inspection and maintenance costs of one stage should not be greater that the budget of that stage. The traffic quality constraint (4) ensures that at each stage and interval, the difference between the demand and the available capacity of each link does not fall below a fixed quality threshold. Whereas Equation (5) states the resource constraint by establishing a maximum on the number of maintenance areas that can be opened. Equations (6), (7), and (8) set the binary decision variables. The introduction of M and M' ensures the definitions of these decision variable to be linear.

In addition to operation constraints that link decisions within the same scenario (e.g., the budget constraint), decisions of different scenarios are linked through non-anticipativity (NA) constraints (9) and (10). The NA constraints ensure that for all scenarios with the same history until stage t, the decision made at stage t should be the same. More specifically, non-anticipativity requires that if scenarios s and s' are indistinguishable after inspection decision in stage t, then maintenance decisions for stage t and inspection decision for stage $t+1$ should be the same for scenarios s and s'. If no inspection was made, then no information is available to differentiate between scenarios s and s', and NA constraints that link decisions in scenarios s and s' should be applied. The NA constraints (10) are formulated as in [4, 5, 6] using the disjunctive programming formulation.

$\sigma_{ss't}$ is a boolean variable equal to $true$ if scenarios s and s' are indistinguishable after inspection decision at stage t, and $false$ otherwise. $\sigma_{ss't}$ is $true$ when no inspection is made and different realizations of random parameters (ρ, θ) occur but from the same deterioration law.

Define the set $E(s,s',t) \subseteq I$ as the set of sections that have different random characteristics (ρ, θ) at stage t, and

$$E_{lr}(s,s',t) = E(s,s',t) \;\cap\; \bigcup_{j \in \overrightarrow{J_{lr}}} I_j$$

We also define the set $F(s,s',t) \subseteq I$ as the set of sections that are under different deterioration law at stage t. Recall that the deterioration law of a section is defined by the last performed maintenance on it, i.e., if $v_{iks\bar{t}}$ is the last performed action till stage t, where $\bar{t}$ is the stage of the last performed maintenance or the first stage if no maintenance has been performed yet: $\bar{t} = \max\{t' \mid t' < t,\ v_{ist'} > 0\}$, then $F(s,s',t) = \{i \in I \mid ; v_{ist} \neq v_{is't}\}$. So we have:

$$F_{lr}(s,s',t) = F(s,s',t) \;\cap\; \bigcup_{j \in \overrightarrow{J_{lr}}} I_j$$

Note that, specific to our RNIMP, because the state transition probabilities are maintenance action dependent, maintenance decisions may resolve partially some uncertainty that was not resolved by inspection by enabling scenarios to be distinguishable.

4 Scenario trees: Illustration of the NA constraints

We illustrate the last statement by the following example illustrated in Fig. 4.1 and 4.2. Fig. 4.1 illustrates the endogenous uncertainty and the non-anticipativity constraints in the RNIMP. We consider a single section that can be in three different deterioration levels representing its LCP and DGR denoted $1 = (\rho_1, \theta_1), 2 = (\rho_2, \theta_2)$, and $3 = (\rho_3, \theta_3)$. A node in the decision tree represents the state of the section after deciding whether to inspect or not.

Two actions are available to the decision -maker: DN or MX. The DN-action does not change the section state, whereas MX reduces its deterioration level according to a deterministic function of the section state before maintenance and the added thickness. Note that, after a DN action, the section deterioration level cannot decrease. We assume that at stage $t = 1$ the section state is known to the decision-maker and is equal to 1. At each state, the decision-maker decides of the maintenance action to perform and whether to inspect the section in the beginning of the next stage. The inspection (I) reveals the true section state and allows different scenarios to be distinguished. Whereas without inspection (NI) (bold arcs), different scenarios cannot be distinguished (empty nodes), and NA constraints should be applied.

To explain further how NA constraints works with endogenous uncertainty and both inspection and maintenance decisions, we consider the part of the scenario tree in the rectangle that is detailed in Fig. 4.2 starting from the first node.

At stage $t = 1$ the section is in state 1 and the decision-maker selects DN and NI in the beginning of the next stage. At $t = 2$, the section state cannot be revealed. If MX is selected for the second stage with I at $t = 3$, and the system is found in state 3 at $t = 3$, then this scenario can correspond to one of the scenarios $9, 20$ or 30, as the intermediate state at $t = 2$ is $1, 2$ or 3. Thus, scenarios $9, 20$ and 30 are indistinguishable (horizontal links).
However, if at stage $t = 2$, DN is selected with I at stage $t = 3$ at which the system is found in state 1 (scenario 1), we can therefore deduce that the section is in state 1 at $t = 2$. The uncertainty is therefore resolved without inspection due to the particular structure of the RNIMP.

5 Conclusion and perspectives

In this work, we consider the problem of road network management under uncertainty. We formulate a MSSP that maximizes the road and network quality subject to budget, capacity and demand constraints. This yields the optimal inspection and maintenance decisions as well as the scheduling of those decisions.

Unlike existing models, we do not only consider the maintenance optimization problem, but also inspection is optimized in a non-periodic inspection. This leads to an endogenous uncertainty as the stages where inspections are performed are not known a priori but dynamically.

With this work, we open research perspectives in stochastic programming with endogenous uncertainty. In the RNIMP problem, the two consecutive decisions, i.e., inspection and maintenance, are both sources of endogenous uncertainty. Hence, even if no uncertainty is revealed by inspection, an additional information can be available to the decision maker after the maintenance decision, which can lead in some cases to resolve the uncertainty. Therefore, it is possible to take advantage from this particular structure to determine the application of the non-anticipativity constraints.

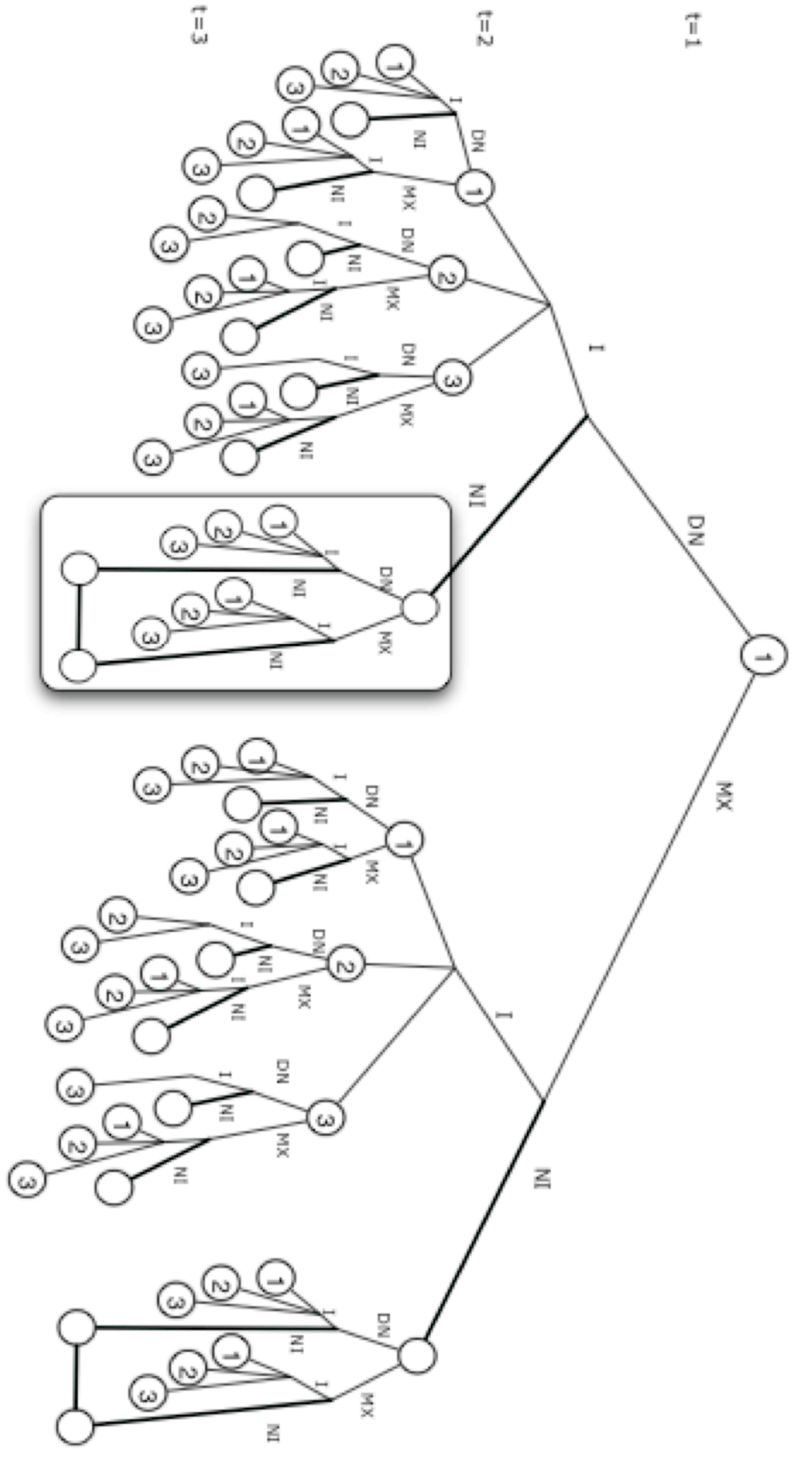

Figure 4.1: Scenario tree for a three-stage RNIMP with 3 states and 2 actions.

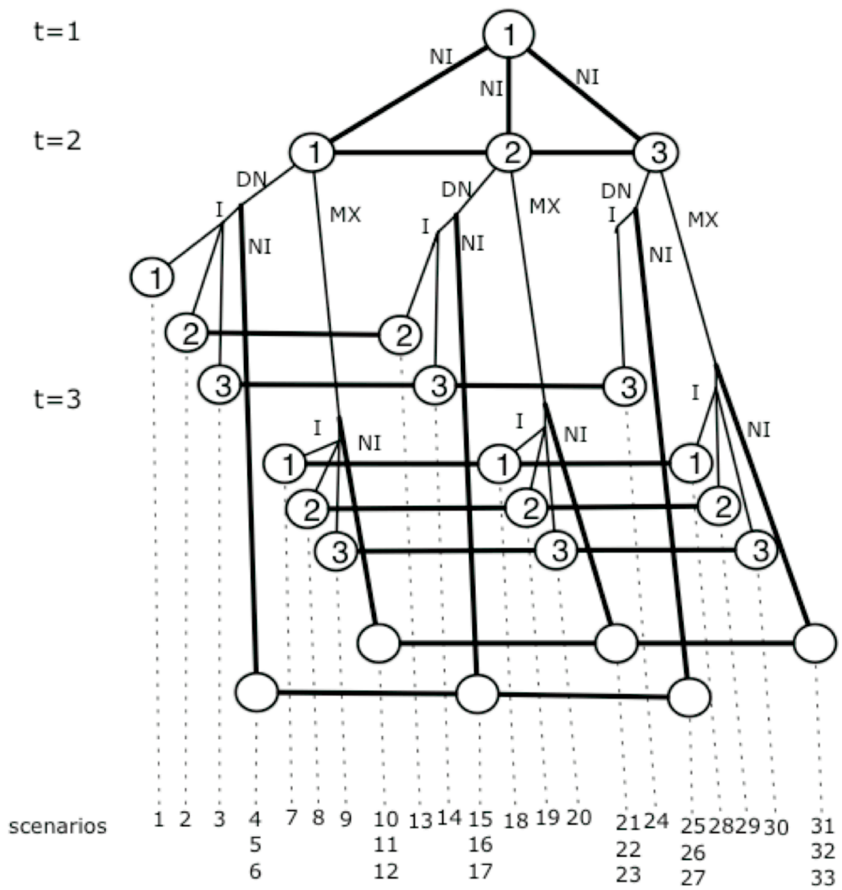

Figure 4.2: Resolving some uncertainty with maintenance decisions and without inspection.

Stochastic programming problems with endogenous uncertainty are some of the most difficult problems of mathematical programming as their size and complexity grow exponentially with the number of the random variables and the number of stages [9].

It is clear that a main perspective of this work is the proposition of an efficient solution procedure that should take advantage of the specific RNIMP tree structure. From this perspective, we linearized all variables and expressions of the model, except the user cost expression. Moreover, scenarios can be generated using the state-dependent deterioration model developed by Zouch *et al.* [12].

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Résumé : L'objectif de cette thèse est de développer des approches de maintenance conditionnelle pour le problème de gestion de l'entretien des réseaux routiers soumis à fissuration. Les indicateurs actuels de dégradation ne permettent pas de rendre compte de l'état structurel de la chaussée ainsi que de l'impact d'un entretien sur le processus de dégradation. Nous avons proposé la construction d'un processus stochastique bivarié pour rendre compte de l'accumulation de dégradation non observable lors de l'initiation de la fissuration ainsi que de l'évolution de la dégradation observable et non-observable pendant la propagation. La construction du modèle de dégradation est basée sur une formulation conditionnelle de la loi jointe d'évolution du processus. Outre les avantages de modélisation, l'intégration d'une nouvelle variable de décision offre une flexibilité de la décision optimale de maintenance ainsi que la prise en compte des effets de maintenance imparfaite sur l'évolution future de la fissuration. La formulation de type processus de décision markovien (MDP) puis partiellement observable (POMDP) pour prendre en compte la non observabilité du nouvel indicateur permet de dégager des propriétés structurelles du problème MDP lorsque le processus de transition change après maintenance. Ces propriétés nous permettent d'accélérer l'algorithme classique de résolution des MDPs ainsi que de construire une heuristique efficace pour le POMDP. Enfin, nous avons proposé une modélisation de type programmation stochastique multi-étapes pour le problème de l'optimisation jointe de l'inspection et de l'entretien du réseau pour lequel la nature endogène de l'incertitude est importante.

Abstract : The objective of this thesis is to develop condition-based maintenance approaches for the management of road networks that are subject to cracking. Deterioration indicators currently used neither represent the structural deterioration level of the road nor model maintenance impacts on the deterioration process. We propose a novel bivariate stochastic process to model the non-observable deterioration during the crack initiation phase as well as the evolution of the observable and non-observable deterioration in the propagation phase. The construction of the deterioration model is based on a condition-based, joint evolution law of the two deterioration parameters. The introduction of a new decision parameter representing the non-observable deterioration offers flexibility in the decision process and allows modeling imperfect maintenance effects on the deterioration process. We formulate the maintenance optimisation problem as a Markov decision process (MDP), and then extend it to a partially observed Markov decision process (POMDP) in order to take into account that the new deterioration parameter is not directly observable. We derive structural properties for the MDP problem with action-dependent transitions and use them to reduce the computation time for the traditional policy iteration algorithm as well as define a heuristic solution procedure for the POMDP. Finally, we address the problem of joint inspection and maintenance optimisation for a road network that we formulate as a multi-stage stochastic program.