l'université nantes angers le mans
COMMUNAUTÉ D'UNIVERSITÉS ET ÉTABLISSEMENTS

NANJING UNIVERSITY 1902

UNIVERSITÉ DE NANTES

Thèse de Doctorat

Zhu Lu

Mémoire présenté en vue de l'obtention du
grade de Docteur de l'Université de Nantes
Docteur de Nanjing University
Label européen
sous le label de l'Université de Nantes Angers Le Mans

École doctorale : Sciences et technologies de l'information, et mathématiques

Discipline : Mathématiques
Unité de recherche : Laboratoire de Mathématiques Jean Leray(LMJL)

Soutenue le 31 Octobre 2014

Comportement en grand temps des solutions de l'équation de Schrödinger dissipative

JURY

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Remerciements

With the coming of the end of my Ph'D life, I have to express my sincere gratitudes to many people for the memorable experience.

First, I am deeply indebted to my Ph.D. supervisor, Prof. Wang Xueping and Prof. Yin Huicheng. With their enthusiasm, their inspiration, and their great efforts to explain things clearly and simply, they helped to make mathematics fun for me. I will bear in mind their instructions and make real progress in the future. In prticular, I would thank Professors Wang and Yin for their warmheart and many supports in my daily life. Many thanks again to Professors Wang and Yin.

I would like to thank my academic brothers and sisters : Dr. Xu Ning, Dr.Zhang Pingzheng, Dr. Cui Dcheng, Dr. Li Jun, Dr. Xu Gang, Dr. Zhou Chunhui, Dr. Chen Xuezhang, Dr. Feiminwen, Dr. Zhang, Kangqun, Dr.Ruan, Zhuoping, Dr. Wang Zhan, Dr. Yang Hui, Dr. Li Gang, Dr. Zhang Fu, Dr. Li gang, Zhang Hen, Dr. Li Yuchen, Ma Hong, Dr. Liaoxian, Ding Bingbing, Cao Zhengzi, He Daoyin, Li Liang, Zhang Lin, Liu Yingbo, Pan Xinghong, Hou Fei, Sui Yuxia, Li Zijin, Yang Jiaqi, Yuan Qian, Zhao Wenbing, etc. for their suggestions during my study in Nanjing University. I also thank my friends in Nantes : Xu Yan, Li Dayu, Han Junfeng, Li Weixi, Zhang Yuesheng, Cha Limei, Liu Guogang, Dong Chundi, Ilaria Mondello, Gilberto Spano, etc. for the enjoyable time.

I am grateful to the secretaries and librarians in the Laboratoire de Math¡äematiques Jean Leray and Academy of Mathematics of System Sciences, for assisting me in many different ways.

Finally, special thanks and deep loves to my parents, my brother and my girl friend Miss Du Shanshan for their constant supports and loves.

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Introduction

This thesis is devoted to studying the large time behavior of the solutions to the Cauchy problem of the dissipative Schrödinger equations

$$\begin{cases} i\frac{\partial}{\partial t}u(t,x) = Hu(t,x),\ x \in \mathbb{R}^n,\ t \geq 0 \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $H = -\Delta + V(x)$ is the linear Schrödinger operator. Here we always assume that $V(x)$ is a complex potential satisfying the short-range condition

$$|V(x)| = O(\langle x\rangle^{-\rho}), \tag{1.2}$$

for some $\rho > 2$ and the dissipative condition $\Im V(x) \leq 0$.

This chapter is organized as follows. In Section 1.1, we present an introduction and some classical results about Schrödinger operators. The main goal of this thesis will be given in Section 1.2. In Section 1.3, we will state the main results of this thesis and then give the sketch of the proof.

1.1 Presentation

In quantum mechanics, the Schrödinger equation is a partial differential equation that describes how the quantum state of some physical system changes with time. It was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrödinger. In classical mechanics, the equation of motion is Newton's second law and which replaces Newton's law in quantum mechanics is Schrödinger's equation. It is not a simple algebraic equation, but a linear partial differential equation in general. This differential equation describes the wave function of the system which is also called the quantum state.

Let $Hu = -\Delta u+V(x)u$. Here u is the wave function representing the position, $-\Delta u = -\sum_{j=1}^{n} \partial_{x_j}^2 u$ is the kinetic energy and V is the potential energy. In [58], it is indicated that there three general mathematical problems arisen in quantum mechanical model : (1). Self-adjointness ; (2). Spectral analysis ; (3). Scattering theory. Our main risk is to try to answer these problems for the model (1.1) if $V(x)$ is a complex potential with a non-positive imaginary part. The self-adjoinness for the Schödinger operator is usually easy to obtain as long as the potential is a real function and satisfies some scale condition at infinity, especially assumption (1.2). So (2) and (3) are more important.

If $V(x)$ is real function satisfying (1.2), then H is selfadjoint on L^2 with domain $\mathcal{D}(H) = H^2(\mathbb{R}^n)$ and the results about its spectral analysis and scattering theory are very classic and complete. The low-energy analysis has been discussed by lots of mathematicians, such as [1], [6], [7], [8], [9], [17], [18], [23], [25], [27], [31], [32], [33], [37], [35], [46], [49], [51], [75], [76], [81], [84] and the references therein. The main difficulty is to analyze the threshold eigenvalue and resonance in some weighted Sobolev space. 0 is called an eigenvalue if there exists a L^2 function u such that $Hu = 0$ and called a resonance if there exists a function u satisfying $Hu = 0$ for some $u \in L^2(\langle x\rangle^s dx) \setminus L^2(dx)$, $s > \frac{1}{2}$. The one-dimensional and two-dimensional cases are discussed in [7], [8], [9], [35] and the references therein. The three-dimensional and four dimensional cases have been studied in [31] and [33] respectively. For dimension larger than four it has been discussed in [32]. Based on the delicate analysis of resolvent on low energies and the limiting absorption, the large-time behavior of the unitary group e^{-itH} can be obtained in weighted Sobolev space. On the other hand, the classical dispersive estimates have been studied in [17], [18], [25], [63] ,[84] and the references therein.

If $V(x)$ is complex, then H is non-selfadjoint. Suppose that H is dissipative. The limiting absorption principle on the positive axis from the upper complex plane was established in [60] by using the Mourré's commutator method. In [77], it was proved that 0 is regular point and the eigenvalues of H can not accumulate to the real axis near 0. Furthermore, if $\Im V(x)$ is sufficiently small, the discrete spectrum of H is a perturbation of the eigenvalues and the resonance of $\Re H$. Based on the spectral analysis, the expansion of e^{-itH} can be obtained in weighted Sobolev space in [78]. Besides these, in [57] J. Rauch proved that the time-decay if $V(x)$ has a exponential decay at infinity and in [26], M. Goldberg proved a dispersive estimate for some non-selfadjoint Schrödinger operator.

The scattering theory for the short-range Schrödinger operators is complete. There are lots of classical methods to treat it, such as Cook's method, Enss' method([56],[58]) and so on. The quantum scattering for non-selfadjoint operators appears in many physical situations such as optical models of nuclear scattering ([19]). Its Hilbert space theory is studied in [40], [50] and [13], [14], [15], [66]. See also [3], [4], [38]. In particular, one can construct the scattering operator for a pair of operators (H, H_0) where H_0 is selfadjoint and H is maximally dissipative, if the perturbation is of short-range in Enss' sense. Several equivalent conditions for the asymptotic completeness of dissipative quantum scattering are discussed in [14].

1.2 Goal of this thesis

In this thesis, we consider the dissipative Schrödinger operators, a class of non-selfadjoint operators.

Let A be a closed operator with the domain $\mathcal{D}(A)$ which is dense in some Hilbert space $\mathcal{H}$. If for each $x \in \mathcal{D}(A)$,

$$\Im\langle Ax, x\rangle \leq 0,$$

then A is called a dissipative operator. Moreover if there is no proper dissipative extension of A, then A is said to be maximal dissipative. By Hille-Yosida Theorem, one can prove that $-iA$ can generate a contraction semi-group on $\mathcal{H}$.

In this thesis, we always assume that $V(x) = V_1(x) - i\varepsilon V_2(x)$ satisfying $V_1(x)$, $V_2(x)$ are real functions and $V_2(x) \geq 0$ and $V_2(x) > 0$ on some open set. Then under assumption (1.2), $H(\varepsilon)$(we emphasize that H depends on ε) is maximally dissipative. Therefore, the solution of the Cauchy problem (1.1) can be represented by the semi-group e^{-itH}. Thus the main goal of this thesis is to study the behavior of e^{-itH} when t tends to infinity. Here $\varepsilon > 0$ is a sufficiently small constant such that we can treat the imaginary part of $H(\varepsilon)$ as a perturbation of the real part $H_1 = -\Delta + V_1(x)$.

An important problem is the completeness of the scattering operator for the pair $(H, -\Delta)$. In [14], E.B. Davies proved the existence of the wave operators and scattering operator, and some equivalent conditions for the completeness for the dissipative scattering. However, to our knowledge, there is still no result on the asymptotic completeness itself in this framework. One of the purposes in our work is to give a result on the asymptotic completeness of dissipative quantum scattering. Here we assume that 0 is a regular point of H_1, which means that 0 is neither a resonance nor an eigenvalue of H_1.

Second, we will discuss the large-time behavior of the semigroup $e^{-itH(\varepsilon)}$ in some weighted Sobolev space. Actually, it is a direct corollary of the low-energy estimate of the resolvent. Here we mainly focus on three cases respectively : zero is only an eigenvalue but not a resonance of H_1 in dimension three ; zero is only a resonance but not an eigenvalue of H_1 in dimension four ; zero is both a resonance and an eigenvalue of H_1 in dimension four. Furthermore we can show that the global estimate of the resolvent we need in the proof of the completeness of the scattering still holds. But unfortunately this estimate can not hold in the selfadjoint case. So we can't prove the completeness of the scattering.

1.3 Completeness of the scattering for $(H(\varepsilon), -\Delta)$

The quantum scattering for non-selfadjoint operators appears in many physical situations such as optical models of nuclear scattering ([19]). The first purpose of this thesis is to give a result on the asymptotic completeness of dissipative quantum scattering.

Denote $H_0 = -\Delta$ and $H_1 = -\Delta + V_1$. The wave operators

$$W_-(H, H_0) = \text{s-}\lim_{t\to-\infty} e^{itH}e^{-itH_0} \tag{1.3}$$

$$W_+(H_0, H) = \text{s-}\lim_{t\to+\infty} e^{itH_0}e^{-itH} \tag{1.4}$$

exist on $L^2(\mathbb{R}^n)$ and on $\mathcal{H}_{ac}$, respectively, where $\mathcal{H}_{ac}$ is the closure of the subspace

$$\mathcal{M}(H) = \{f \in L^2; \exists C_f \quad s.t. \quad \int_0^\infty |\langle e^{-itH}f, g\rangle|^2 \, dt \le C_f\|g\|^2, \forall g \in L^2\}.$$

See [14, 66]. It is known that Ran $W_-(H, H_0) \subset \mathcal{H}_{ac}$ (see Lemma 2 of [14]). The dissipative scattering operator $S(H, H_0)$ for the pair (H, H_0) is then defined as

$$S(H, H_0) = W_+(H_0, H)W_-(H, H_0). \tag{1.5}$$

$W_+(H_0, H)$ should be compared with the adjoint of the outgoing wave operator in selfadjoint cases, because for the pair of selfadjoint operators (H_1, H_0), the scattering operator $\widetilde{S}(H_1, H_0)$ is defined as

$$\widetilde{S}(H_1, H_0) = W_+(H_1, H_0)^* W_-(H_1, H_0).$$

A fundamental question for quantum scattering for a pair of selfadjoint operators is to study the asymptotic completeness of wave operators which implies that the scattering operator is unitary. In dissipative quantum scattering, the scattering operator $S(H, H_0)$ is a contraction : $\|S(H, H_0)\| \le 1$. The completeness of dissipative scattering can be interpreted as the bijectivity of $S(H, H_0)$. The equivalence of the following two conditions is due to E. B. Davies (Theorem 7, [14]) :

1. The range of $W_-(H, H_0)$ is closed ;
2. The scattering operator $S(H, H_0)$ is bijective on L^2.

In fact, E.B. Davies proves more general results in an abstract setting which can be applied to our case under the assumption (1.2) with $\rho > 1$.

Denote

$$W_-(\varepsilon) = W_-(H(\varepsilon), H_0) \text{ and } S(\varepsilon) = S(H(\varepsilon), H_0)$$

the wave and scattering operators defined as above with $H = H(\varepsilon)$. Denote

$$R(z) = (H - z)^{-1};$$

$$R_j(z) = (H_j - z)^{-1}, \text{ for } j = 0, 1$$

and the working spaces

$$H^{r,s}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \langle x\rangle^s (1-\Delta)^{\frac{r}{2}} u \in L^2(\mathbb{R}^n)\}$$

Let $\mathcal{L}(r, s; r', s')$ be the bounded operators from $H^{r,s}(\mathbb{R}^n)$ to $H^{r',s'}(\mathbb{R}^n)$.

Theorem 1.3.1. *Assume the condition (1.2) with $\rho > 2$ and $n \geq 3$. Suppose that 0 is neither an eigenvalue nor a resonance of H_1. Then one has for some $\varepsilon_0 > 0$*

$$\text{Ran } W_-(\varepsilon) = \text{Ran } \Pi'(\varepsilon), \quad 0 < \varepsilon \leq \varepsilon_0, \tag{1.6}$$

where $\Pi'(\varepsilon) = 1 - \Pi(\varepsilon)$ and $\Pi(\varepsilon)$ is the Riesz projection associated with discrete spectrum of $H(\varepsilon)$.

Since one can prove that $\Pi(\varepsilon)$ is of finite rank, thus Ran $W_-(\varepsilon)$ is closed and the scattering is complete. The proof of Theorem 2.1.1 is based on a uniformly global limiting absorption principle for the resolvent of $H(\varepsilon)$ on the range of $\Pi'(\varepsilon)$

Theorem 1.3.2. *Under the assumptions of Theorem 2.1.1, one has the uniform global resolvent estimate*

$$\|\langle x\rangle^{-s}\Pi'(\varepsilon)R(\lambda + i0, \varepsilon)\Pi'(\varepsilon)\langle x\rangle^{-s}\| \leq C_s\langle\lambda\rangle^{-1/2}, \lambda \in \mathbb{R} \tag{1.7}$$

uniformly in ε, where $R(\lambda+i0, \varepsilon) = \lim_{\mu\to 0_+} R(\lambda+i\mu, \varepsilon)$ in $\mathcal{L}(0, s; 0, -s)$, $s > 1$. Here $\Pi'(\varepsilon) = 1-\Pi(\varepsilon)$, $\Pi(\varepsilon) = \sum_j \Pi_j(\varepsilon)$ being the Riesz projection of $H(\varepsilon)$ associated to $\sigma_{\rm disc}(H(\varepsilon))$.

By the technique of selfadjoint dilation for dissipative operators([55]), this gives a uniform Kato smoothness estimate for the semigroup $e^{-itH(\epsilon)}$. The condition that 0 is neither an eigenvalue nor a resonance of H_1 is necessary for such uniform estimates. We identify the range of $W_-(\epsilon)$ for $\epsilon > 0$ small, making use of the asymptotic completeness of the wave operators for the selfadjoint pair (H_1, H_0).

1.4 Asymptotic expansion in time of $e^{-itH(\varepsilon)}$

Secondly, we consider the Cauchy problem of the following dissipative Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) = H(\varepsilon)u(t,x),\ t \geq 0,\ x \in \mathbb{R}^n,\ n \geq 3, \\ u(0,x) = u_0(x), \end{cases} \tag{1.8}$$

The main task in the second part is to get the asymptotic expansion of $e^{-itH(\varepsilon)}$ in some weighted L^2 space as t tends to infinity.

So far there have been many works on the low-energy spectral analysis for the self-adjoint Schrödinger operator and time-decay of the resulting unitary group (cf. [1], [7], [8], [9], [31], [32], [33], [35], [46], [49], [51], [75], [76], [81], [84] and the references therein). Among these works, the low-energy analysis can be done in the operator space $\mathcal{L}(0, s; 0, -s)$ for some $s > 1$. It is well known that the large-time expansion of the unitary group $U_1(t) = e^{-itH_1}$ in $\mathcal{L}(-1, s; 1, -s)$ is closely related to the behavior of the resolvent $R_1(z) = (H_1 - z)^{-1}$ for z near 0. The main difficulty in studying the behavior of $R_1(z)$ near 0 comes from the existence of the zero eigenvalue and the zero resonance. Let $\mathcal{M} = \{\phi \in H^{1,-s} : H_1\phi = 0, \text{ for any } s > \frac{1}{2}\}$ be the null space of H_1 in $H^{1,-s}$ and then $\mathcal{M} \cap L^2$ is called the eigenspace of H_1 at zero. If $\mathcal{M} \setminus L^2$ is nontrivial, 0 is called a resonance of H_1 and $\phi \in \mathcal{M} \setminus L^2$ is called a zero resonant state of H_1. Under the assumption (1.2) for $\rho_0 > 2$, it is known that $\dim(\mathcal{M}/(\mathcal{M} \cap L^2)) \leq 1$ in dimension three (see [31]) and dimension four (see [33]), and $\mathcal{M} \subset L^2$ in dimension $n \geq 5$ (see [32]). The one-dimensional and two-dimensional cases are discussed in [7], [8], [9], [35] and the references therein. In these works, V_1 is treated as a perturbation of H_0. In [75] and [76], there permits a decaying of critical order $O(\frac{1}{|x|^2})$ as $|x| \to \infty$ on $V_1(x)$. It is clear that this kind of potential can not be seen as a perturbation of H_0 in the low-energy analysis. It must be treated together with H_0 and the zero resonance can appear in any dimensional case.

In this paper, we focus on three cases : zero is only an eigenvalue but not a resonance of H_1, i.e. $\mathcal{M} \subset L^2$ in dimension three ; zero is only a resonance but not an eigenvalue of H_1, i.e. $\mathcal{M} \cap L^2 = \emptyset$ in dimension four ; zero is both a resonance and an eigenvalue of H_1 in dimension four. Actually, the first case can be extended to $n \geq 4$ if zero is only an eigenvalue of H_1. Since for $n \geq 5$, 0 can be only an eigenvalue of H_1, the result is complete for $n \geq 5$. On the other hand, this method we use in the eigenvalue case can be applied in the four-dimensional resonance case but it is invalid for the three-dimensional resonance case. This will be explained in detail below.

Similar to the selfadjoint case, the large-time behavior of the semigroup generated by the dissipative Schrödinger operator also depends on the low-energy spectral analysis. There are some works about the non-selfadjoint case such as [26] and [57]. In our case, under the assumption (1.2) and ε small enough, the crucial point is also to get the asymptotic behavior of the resolvent $R(z, \varepsilon) = (H(\varepsilon) - z)^{-1}$ for z near 0. In [77], it is proved that 0 is a regular point and

$$R(\lambda \pm i0, 1) = \lim_{\kappa \to 0_+} R(\lambda \pm i\kappa, 1)$$

exist in $\mathcal{L}(-1, s; 1, -s)$ for $s > 1$ on $[-c', c']$ for some $c' > 0$ under the assumption (1.2) for $\rho_0 > 2$. On the other hand, in [60], we know that $R(\lambda + i0, 1) \in C^1(]0, \infty[; \mathcal{L}(0, s; 0 - s))$ for some $\rho_0 > 2$ and $s > \frac{3}{2}$. Then by the formula proved in [78]

$$\langle U(t, 1)f, g\rangle = \frac{1}{2\pi i} \int_{\mathbb{R}} \langle R(\lambda + i0, 1)f, g\rangle d\lambda, \ t > 0 \tag{1.9}$$

for $f, g \in L^{2,s}$ for $s > 1$, the author gave an expansion of the semigroup $U(t, 1)$ for the large time in $\mathcal{L}(0, s; 0, -s)$ under some additional conditions on the derivatives of $V = V_1 - iV_2$. Meanwhile, the author constructed a dissipative example such that there exists a positive resonance and $R(\lambda - i0, 1)$ does not exist at this point.

For later use, we denote $\sigma(H(\varepsilon))$ ($\sigma_{disc}(H(\varepsilon))$ and $\sigma_{ess}(H(\varepsilon))$) by the spectrum (the discrete and essential spectrum) of $H(\varepsilon)$ respectively. By Weyl's essential spectrum theorem, one has $\sigma_{ess}(H(\varepsilon)) = \mathbb{R}_+ \triangleq [0, \infty[$ and $\sigma_{disc}(H(\varepsilon)) \subset \mathbb{C}_- = \{z \in \mathbb{C} : \Im z < 0\}$ which is a set of the eigenvalues with finite multiplicity.

In this thesis, for $\varepsilon > 0$ sufficiently small and some $\rho_0 > 2$ in (1.2), we can obtain the existence of $R(\lambda \pm i0, \varepsilon)$ by Grushin method for the low energies and by the method of perturbation for $\lambda \in [\lambda_0, \infty[$ in $\mathcal{L}(0, s; 0, -s)$ for some $s > 1$ and some fixed positive λ_0. Thus we can use the relation

$$e^{-itH(\varepsilon)}\Pi'(\varepsilon) = \frac{1}{2\pi i} \int_0^{+\infty} (R(\lambda + i0, \varepsilon) - R(\lambda - i0, \varepsilon))e^{-it\lambda} d\lambda, \ t > 0 \tag{1.10}$$

in $\mathcal{L}(0, s; 0, -s)$ for some $s > 1$ and any fixed $\varepsilon > 0$ sufficiently small. (1.10) will be checked in Section 4. Here $\Pi(\varepsilon)$ is the Riesz projection associated with the discrete spectrum of $H(\varepsilon)$ and $\Pi'(\varepsilon) = 1 - \Pi(\varepsilon)$. The distribution of the discrete spectrum of $H(\varepsilon)$ for ε sufficiently small has been discussed in [77]. It is different from the self-adjoint case in which the singularity of $R_1(\lambda \pm i0)$ in $\mathcal{L}(0, s; 0, -s)$, $s > \frac{1}{2}$ only occurs at $\lambda = 0$ such as in [31], [35], [32], [33], [76]. Here the eigenvalues of $H(\varepsilon)$ are all located on the lower complex plane. But the accurate position of these eigenvalues can not be obtained. So the expansion of the resolvent near these eigenvalues may not be computed directly. Fortunately, it is proved in [77] that the distance from these eigenvalues to the positive real axis has a positive lower bound dependent on ε. Thus based on this fact, we can deduce the expansion for low energies of $R(z, \varepsilon)$ outside some discs on the lower plane which contain the eigenvalues and the radii of which depend on ε. Since $H(\varepsilon)$ is non-selfadjoint, there maybe exists some Jordan block structure at each eigenvalue of $H(\varepsilon)$. From [77], one can see that the number of the eigenvalues of $H(\varepsilon)$ counted according to their algebraic multiplicities equals to the number of eigenvalues of H_1 which is finite. Thus $U(t, \varepsilon)\Pi(\varepsilon)$ is of finite rank. Furthermore since $\sigma_{disc}(H(\varepsilon)) \subset \mathbb{C}_-$, then $U(t, \varepsilon)\Pi(\varepsilon)$ has exponential decay rate dependent on ε. Some properties of the Riesz projection $\Pi(\varepsilon)$ will be discussed in Section 3.3 for three-dimensional case.

In this chapter, we will first consider the 3-dimensional case under the assumption that 0 is an eigenvalue but not a resonance of H_1.

Theorem 1.4.1. *Let $n = 3$ and $N \geq 3$ be a positive integer. Suppose that assumption (1.2) holds for some $\rho_0 > 2N + 1$ and that 0 is only an eigenvalue but not a resonance of H_1. Then for $s \in]N + \frac{1}{2}, \infty[$ and $\alpha \in]0, \min\{1, s - N - \frac{1}{2}\}[$, there exists $\varepsilon_0 > 0$ small enough such that for $\varepsilon \in]0, \varepsilon_0]$, the expansion of the semigroup generated by the dissipative Schrödinger operator $H(\varepsilon)$ takes the form*

$$e^{-itH(\varepsilon)}\Pi'(\varepsilon) \quad = \quad t^{-\frac{3}{2}}T_1(\varepsilon) + \sum_{j=2}^{[\frac{N+1}{2}]} \frac{t^{-j-\frac{1}{2}}}{\varepsilon^j} T_j(\varepsilon) + \varepsilon^{-\frac{N+\alpha+1}{2}} t^{-\frac{N+\alpha}{2}-1} L(t, \varepsilon) \tag{1.11}$$

in $\mathcal{L}(0, s; 0, -s)$. Here $T_j(\varepsilon)$ is a uniformly bounded operator on ε in $\mathcal{L}(-1, s_j; 1, -s_j)$ for $s_j > 2j - \frac{1}{2}$, $j = 1, \ldots, [\frac{N+1}{2}]$ and $L(t, \varepsilon)$ is uniformly bounded on ε, t in $\mathcal{L}(0, s; 0, -s)$. Moreover each $T_j(\varepsilon)$ is of finite rank.

Remark 1.4.2. *One can compare Theorem 1.4.1 with the selfadjoint case in [31] and the dissipative case in [78]. First if 0 is an eigenvalue but not a resonance of H_1, then for $\rho_0 > 5$ and $s > \frac{5}{2}$, the expansions of $R_1(z)$ and $U_1(t) = e^{-itH_1}$ have the form*

$$R_1(z) = -z^{-1}P_0 - iz^{-\frac{1}{2}}B_{-1} + O(z^{-\frac{1}{2}+\sigma}),$$

in $\mathcal{L}(-1, s; 1, -s)$ where $z = |z|^{\frac{1}{2}}e^{i \arg z}$ with $\arg z \in]0, 2\pi[$, $|z| \to 0$ and

$$U_1(t)\Pi_{ac} = -(\pi i t)^{-\frac{1}{2}}B_{-1} + O(t^{-\frac{1}{2}-\sigma}), \tag{1.12}$$

in $\mathcal{L}(0, s; 0, -s)$ as $t \to \infty$, where P_0 is the eigenprojection with respect to 0 and Π_{ac} is the orthogonal projection onto the absolutely continuous space of H_1. Here, $\sigma > 0$ is a positive constant dependent on s. Moreover, B_{-1} is at most of rank 3.

Second in [78], the author discussed the n-dimensional dissipative Schrödinger operator. There the imaginary part of the dissipative operator is not necessarily small. Then one can obtain the expansion of the semigroup in $\mathcal{L}(0, s; 0, -s)$ for $\rho_0 > n$ and $s > [\frac{n}{2}] + 2$ described as follows

$$U(t) = t^{-\frac{n}{2}}C_0 + O(t^{-\frac{n}{2}-\delta}),$$

where C_0 is of rank one. Here it needs some additional conditions on the derivatives of $V(x) = V_1(x) - iV_2(x)$.

In the selfadjoint case, the unitary group acting on the orthogonal complement space of the eigenspace of H_1 has a decay of rate $t^{-\frac{1}{2}}$ in $\mathcal{L}(0, s_1; 0, -s_1)$ for $s_1 > \frac{3}{2}$. This destroys the decay-rate $t^{-\frac{3}{2}}$ although the eigenspace has been excluded. It is different from the dissipative case in which $U(t, \varepsilon)\Pi'(\varepsilon)$ also decays with rate $O(t^{-\frac{3}{2}})$. On the other hand, since the imaginary part of each eigenvalue of $H(\varepsilon)$ is equal to $-c\varepsilon + o(\varepsilon)$ for some $c > 0$, then $U(t, \varepsilon)\Pi(\varepsilon)$ decays with an exponential rate. One can see from Remark 3.3.9 that the principal term $T_1(\varepsilon)$ is of rank one, which coincides with the result in [78]. In particular in [78], since in the formula (1.9) applied to get the expansion the author used the limit of the resolvent from the upper plane, the effect of the eigenvalues can not be observed. And due to (1.10), we can obtain the expansion in the complete subspace of the eigenspace, which can be compared to the absolutely continuous spectral subspace of H_1.

In the expansion (1.11), there exists singularities on ε. It is because the existence of the eigenvalues of $H(\varepsilon)$ near 0. The singularities come from the distance between these eigenvalues and the positive axis which is the essential spectrum of $H(\varepsilon)$. In particular, one can see that the expansion (1.12) for the selfadjoint case cannot be seen as a limit of (1.11) when ε tends to 0.

Remark 1.4.3. *We note that the expansion (1.11) holds for any $t > 0$ and the singularity of each term on ε has been described explicitly.*

Remark 1.4.4. *It is interesting that the principal term is $t^{-\frac{3}{2}}T_1(\varepsilon) = t^{-\frac{3}{2}}O(1)$ which is uniformly bounded on ε. If ε tends to 0, then the limit of the principal term exists in $\mathcal{L}(-1, s_1; 1, -s_1)$ for $s_1 > \frac{3}{2}$ and it is nontrivial. In particular, the limit $T_1(0)$ is dependent on V_2 and its explicit representation can be obtained in Section 3.4.*

Then we state the theorem for the 4-dimensional resonance case.

Theorem 1.4.5. *Let $n = 4$ and $N \geq 3$ be a positive integer. Suppose that assumption (1.2) holds for $\rho_0 > 4N + 2$ and that 0 is only a resonance but not an eigenvalue of H_1. Then for $s \in]2N + 1, \frac{\rho_0}{2}]$ and $\alpha \in]0, \min\{1, \frac{s}{2} - N - \frac{1}{2}\}[$, there exists $\varepsilon_0 > 0$ small enough such that for $\varepsilon \in]0, \varepsilon_0]$, the expansion in $\mathcal{L}(0, s; 0, -s)$ of the semigroup generated by the dissipative Schrödinger operator $H(\varepsilon)$ takes the form*

$$e^{-itH(\varepsilon)}\Pi'(\varepsilon) = \sum_{j=1}^{N}(\varepsilon t)^{-1-j}\sum_{k=0}^{j-1}\ln^k tT_j^k(\varepsilon) + (\varepsilon t)^{-N-1-\alpha}L(t, \varepsilon), \tag{1.13}$$

where $T_j^k(\varepsilon)$ is uniformly bounded operator on ε in $\mathcal{L}(-1, s_j; 1, -s_j)$, $s_j > 2j + 1$ for $j = 1, \dots, N$, $k = 0, 1, \dots, j-1$ and $L(t, \varepsilon)$ is a uniformly bounded operator on t, ε in $\mathcal{L}(0, s; 0, -s)$. Furthermore, each $T_j^k(\varepsilon)$ is of finite rank.

Remark 1.4.6. *Similar to the three-dimensional case, we can also compare our result with the selfadjoint case in [33] and the dissipative case in [78]. In [33], if 0 is a resonance but not an eigenvalue of H_1, then one has the expansion in $\mathcal{L}(0, s; 0, -s)$ for $s > 6$ and $\rho_0 > 12$*

$$U_1(t)\Pi_{ac} = \Phi(t)\langle\cdot, \phi\rangle\phi + O(t^{-1}),$$

as $t \to \infty$, where

$$\Phi(t) = \int_0^\infty \frac{1}{\lambda}\frac{e^{-it\lambda}}{\pi^2 + (a - \ln\lambda)^2}d\lambda = O(\ln^{-1} t)$$

for some real constant a dependent on $V_1(x)$. In the dissipative case, one can see that the principal term has decay rate of t^{-2}. In particular, we can compute that the principal term $T_1^0(\varepsilon)$ is an operator of rank one (see Remark 3.5.5). This coincides with the result in [78].

Meanwhile, we can also obtain the similar conclusion in four-dimensional case in which we assume that zero is both a resonance and an eigenvalue of H_1.

Theorem 1.4.7. *Let $n = 4$ and $N \geq 3$ be a positive integer. Suppose that assumption (1.2) holds for $\rho_0 > 4N+2$ and that 0 is both a resonance and an eigenvalue of H_1. Then there exists $\varepsilon_0 > 0$ small enough such that for $\varepsilon \in]0, \varepsilon_0]$, the expansion (1.13) in $\mathcal{L}(0, s; 0, -s)$ for $s > 2N + 1$ and the properties of the terms stated in Theorem 1.4.5 still hold.*

Remark 1.4.8. *The explicit expression of the terms in the expansion of $e^{itH(\varepsilon)}\Pi'(\varepsilon)$ will be calculated in Section 6. In particular, the principal term $T_1^0(\varepsilon) \in \mathcal{L}(-1, s_1; 1, ; -s_1)$ for $s > 3$ is of rank one which coincides the result in [78]. This will be obtained in Remark 3.6.5.*

Remark 1.4.9. *The main part of the proof is to obtain the expansion of the resolvent near 0 and the key point is the observation that the eigenvalues of $H(\varepsilon)$ near 0 has distance $c\varepsilon + o(\varepsilon)$ in the eigenvalue case in three dimension case and $c\varepsilon|\ln\varepsilon|^{-1} + o(\varepsilon|\ln\varepsilon|^{-1})$ for the resonance case in four dimensional case from the real axis. Here $c > 0$ is some generic constant. We note that in the eigenvalue case for dimension $n \geq 4$ the distance between the eigenvalues of $H(\varepsilon)$ near 0 and the real axis is also $c\varepsilon + o(\varepsilon)$(See [77]). So the methods we apply here can be also used in the eigenvalue case for dimension $n \geq 4$. Since there exists no 0-resonance for $n \geq 5$, thus the results are complete for $n \geq 3$ except the resonance case in three dimensional case.*

We note that in [77], Prof. Wang proved the number of the eigenvalues of $H(\varepsilon)$ near zero for dimension $n \geq 3$ under some additional condition (1.8) for the case that 0 is only a resonance but not an eigenvalue of H_1(See Theorem 1.2(b)). Here the case $n = 4$ which we consider coincides with the case $\nu_1 = 1$ in [77]. But we can prove the same conclusion without the condition (1.8) in [77].

The proof of these three theorems are based on the low-energy analysis. In particular, we will also discuss some properties of the Riesz projection of $H(\varepsilon)$ associated with the eigenvalues near 0.

1.5 Notations

$V_\varepsilon(x) = V_1(x) - i\varepsilon V_2(x),\ V_1, V_2$ real, $\varepsilon > 0$ sufficiently small;
$H(\varepsilon) = -\Delta + V_\varepsilon(\varepsilon)$;
$H_0 = -\Delta,\ H_1 = H_0 + V_1$;
$R(z, \varepsilon) = (H(\varepsilon) - z)^{-1},\ R_j(z) = (H_j - z)^{-1},\ j = 0, 1$;
$H(\varepsilon) = -\Delta + V_1(x) - i\varepsilon V_2(x),\ R(z) = (H - z)^{-1}$;
$H^{k,s}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \langle x\rangle^s \langle -i^{-1}\nabla\rangle^k f \in L^2(\mathbb{R}^n)\}$;
$L \in \mathcal{L}(k, s; k', s') : H^{k,s}(\mathbb{R}^n) \to H^{k',s'}(\mathbb{R}^n)$ linear bounded.

2

On the wave operator for dissipative potentials with small imaginary part

2.1 Main results

The quantum scattering for non-selfadjoint operators appears in many physical situations such as optical models of nuclear scattering ([19]). Its Hilbert space theory is studied in [40, 50] and [13, 14, 15, 66]. See also [3, 4, 38]. In particular, one can construct the scattering operator for a pair of operators (H, H_0) where H_0 is selfadjoint and H is maximally dissipative, if the perturbation is of short-range in Enss' sense. Several equivalent conditions for the asymptotic completeness of dissipative quantum scattering are discussed in [14]. However, to our knowledge, there is still no result on the asymptotic completeness itself in this framework. The purpose of this chapter is to give a result on the asymptotic completeness of dissipative quantum scattering under some conditions.

In this chapter, we study the dissipative quantum scattering under the assumption that the imaginary part of the potential is small. The main result is described as follows.

Theorem 2.1.1. *Assume the condition (1.2) with $\rho > 2$ and $n \geq 3$. Suppose that 0 is neither an eigenvalue nor a resonance of H_1. Then one has for some $\varepsilon_0 > 0$*

$$\text{Ran } W_-(\varepsilon) = \text{Ran } \Pi'(\varepsilon), \quad 0 < \varepsilon \leq \varepsilon_0, \tag{2.1}$$

where $\Pi'(\varepsilon) = 1 - \Pi(\varepsilon)$ and $\Pi(\varepsilon)$ is the Riesz projection associated with discrete spectrum of $H(\varepsilon)$.

Theorem 2.1.1 can be compared with the asymptotic completeness of wave operators in the selfadjoint case which says that

$$\text{Ran } W_\pm(H_1, H_0) = \text{Ran } \Pi_{ac},$$

where Π_{ac} is the projection onto the absolutely continuous spectra subspace of H_1. Under the condition $\rho > 2$, $\Pi(\varepsilon)$ is of finite rank and $\text{Ran } \Pi'(\varepsilon) = \text{Ker } \Pi(\varepsilon)$ is closed. As a consequence of Theorem 2.1.1 and Theorem 7 of [14], the scattering operator $S(\epsilon)$ is bijective for $\epsilon > 0$ small enough. The asymptotic completeness of dissipative quantum scaterring has the following consequence on the dynamics of the semigroup of contractions. For any $f \in L^2$, one can decompose it as $f = f_1 + f_2$ with $f_1 \in \text{Ran } \Pi(\varepsilon)$ and $f_2 \in \text{Ran } \Pi'(\varepsilon)$. Since $H(\varepsilon)$ has a finite number of eigenvalues, all with negative imaginary part, $e^{-itH(\varepsilon)} f_1$ decreases exponentially as $t \to +\infty$. The existence of the scattering operator $S(\varepsilon)$ implies that there exists $f_\infty \in L^2$ such that

$$\lim_{t\to+\infty} \|e^{-itH(\varepsilon)} f_2 - e^{-itH_0} f_\infty\| = 0 \tag{2.2}$$

and the asymptotic completeness of the wave operator $W_-(\epsilon)$ ensures that $f_\infty \neq 0$ if $f_2 \neq 0$. Theorem 2.1.1 shows that either $\|e^{-itH(\epsilon)}f\|$ decreases exponentially (when $f \in \text{Ran}\,\Pi(\varepsilon)$) or it tends to some non-zero limit as t goes to the infinity (when $f \notin \text{Ran}\,\Pi(\varepsilon)$).

The proof of Theorem 2.1.1 is based on a uniform global limiting absorption principle for the resolvent of $H(\varepsilon)$ on the range of $\Pi'(\varepsilon)$ which is proved in Section 2.3. By the technique of selfadjoint dilation for dissipative operators([55]), this gives a uniform Kato smoothness estimate for the semigroup $e^{-itH(\epsilon)}$. The condition that 0 is neither an eigenvalue nor a resonance of H_1 is necessary for such uniform estimates. Then, we identify the range of $W_-(\varepsilon)$ for $\varepsilon > 0$ small, making use of the asymptotic completeness of the wave operators for the selfadjoint pair (H_1, H_0).

2.2 Some preliminaries

In this section, we will first introduce some basic properties about the dissipative operator (cf. [13],[14], [15],[41],[58]), and the scattering theory for the self-adjoint case in [56]. Let $\mathcal{H}$ be a Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$, and first we give the definition of dissipative operator.

Definition 2.2.1. *Let A be a closed operator with the domain $\mathcal{D}(A)$ which is dense in $\mathcal{H}$. If for each $x \in \mathcal{D}(A)$,*

$$\Im\langle Ax, x\rangle \leq 0,$$

then A is called a dissipative operator.

Immediately, we can get a property for the dissipative operator only using the definition.

Proposition 2.2.2. *Let A be a dissipative operator on $\mathcal{H}$. Then we have the estimate*

$$\forall\lambda \in \mathbb{C}_+, x \in \mathcal{D}(A),\ \| x \| \leq \frac{1}{\Im\lambda} \| (A-\lambda)x \| . \tag{2.3}$$

Démonstration. $\forall x \in \mathcal{D}(A)$, $\lambda = \alpha + i\beta \in \mathbb{C}_+$, where $\alpha \in \mathbb{R}$ and $\beta > 0$, then

$$\begin{aligned} \|(A-\lambda)x\|^2 &= \langle (A-\lambda)x, (A-\lambda)x\rangle \\ &= \|(A-\alpha)x\|^2 + \beta^2\|x\|^2 - 2\Re\langle (A-\alpha)x, i\beta x\rangle \\ &= \|(A-\alpha)x\|^2 + \beta^2\|x\|^2 - 2\beta\Im\langle Ax, x\rangle \\ &\geq \beta^2\|x\|^2. \end{aligned}$$

□

Remark 2.2.3. *Furthermore, it is easy to see that if* Ran $(A-\lambda)$ *is dense in $\mathcal{H}$, then $\mathbb{C}_+ \triangleq \{z \in \mathbb{C} \mid \Im z > 0\} \subset \rho(A)$. In fact, if* Ran $(A-\lambda)$ *is dense in $\mathcal{H}$, then for each $x \in \mathcal{H}$, there exist $x_n \subset$* Ran $(A-\lambda)$ *for $\lambda \in \mathbb{C}^+$ such that $x_n \to x$. Set $y_n \in \mathcal{D}(A)$ satisfying $(A-\lambda)y_n = x_n$. Then by proposition 1.2, we have that*

$$\|y_n\| \leq \frac{1}{\Im\lambda}\|x_n\|.$$

Therefore, there exists a y such that $y_n \to y$ in $\mathcal{H}$. Because A is a closed operator, then $x = (A-\lambda)y$. So $A-\lambda$ is invertible and for $\forall x \in \mathcal{H}$, one has

$$\|(A-\lambda)^{-1}x\| \leq \frac{1}{\Im\lambda}\|x\|.$$

Definition 2.2.4. *Let A be a dissipative operator in $\mathcal{H}$. Moreover if there is no dissipative extension of A, then A is said to be maximal dissipative.*

There are some equivalent conditions to the maximal dissipative operator :

Proposition 2.2.5. *Let A be a closed dissipative operator in $\mathcal{H}$. Then the following assertions are equivalent :*
(1). $\exists\lambda \in \rho(A)\cap\mathbb{C}_+$;
(2). $\mathbb{C}_+ \subset \rho(A)$;
(3). A is a maximal dissipative operator.

Démonstration. We will complete this proposition by proving that (2) and (3) are both equivalent to (1).

"$(1) \Longleftrightarrow (2)$". It is obvious that (2) includes (1), and so we only to prove that (1) implies (2). We claim that if $\lambda \in \rho(A)\cap\mathbb{C}_+$, then $D(\lambda, \frac{\Im\lambda}{2}) \subset \rho(A)$. In fact, if there exists $\eta \in D(\lambda, \frac{\Im\lambda}{2})$ but $\eta \notin \rho(A)$. Then Ran $(A-\eta)$ is not dense in $\mathcal{H}$. So there exists a $\varphi \in \overline{\text{Ran }(A-\eta)}^{\perp} \cap \mathcal{D}(A^*)$, and for each $\phi \in \mathcal{H}$,

$$\langle (A^*-\bar{\eta})\varphi, \phi\rangle = \langle \varphi, (A-\eta)\phi\rangle = 0.$$

Thus, $\varphi \in \ker(A^*-\bar{\eta})$. On the other hand, by the invertibility of $A-\lambda$, there exists $\psi \in \mathcal{D}(A)$ such that $\varphi = (A-\lambda)\psi$, and then one has

$$0 = \langle (A^*-\bar{\eta})\varphi, \psi\rangle = \langle (A^*-\bar{\lambda})\varphi, \psi\rangle + \overline{(\lambda-\eta)}\langle \varphi, \psi\rangle = \|\varphi\|^2 + \overline{(\lambda-\eta)}\langle \varphi, (A-\lambda)^{-1}\varphi\rangle.$$

Hence

$$\begin{aligned}
\|\varphi\|^2 &= \overline{(\eta-\lambda)}\langle \varphi, (A-\lambda)^{-1}\varphi\rangle \\
&\leq |\eta-\lambda|\|\varphi\|\|(A-\lambda)^{-1}\|\|\varphi\| \\
&\leq \frac{\Im\lambda}{2}\frac{1}{\Im\lambda}\|\varphi\|^2 \\
&= \frac{1}{2}\|\varphi\|^2
\end{aligned}$$

It is a contradiction. So $D(\lambda, \frac{\Im\lambda}{2}) \subset \rho(A)$ and then by this one can obtain that $\{\eta \in \mathbb{C}_+ : \Re\eta = \Re\lambda \text{ or } \Im\eta = \Im\lambda\} \subset \rho(A)$. Consequently, one can obtain $\mathbb{C}_+ \subset \rho(A)$.

"$(1) \Longleftrightarrow (3)$" If there exists a $\lambda \in \mathbb{C}_+\cap\rho(A)$ and a dissipative extension B of A, i.e. $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $B|_{\mathcal{D}(A)} = A$. For $\forall\varphi \in \mathcal{D}(B)$, set $\phi = (A-\lambda)^{-1}(B-\lambda)\varphi \in \mathcal{D}(A)$. Then $(B-\lambda)\phi = (A-\lambda)\phi = (B-\lambda)\varphi$. Since $B-\lambda$ is a bijection from $\mathcal{D}(B)$ to $\mathcal{H}$, $\varphi = \phi \in \mathcal{D}(A)$. So $\mathcal{D}(A) = \mathcal{D}(B)$. Thus $A = B$. It means that A is a maximal dissipative operator.

On the other hand, assume that A is a maximal dissipative operator. If (1) is not true, i.e. there exists a $\lambda \in \mathbb{C}_+$ but $\lambda \notin \rho(A)$. Then there exists a $\psi \in \ker(A^*-\bar{\lambda}) \setminus \{0\}$. One has

$$\Im\langle A\psi, \psi\rangle = \Im\langle \psi, A^*\psi\rangle = (\Im\lambda)\|\psi\|^2 \geq 0.$$

Thus, $\psi \notin \mathcal{D}(A)$. We define an operator B with domain $\mathcal{D}(B) = \mathcal{D}(A) \oplus \{\psi\}$ as $B\varphi = A\varphi$, if $\varphi \in \mathcal{D}(A)$ and $B\psi = \bar{\lambda}\psi$. Then for each $\phi = \varphi + \mu\psi$ with $\varphi \in \mathcal{D}(A)$ and $\mu \in \mathbb{C}_+$, one has

$$\begin{aligned}
\langle B\phi, \phi\rangle &= \langle A\varphi, \varphi\rangle - \bar{\lambda}\mu\langle \psi, \varphi\rangle + \bar{\mu}\langle \varphi, A^*\psi\rangle + \bar{\lambda}|\mu|^2\|\psi\|^2 \\
&= \langle A\varphi, \varphi\rangle - \bar{\lambda}\mu\langle \psi, \varphi\rangle - \lambda\bar{\eta}\langle \varphi, \psi\rangle + \bar{\lambda}|\mu|^2\|\psi\|^2 \\
&= \langle A\varphi, \varphi\rangle - \Re(\bar{\lambda}\mu\langle \psi, \varphi\rangle) + \bar{\lambda}|\mu|^2\|\psi\|^2
\end{aligned}$$

Therefore

$$\Im\langle B\phi, \phi\rangle = \Im\langle A\varphi, \varphi\rangle - (\Im\lambda)|\mu|^2\|\psi\|^2 \leq 0.$$

So B is a nontrivial dissipative extension of A. This is in contradiction with (3).

□

Remark 2.2.6. *From proposition 2.2.2 and 2.2.5, We know that for a maximal dissipative operator A, (2.3) actually gives us a resolvent estimate*

$$\forall \lambda \in \mathbb{C}_+ \subset \rho(A), \|(A-\lambda)^{-1}\| \leq \frac{1}{\Im\lambda}. \tag{2.4}$$

Then we come back to the schrödinger operator we consider. Due to condition $V_2 \geq 0$, this kind of the schrödinger operator is obviously dissipative. In fact, it is a maximal dissipative operator, and for this we only need to check the following lemma.

Lemma 2.2.7. $\exists \lambda \in \rho(H) \cap \mathbb{C}_+$.

Démonstration. We set $\varepsilon = 1/2$ in (**??**) and let $\lambda = i\delta \in i\mathbb{R}_+$, and then

$$\begin{aligned} \|V_2(H_1+i\delta)^{-1}\| &\leq \frac{1}{2}\|H_1(H_1+i\delta)^{-1}\| + C\|(H_1+i\delta)^{-1}\| \\ &\leq \frac{1}{2}(1+\frac{1}{\delta}) + \frac{C}{\delta} \\ &= \frac{\delta+1+2C}{2\delta} \end{aligned}$$

Set $\delta > 2C+1$. Then $\|V_2(H_1+i\delta)^{-1}\|$ is strictly less than 1 and the same is true for $\|(H_1-i\delta)^{-1}V_2\|$. So by Neumann's series, $1 - i(H_1-i\delta)^{-1}V_2$ has a bounded inverse. Therefore,

$$H - i\delta = (H_1 - i\delta)(1 - i(H_1-i\delta)^{-1}V_2)$$

also has a bounded inverse. It completes the proof. □

Thus by Hille-Yoshida Theorem in [58], $-iH$ generates a contraction semigroup $\{e^{-iHt}\}_{t\geq 0}$. Then we consider the notions of the wave operators and the scattering operator for the pair (H, H_0).

Denote the wave operators

$$W_-(H, H_0) = s - \lim_{t\to+\infty} e^{-itH}e^{itH_0} \tag{2.5}$$

$$W_+(H_0, H) = s - \lim_{t\to-\infty} e^{itH_0}e^{-itH} \tag{2.6}$$

Then we consider the abstract version that V is a closed operator with domain $D(H_0)$ on $\mathcal{H}$. There is an important condition called Enss condition in the study of scattering operator.

Definition 2.2.8. *Let $R > 0$ and $F(|x| > R)$ be a characteristic function of $\mathbb{R}^n \setminus D(0,R)$. Denote*

$$h(R) = \|V(H_0 - i)^{-1}F(|x| > R)\|.$$

If

$$\int_0^\infty h(R)dR < \infty, \text{ and } h(0) < \infty,$$

then we call V satisfies Enss condition.

With the help of the Enss condition, The following theorem provides the existence of the wave operators.

Theorem 2.2.9 (Theorem 9.3 in [66])**.** *Let V be a closed operator on L^2 with :*

1. $\|V\phi\| \leq a\|H_0\phi\| + b\|\phi\|$*, for some* $a < 1$ *;*
2. $\Im\langle\phi, V\phi\rangle \leq 0$*, for all* $\phi \in D(H_0)$ *;*
3. *The Enss condition holds for* V*.*

Let $H = H_0 + V$. Then

1. *W_- defined by (1.3) exists on L^2 ;*

2. *W_+ defined by (1.4) exists on $\mathcal{H}_b^\perp$, where $\mathcal{H}_b = \{\varphi \in L^2 : \ H\varphi = \lambda\varphi \ with \ \lambda \in \mathbb{R}\}$;*

3. *The only possible limit point of real eigenvalue of H is 0 and any non-zero real eigenvalue has finite multiplicity.*

For the dissipative Schrödinger operator we discuss, we only need to prove the following proposition which provides us the existence of the wave operators.

Proposition 2.2.10 (Proposition 4.1 [56]). *The complex potential $V = V_1 - iV_2$ with the assumption (**??**) satisfies Enss condition.*

Démonstration. Choosing a cutoff function $\chi(x) \in C^\infty(\mathbb{R}^n)$ with $\chi(x) = 1$ if $|x| \geq 1$, $\chi(x) = 0$ if $|x| \leq \frac{1}{2}$ and $|\nabla\chi| \leq C$. Let $\chi_R(x) = \chi(x/R)$. Then

$$\begin{aligned} h(R) &= \|V(H_0 - i)^{-1}F(|x| > R)\| \\ &= \|F(|x| > R)(H_0 + i)^{-1}V^*\| \\ &\leq \|\chi_R(H_0 + i)^{-1}V^*\| \\ &\leq \|(H_0 + i)^{-1}\chi_R V^*\| + \|[\chi_R, (H_0 + i)^{-1}]V^*\| \\ &\triangleq (1) + (2) \end{aligned}$$

And for (1),

$$(1) \leq \|(H_0 + i)^{-1}\|\|\chi_R V^*\|_\infty \leq R^{-\rho_0}, \text{ for } R > 0 \text{ large enough.}$$

On the other hand,

$$\begin{aligned} [\chi_R, (H_0 + i)^{-1}] &= -(H_0 + i)^{-1}[\chi_R, (H_0 + i)](H_0 + i)^{-1} \\ &= -(H_0 + i)^{-1}(2\nabla \cdot (\nabla\chi_R \cdot) - \Delta\chi_R)(H_0 + i)^{-1} \end{aligned}$$

By $\|\nabla\chi_R\| \leq CR^{-1}$, $\|\Delta\chi_R\| \leq CR^{-2}$ and $\|(H_0 + i)^{-1}\nabla\| \leq C$, then

$$\begin{aligned} (2) &\leq 2\|(H_0 + i)^{-1}\nabla\|\|\nabla\chi_R(H_0 + i)^{-1}V^*\| + \|(H_0 + i)^{-1}\|\|\Delta\chi_R(H_0 + i)^{-1}V^*\| \\ &\leq C(R^{-1} + R^{-2})\|F(|x| > R/2)(H_0 + i)^{-1}V^*\|. \end{aligned}$$

Hence, $h(R)$ is an integrable function plus some terms with order R^{-1} near the infinite, and then by putting this estimate back to the above formula, we get $h(R)$ is integrable. □

Denote

$$\mathcal{M}(H) = \{\psi \in L^2(\mathbb{R}^n); \exists C_\psi > 0, s.t. \int_0^\infty |\langle e^{-itH}\psi, \phi\rangle|^2 dt \leq C_\psi\|\phi\|^2, \forall\phi \in L^2(\mathbb{R}^n)\}, \tag{2.7}$$

and $\mathcal{H}_{ac}(H) = \overline{\mathcal{M}(H)}$. Then we have the following propositions :

Proposition 2.2.11. *1. $\mathcal{H}_{ac}(H) \subset \mathcal{H}_b^\perp$;*

2. *W_- maps L^2 to $\mathcal{H}_{ac}(H)$. So $\mathcal{H}_{ac}(H)$ is an invariant space of $W_-(H, H_0)$.*

Démonstration. (1). First, we claim that for each ϕ satisfying $H\phi = \lambda\phi$ with $\lambda \in \mathbb{R}$ we have $V_2\phi = 0$ and then $H^*\phi = \lambda\phi$. In fact,

$$\lambda\|\phi\|^2 = \langle H\phi, \phi\rangle = \langle H_1\phi, \phi\rangle - i\langle V_2\phi, \phi\rangle,$$

so $\langle V_2\phi, \phi\rangle = 0$. Since $V_2 \geq 0$ it follows that $V_2\phi = 0$. $\forall \phi \in \mathcal{M}(H)$, and $\varphi \in \mathcal{H}_b^\perp$, by the definition of $\mathcal{M}(H)$, then

$$\begin{aligned} 0 &= \lim_{t\to+\infty} |\langle e^{-itH}\phi, \varphi\rangle| \\ &= \lim_{t\to+\infty} |\langle \phi, e^{itH^*}\varphi\rangle| \\ &= \lim_{t\to+\infty} |\langle \phi, e^{it\lambda}\varphi\rangle| \\ &= |\langle \phi, \varphi\rangle|. \end{aligned}$$

It follows that $\mathcal{M}(H) \subset \mathcal{H}_b^\perp$. Thus $\mathcal{H}_{ac}(H) = \overline{\mathcal{M}(H)} \subset \mathcal{H}_b^\perp$.

(2). $\forall \varphi \in \mathcal{M}(H_0)$, Let $\phi = W_-\varphi$. And for $\forall \psi \in {}^2$,

$$\begin{aligned} \int_0^\infty |\langle e^{-itH}\phi, \psi\rangle|^2 dt &= \int_0^\infty |\langle e^{-itH}W_-\varphi, \psi\rangle|^2 dt \\ &= \int_0^\infty |\langle W_- e^{-itH_0}\varphi, \psi\rangle|^2 dt \\ &= \int_0^\infty |\langle e^{-itH_0}\varphi, W_-^*\psi\rangle|^2 dt \\ &\leq C_\varphi \|W_-^*\psi\|^2 \\ &\leq C_\varphi \|\psi\|^2, \end{aligned}$$

since $\|W_-^*\| = \|W_-\| \leq 1$. On the other hand, we will check that $\overline{\mathcal{M}(H_0)} = L^2(\mathbb{R}^n)$. Let $\mathcal{H}_{ac}(H_0)$ be the absolutely continuous spectral space of H_0. Due to theorem 1.3 in [56], there exist a dense set $\mathcal{K}(H_0) = \{\psi \in \mathcal{H}_{ac}(H_0) : \frac{d\mu_\psi}{d\lambda} \in L^\infty$ and suppf_ψ is compact$\}$ where μ_ψ of $\mathcal{H}_{ac}(H_0)$ is the spectral measure associated to ψ and $\frac{d\mu_\psi}{d\lambda}$ is Randon-Nikodym derivative. Moreover, also by this theorem and for any Hilbert-Schmidt operator and $\psi \in \mathcal{K}(H_0)$, we have

$$\int_{-\infty}^\infty \|Ae^{-itH_0}\psi\|^2 dt \leq \|A\|_2^2 \|\psi\|^2,$$

where $\|A\|_2$ is the Hilbert-Schmidt norm of A. By taking $A = \langle \cdot, \varphi\rangle\varphi$, then $\mathcal{K}(H_0) \subset \mathcal{M}(H_0)$ and it follows that $\mathcal{H}_{ac}(H_0) \subset \overline{\mathcal{M}(H_0)}$. In light of $\mathcal{H}_{ac}(H_0) = L^2$ we have that $\mathcal{M}(H_0)$ is dense in L^2. So this proves the proposition.. □

Remark 2.2.12. *In our problem, it is known that there is no real eigenvalues of the dissipative Schrödinger operator and they are all on the lower-half complex plane(cf. [77]).*

So by the above, we define the dissipative scattering operator for the pair (H, H_0) as

$$S(H, H_0) = W_+(H_0, H)W_-(H, H_0). \tag{2.8}$$

An importance problem is the asymptotic completeness of the wave operator. In the self-adjoint case, the definition of the asymptotic completeness in the self-adjoint case is given by

$$\text{Ran } W_+(H_1, H_0) = \text{Ran } W_-(H_1, H_0) = \mathcal{H}_{ac}(H_1), \tag{2.9}$$

where $\mathcal{H}_{ac}(H_1)$ is the absolutely continuous spectral space of H_1. We have seen that Enss condition plays an important role in the existence of the wave operators, and in fact, it is also sufficient to the completeness in the self-adjoint case.

Theorem 2.2.13 ([56]). *Let $H_0 = -\Delta$ and let H_1 be the self-adjoint operator $H_1 = H_0 + V$ where V satisfies the Enss condition. Then*

1. *H_1 has empty singular continuous spectum ;*
2. *The wave operator $W_\pm(H_1, H_0)$ are complete ;*
3. *Eigenvalues of H_1 can accumulate only at 0 and nonzero eigenvalues have finite multiplicity.*

Due to the assumption (**??**) of V_1 and similar to Proposition 2.2.10, V_1 satisfies Enss condition and then by Theorem 2.2.13 the wave operators of H_1 are complete. So we can define the the scattering operator in the self-adjoint case for the pair (H_1, H_0) as

$$\widetilde{S}(H_1, H_0) = W_+(H_1, H_0)^* W_-(H_1, H_0). \tag{2.10}$$

For the dissipative Schrödinger operator, an important problem is the invertibility of the dissipative scattering operator $S(H, H_0)$ on L^2. The relation between the invertibility of $S(H, H_0)$ and the asymptotic completeness of $W_-(H, H_0)$ is described by the next theorem.

Theorem 2.2.14 (Theorem 7 [14]). *Assume that there exists a set $\mathcal{D}$ dense in $L^2(\mathbb{R}^n)$ such that*

$$\int_0^\infty \|Ve^{itH_0}\phi\| dt < \infty \tag{2.11}$$

for all $\phi \in \mathcal{D}$. The following conditions are equivalent :

1. *The range of $W_-(H, H_0)$ is a closed subspace ;*
2. *The scattering operator $S(H, H_0) = W_+(H_0, H)W_-(H, H_0)$ is invertible on L^2.*

Lemma 2.2.15. *If V satisfies Enss conditions, then (2.11) holds for V.*

Démonstration. Denote $\mathcal{D} = \{\varphi \in H^2(\mathbb{R}^n) \cap C^\infty : \text{ spt } \hat{\varphi} \subset \mathbb{R}^n \setminus \{0\}$ is compact. $\}$. It is easy to check that $\mathcal{D}$ is dense in $L^2(\mathbb{R}^n)$. For each $\varphi \in \mathcal{D}$, there exists a constant $a > 0$ such that $\text{spt } \hat{\varphi} \subset \mathbb{R}^n \setminus D(0, a)$. For each $t \in \mathbb{R}_+$, and $x \in D(0, at)$,

$$\begin{aligned} (e^{itH_0}(H_0 - i)\varphi)(x) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{it|\xi|^2} \mathcal{F}((H_0 - i)\varphi) d\xi \\ &= \frac{e^{-\frac{i|x|^2}{4t}}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\frac{i|x+2t\xi|^2}{4t}} \mathcal{F}((H_0 - i)\varphi) d\xi \end{aligned}$$

Since $|x + 2t\xi| \geq 2t|\xi| - |x| \geq 2at - at = at$, then using the stationary phase method, one has

$$\begin{aligned} |(e^{itH_0}(H_0 - i)\varphi)(x)| &= |\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\theta\in\mathbf{S}^{n-1}} \int_a^\infty e^{-itr^2} \mathcal{F}((H_0 - i)\varphi)(r,\theta) r^{n-1} dr d\theta| \\ &= |\frac{1}{2(2\pi)^{\frac{n}{2}} t} \int_{\theta\in\mathbf{S}^{n-1}} \int_a^\infty \mathcal{F}((H_0 - i)\varphi)(r,\theta) r^{n-2} d e^{\frac{-ir^2}{4t}} d\theta| \\ &= |\frac{1}{2(2\pi)^{\frac{n}{2}} t} \int_{\theta\in\mathbf{S}^{n-1}} \int_a^\infty e^{\frac{-ir^2}{4t}} \partial_r(\mathcal{F}((H_0 - i)\varphi)(r,\theta) r^{n-2}) dr d\theta|. \end{aligned}$$

Repeating several times by the same way, one has

$$|(e^{itH_0}(H_0 - i)\varphi)(x)| \leq \frac{C}{(1+t)^{n+1}}$$

for some $C > 0$ depending on φ. Consequently,

$$\|F(|x| \leq at)(e^{itH_0}(H_0 - i)\varphi)(x)\| \leq \frac{C}{(1+t)^{\frac{n}{2}+1}},$$

and then

$$\int_0^\infty \|F(|x| \leq at)(e^{itH_0}(H_0 - i)\varphi)(x)\| dt < \infty.$$

So

$$\begin{aligned}\int_0^\infty \|Ve^{itH_0}\varphi\| dt &= \int_0^\infty \|V(H_0 - i)^{-1}(F(|x| \leq at) + F(|x| \geq at))e^{itH_0}(H_0 - i)\varphi\| dt \\ &\leq \|V(H_0 - i)^{-1}\| \int_0^\infty \|F(|x| \leq at)e^{itH_0}(H_0 - i)\varphi\| dt \\ &\quad + \int_0^\infty \|V(H_0 - i)^{-1}F(|x| \geq at)\| \|(H_0 - i)\varphi\| dt \\ &\leq C_\varphi.\end{aligned}$$

Here the second term is dominated by Enss condition. □

Therefore, by Proposition 2.2.10 and Lemma 2.2.15, (2.11) holds for V. So we only need to prove that Ran $W_-(H, H_0)$ is closed and then the invertibility of the scattering operator $S(H, H_0)$ is achieved. Below we will prove that the Riesz projection of all the eigenvalues is of finite rank, and then we hope that the conclusion which is similar to (2.9) for $W_-(H_1, H_0)$ in the self-adjoint case is also true.

Below we consider that the imaginary of H is small enough. That means $H = H(\varepsilon) = -\Delta + V_1 - i\varepsilon V_2$, where $\varepsilon > 0$ is small enough. And the resolvent $R(z)$ is replaced by $R(z, \varepsilon)$.

2.3 Proof of Theorem 2.1.1

In this section, we consider a simple case that 0 is a regular point of H_1, which means that 0 is neither eigenvalue nor resonance of H_1. Here we call 0 is a resonance if the equation $Hu = 0$ has a solution $u \in H^{1,-s} \backslash L^2$ for $s > 1$. Then by the assumption (**??**) and Weyl's theorem, one has $\sigma_{ess}(H_1) = \sigma_{ess}(H_0) = [0, \infty)$ and there may be some eigenvalues of H_1 on $(-\infty, 0)$. In particular, there is no positive eigenvalue(cf. [58]). If $\rho_0 > 2$ in (0.1), then eigenvalues of H_1 can not accumulate to 0(cf. [58]). And in [77] the author confirmed this conclusion for H. Throughout this work, we count the eigenvalues according to their algebraic multiplicity.

So there exists a constant $c_0 > 0$ such that H_1 has a finite number of eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_l < -c_0$. Let $N_1 = \sum_{j=1}^l n_j$, where n_j is the multiplicity of λ_j. Let Π_j be the projection of H_1 associated with the λ_j.Then we have the following lemma that the eigenvalues of $H(\varepsilon)$ are the perturbation to those of H_1. Below, we consider the eigenvalues of $H(\varepsilon)$ without multiplicity, which means that if λ is an eigenvalue of H with multiplicity m, then we call these are m eigenvalues of H.

Lemma 2.3.1. *For $\varepsilon > 0$ small enough, $N(\varepsilon)$ denote the number of eigenvalues of $H(\varepsilon)$, Then there exists some $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$,*

$$N(\varepsilon) = N_1.$$

More precisely, for each λ_j, there are n_j eigenvalues in

$$\mathcal{F}_j \triangleq \{z \in \mathbb{C} : -c\varepsilon \leq \Im z \leq -C\varepsilon, |\Re z - \lambda_j| \leq C\varepsilon\}, \tag{2.12}$$

for some $c, C > 0$. Let $\mathcal{F} \triangleq \bigcup_{j=1}^l \mathcal{F}_j$. One has

$$\|R(z, \varepsilon)\| \leq C_1 \varepsilon^{-1} \tag{2.13}$$

for some $C_1 > 0$ and $z \notin \mathcal{F}$ with $\Re z \leq -c_0$.

Démonstration. : We only need to consider the spectral of $H(\varepsilon)$ near λ_j. Let $\Pi'_j = 1 - \Pi_j$. Then $L^2(\mathbb{R}^n) = \text{Ran } \Pi_j \oplus \text{Ran } \Pi'_j$. We define

$$E_1(z) = (\Pi'_j H_1 \Pi'_j - z)^{-1}\Pi'_j = \Pi'_j(\Pi'_j H_1 \Pi'_j - z)^{-1}\Pi'_j.$$

Since that H_1 is a self-adjoint operator, for $|z - \lambda_j| > 0$ small enough, there exists a positive constant C_j depending on $|z - \lambda_j|$ such that

$$\|E_1(z)\| \le C_j.$$

Let $E(z,\varepsilon) = (\Pi'_j H(\varepsilon)\Pi'_j - z)^{-1}\Pi'_j$, Then

$$\begin{aligned} E(z,\varepsilon) - E_1(z) &= \Pi'_j((\Pi'_j H(\varepsilon)\Pi'_j - z)^{-1} - (\Pi'_j H_1 \Pi'_j - z)^{-1})\Pi'_j \\ &= i\varepsilon \Pi'_j(\Pi'_j H_1 \Pi'_j - z)^{-1}\Pi'_j V_2(x)\Pi'_j(\Pi'_j H(\varepsilon)\Pi'_j - z)^{-1}\Pi'_j \\ &= i\varepsilon E_1(z)V_2(x)E(z,\varepsilon). \end{aligned}$$

Thus

$$E(z,\varepsilon) = (1 - i\varepsilon V_2(x)E_1(z))^{-1}E_1(z)$$

So for $\varepsilon > 0$ and $|z - \lambda_j|$ small enough, $E(z,\varepsilon)$ is holomorphic and uniformly bounded in $\mathcal{L}(L^2)$.

Let $\{\varphi_k^{(j)}\}_{k=1}^{n_j}$ be a basis of Ran Π_j. We define the mapping $R_+ : L^2(\mathbb{R}^n) \to \mathbb{C}^{n_j}$ and $R_- : \mathbb{C}^{n_j} \to \text{Ran } \Pi_j$ by

$$R_+\varphi = \{< \varphi, \varphi_k^{(j)} >\}_{k=1}^{n_j},\ \forall \varphi \in L^2(\mathbb{R}^n);\ R_- a = \sum_{k=1}^{n_j} a_k \varphi_k^{(j)},\ \forall a = \{a_k\}_{k=1}^{n_j} \in \mathbb{C}^{n_j}.$$

Then they satisfy that $R_+R_- = Id_{\mathbb{C}^{n_j}}$ and $R_-R_+ = \Pi_j$. We can construct the Grushin problem :

$$\mathcal{P}(z,\varepsilon) = \begin{pmatrix} H(\varepsilon) - z & R_- \\ R_+ & 0 \end{pmatrix} : L^2(\mathbb{R}^n) \times \mathbb{C}^{n_j} \to L^2(\mathbb{R}^n) \times \mathbb{C}^{n_j}.$$

Thus we can find a approximate inverse matrix :

$$\mathcal{Q}(z,\varepsilon) = \begin{pmatrix} E(z,\varepsilon) & R_- \\ R_+ & R_-(H(\varepsilon) - z)R_+ \end{pmatrix}.$$

Then

$$\begin{aligned} \mathcal{P}(z,\varepsilon)\mathcal{Q}(z,\varepsilon) &= \begin{pmatrix} (H(\varepsilon) - z)E(z,\varepsilon) + \Pi_j & \Pi'_j(H(\varepsilon) - z)R_- \\ 0 & Id_{\mathbb{C}^{n_j}} \end{pmatrix} \\ &= \begin{pmatrix} (H(\varepsilon) - z)E(z,\varepsilon) - \Pi'_j & \Pi'_j(H(\varepsilon) - z)R_- \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &\triangleq \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Here

$$\begin{aligned} A &= (H(\varepsilon) - z)\Pi'_j(\Pi'_j H(\varepsilon)\Pi'_j - z)^{-1}\Pi'_j - \Pi'_j \\ &= (\Pi'_j H(\varepsilon)\Pi'_j - z)(\Pi'_j H(\varepsilon)\Pi'_j - z)^{-1}\Pi'_j + \Pi_j(H(\varepsilon) - z)(\Pi'_j H(\varepsilon)\Pi'_j - z)^{-1}\Pi'_j - \Pi'_j \\ &= \Pi_j(H(\varepsilon) - z)(\Pi'_j H(\varepsilon)\Pi'_j - z)^{-1}\Pi'_j \end{aligned}$$

So $A^2 = 0$. By Neumann series, $\mathcal{P}(z,\varepsilon)\mathcal{Q}(z,\varepsilon)$ is invertible, and the inverse matrix is

$$(\mathcal{P}(z,\varepsilon)\mathcal{Q}(z,\varepsilon))^{-1} = \begin{pmatrix} 1 - A & -B + AB \\ 0 & 1 \end{pmatrix}$$

That means

$$\begin{aligned}\mathcal{P}^{-1}(z,\varepsilon) &= \begin{pmatrix} E(z,\varepsilon) & R_- \\ R_+ & R_-(H(\varepsilon)-z)R_+ \end{pmatrix}\begin{pmatrix} 1-A & -B+AB \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} E(z,\varepsilon) & -E(z,\varepsilon)B+R_- \\ R_+(1-A) & R_+AB-R_+(H(\varepsilon)-z)R_- \end{pmatrix} \\ &= \begin{pmatrix} E(z,\varepsilon) & R_- - E(z,\varepsilon)(H(\varepsilon)-z)R_- \\ R_+ - R_+(H(\varepsilon)-z)E(z,\varepsilon) & R_+((H(\varepsilon)-z)E(z,\varepsilon)(H(\varepsilon)-z)-(H(\varepsilon)-z))R_- \end{pmatrix} \\ &\triangleq \begin{pmatrix} E(z,\varepsilon) & E_+(z,\varepsilon) \\ E_-(z,\varepsilon) & E_{-+}(z,\varepsilon) \end{pmatrix}.\end{aligned}$$

By $\mathcal{P}(z,\varepsilon)\mathcal{P}^{-1}(z,\varepsilon) = 1$ and $\mathcal{P}^{-1}(z,\varepsilon)\mathcal{P}(z,\varepsilon) = 1$, if $E_{-+}^{-1}(z,\varepsilon)$ exists, then we obtain the following expression of the resolvent

$$R(z,\varepsilon) = E(z,\varepsilon) - E_+(z,\varepsilon)E_{-+}^{-1}(z,\varepsilon)E_-(z,\varepsilon), \tag{2.14}$$

and $E_+(z,\varepsilon)$ and $E_-(z,\varepsilon)$ are holomorphic and uniformly bounded in $\mathcal{L}(L^2)$ for $0<\varepsilon<\varepsilon_0$ and $|z-\lambda_j|$ small enough. We can deduce that the eigenvalues of $H(\varepsilon)$ in a small neighborhood for λ_j coincide with the zeros of $F(z,\varepsilon) = \det E_{-+}(z,\varepsilon)$. It is easy to check that

$$\begin{aligned} E_{-+}(z,\varepsilon) &= R_+(z-\lambda_j+i\varepsilon V_2 + (\lambda_j - z - i\varepsilon V_2)E(z,\varepsilon)(\lambda_j - z - i\varepsilon V_2))R_- \\ &= (z-\lambda_j)Id_{\mathbb{C}^{n_j}} + i\varepsilon R_+V_2(x)R_- - \varepsilon^2 R_+V_2E(z,\varepsilon)V_2R_- \end{aligned}$$

Denote

$$E_{-+}^0 = (z-\lambda_j)Id_{\mathbb{C}^{n_j}} + i\varepsilon R_+V_2(x)R_- = i\varepsilon(\frac{z-\lambda_j}{i\varepsilon}Id_{\mathbb{C}^{n_j}} + R_+V_2(x)R_-),$$

and

$$F^0(z,\varepsilon) = \det E_{-+}^0(z,\varepsilon).$$

On the other hand, $R_+V_2(x)R_-$ is a positive definite matrix in $\mathbb{C}^{n_j}$, due to the assumptions of $V_2(x)$. Let $\mu_1^j,\dots,\mu_{n_j}^j \in \mathbb{R}_+$ be the eigenvalues of $R_+V_2(x)R_-$, then $F_0(z,\varepsilon)$ has n_j zeros

$$z_k^j = \lambda_j - i\varepsilon\mu_k^j, \text{ where } k=1,\dots,n_j.$$

Let $\lambda_j - i\varepsilon\mu_k^j$ be one of the zeros of $F_0(z,\varepsilon)$ with order p_0. For a appropriate a_k, C_1 and $C_2>0$,

$$|F_0(z,\varepsilon)| \geq C_1\varepsilon^{n_j},\ |F(z,\varepsilon)-F_0(z,\varepsilon)| \leq C_2\varepsilon^{n_j+1}$$

for $|z-\lambda_j+i\varepsilon\mu_k^j| = a_k\varepsilon$. For ε small enough, then we apply Rouché's theorem to conclude that there are also p_0 zeros of $F(z,\varepsilon)$ in $B(\lambda_j - i\varepsilon\mu_k^j, a_k\varepsilon) \subset \mathbb{C}_- \triangleq \{z\in\mathbb{C} : \Im z<0\}$. So $F(z,\varepsilon)$ at least has n_j zeros $z_1,\dots,z_{n_j}$ in $\mathcal{F}_j$ with $|z_k-\lambda_k| \leq c_k\varepsilon$ for $c_k>0$. Conversely, let z_0 is a zero of $F(z,\varepsilon)$ with multiplicity p in $\mathbb{C}_-$. Then it is easy to check that $-M\varepsilon \leq \Im z_0 < 0$ for some $M>0$ depending on V_2. By the same method, we can also get that there exist p zeros of $F_0(z,\varepsilon)$. This shows that $F(z,\varepsilon)$ has n_j zeros in $\mathbb{C}_-$.

From the proof, we know that for $\varepsilon>0$ small enough, there exists a constant $C>c>0$ such that the zeros of $F(z,\varepsilon)$ are all in $\mathcal{F}_j$ defined by (2.3.17). On the other hand, for $z\notin\mathcal{F}_j$, one has $|z-z_k^j| \geq c_1\varepsilon$ for some $c_1>0$ and the inverse of E_{-+}^0 by

$$(E_{-+}^0)^{-1} = \sum_{k=1}^{n_j} \frac{P_k^j}{z-z_k^j},$$

where P_k^j is the eigenprojection of E_{-+}^0 associated to μ_k^j. So

$$E_{-+}(z,\varepsilon) = E_{-+}^0(1-\varepsilon^2(E_{-+}^0)^{-1}R_+V_2E(z,\varepsilon)V_2R_-) = E_{-+}^0(1+O(\varepsilon)),$$

for $z \notin \mathcal{F}_j$. Then in light of Neumann's series, one can obtain that

$$E_{-+}^{-1}(z,\varepsilon) = (E_{-+}^0)^{-1} + O(1),$$

and

$$\|E_{-+}^{-1}(z,\varepsilon)\| \leq \frac{C}{\varepsilon}, \tag{2.15}$$

for $z \notin \mathcal{F}_j$ and some $C > 0$.

It follows $\|R(z,\varepsilon)\| \leq \frac{C}{\varepsilon}$, for $z \notin \mathcal{F}$ with $\Re z \leq c_0$. □

Using the property of eigenvalues, we can get the limiting absorption principle for the dissipative Schrödinger operator, which is a perturbation to the self-adjoint case.

Lemma 2.3.2. *Set $\Pi(\varepsilon) \triangleq \sum_{j=1}^m \Pi_j(\varepsilon)$ and $\Pi'(\varepsilon) \triangleq 1 - \Pi(\varepsilon)$, where $\Pi_j(\varepsilon)$ is the Riesz projection associated to the eigenvalues which are in $\mathcal{F}_j$. Then*

$$\|\Pi(\varepsilon)\| \leq C,$$

for some $C > 0$ and $\varepsilon > 0$ small enough. If 0 is neither an eigenvalue nor a resonance of H_1, then

$$R(\lambda + i0, \varepsilon) = \lim_{\mu \to 0_+} (H(\varepsilon) - (\lambda + i\mu))^{-1}$$

exists in $\mathcal{L}(0,s;0,-s)$ for $s > 1$ with the estimate

$$\|\langle x\rangle^{-s}\Pi'(\varepsilon)R(\lambda + i0,\varepsilon)\Pi'(\varepsilon)\langle x\rangle^{-s}\| \leq C_s\langle\lambda\rangle^{-\frac{1}{2}},\ \lambda \in \mathbb{R} \tag{2.16}$$

uniformly in ε. Here for $\lambda \in]-\infty, -c_0]$, $R(\lambda + i0,\varepsilon) = R(\lambda,\varepsilon)$.

Démonstration. Fixed a $\delta > 0$, there exists $\varepsilon_0 > 0$ small enough such that $\mathcal{F}_j \subset \{z : |z - \lambda_j| \leq \delta\}$ for $\varepsilon \in]0, \varepsilon_0]$. Then the Riesz projection associated to λ_j can be represented by

$$\Pi_j(\varepsilon) = \frac{1}{2\pi i}\oint_{|z-\lambda_j|=\delta} R(z,\varepsilon)dz. \tag{2.17}$$

By lemma 2.3.1 and the perturbation method, we can deduce that

$$\begin{aligned} \|R(z,\varepsilon)\| &= \|R_1(z)(1 + i\varepsilon V_2(x)R(z,\varepsilon))\| \\ &\leq \|R_1(z)\|(1 + \varepsilon\|V_2\|_{L^\infty}\|R(z,\varepsilon)\|) \\ &\leq \frac{C}{\delta}(1 + \varepsilon\frac{C}{\varepsilon}) \\ &\leq \frac{C(1+C)}{\delta} \\ &\triangleq C'\delta^{-1} \end{aligned}$$

for $z \in \{z : |z - \lambda_j| = \delta\}$. Therefore, together with (2.17), we have $\|\Pi_j(\varepsilon)\| \leq C_1$ for a constant $C_1 > 0$. Thus, $\Pi(\varepsilon)$ is a bounded operator on $L^2(\mathbb{R}^n)$.

Case $\lambda > -c_0$.

For the selfadjoint Schrödinger operator, fix a small constant $c > 0$ small enough and then for $\lambda \geq c$, it is proved in [60] that

$$\|\langle x\rangle^{-s}R(\lambda + i0,\varepsilon)\langle x\rangle^{-s}\| \leq C_{c,s}\langle\lambda\rangle^{-\frac{1}{2}}, \frac{\rho_0}{2} > s. \tag{2.18}$$

For $\lambda \in (-c, c)$, due to [76] and the assumption that 0 is a regular point of H_1, there is a constant $C > 0$ independent on λ such that

$$\|\langle x\rangle^{-s}R_1(\lambda + i0)\langle x\rangle^{-s}\| \leq C\langle\lambda\rangle^{-\frac{1}{2}},\ \frac{\rho_0}{2} > s,$$

where
$$R_1(\lambda+i0)=\lim_{\mu\to 0_+}(H_1-(\lambda+i\mu))^{-1}.$$
For $\lambda\in(-c_0,-c]$, it is easy to check that
$$\|\langle x\rangle^{-s}R_1(\lambda+i0)\langle x\rangle^{-s}\|\leq\|\langle x\rangle^{-s}\|_{L^\infty}\|R_1(\lambda+i0)\|\|\langle x\rangle^{-s}\|_{L^\infty}\leq C_c\langle\lambda\rangle^{-\frac{1}{2}}.$$
Therefore by the formula
$$\langle x\rangle^{-s}R(\lambda+i0,\varepsilon)\langle x\rangle^{-s}=(1-i\varepsilon\langle x\rangle^{-s}R_1(\lambda+i0)\langle x\rangle^{-s}\langle x\rangle^{2s}V_2(x))^{-1}\langle x\rangle^{-s}R_1(\lambda+i0)\langle x\rangle^{-s}$$
we have
$$\|\langle x\rangle^{-s}R(\lambda+i0,\varepsilon)\langle x\rangle^{-s}\|\leq C\langle\lambda\rangle^{-\frac{1}{2}},\ \lambda\in(-c_0,c),\frac{\rho_0}{2}>s, \tag{2.19}$$
for $\varepsilon>0$ small enough. Consequently, together with (2.3.23)and (2.3.24), we have that for $\lambda>-c_0$,
$$\|\langle x\rangle^{-s}R(\lambda+i0,\varepsilon)\langle x\rangle^{-s}\|\leq C\langle\lambda\rangle^{-\frac{1}{2}},\frac{\rho_0}{2}>s. \tag{2.20}$$
On the other hand, for $\lambda>-c_0$,
$$\begin{aligned}R(\lambda+i0,\varepsilon)\Pi_j(\varepsilon) &= \frac{R(\lambda+i0,\varepsilon)}{2\pi i}\oint_{|z-\lambda_j|=\delta}R(z,\varepsilon)dz\\ &= \frac{1}{2\pi i}\oint_{|z-\lambda_j|=\delta}\frac{R(\lambda+i0,\varepsilon)-R(z,\varepsilon)}{z-\lambda}dz\\ &= -\frac{1}{2\pi i}\oint_{|z-\lambda_j|=\delta}\frac{R(z,\varepsilon)}{z-\lambda}dz\end{aligned}$$
Note that if $\lambda>-c_0$, $|z-\lambda|$ has a positive lower bound for $|z-\lambda_j|=\delta$, which is independent on δ and ε, so
$$\|R(\lambda+i0,\varepsilon)\Pi_j(\varepsilon)\|\leq\frac{1}{2\pi i}\oint_{|z-\lambda_j|=\delta}\frac{\|R(z,\varepsilon)\|}{|z-\lambda|}|dz|\leq C\oint_{|z-\lambda_j|=\delta}\delta^{-1}|dz|\leq C'.$$
Since
$$\Pi'(\varepsilon)R(\lambda+i0,\varepsilon)\Pi'(\varepsilon)=R(\lambda+i0,\varepsilon)-R(\lambda+i0,\varepsilon)\Pi(\varepsilon)$$
and together with (2.3.23), (2.3.21) is true for $\lambda>-c_0$.

Case $\lambda\leq -c_0$.

First,
$$\begin{aligned}\Pi_j(\varepsilon)-\Pi_j &= -\frac{1}{2\pi i}\oint_{|z-\lambda_j|=\delta}R(z,\varepsilon)-R_1(z)dz\\ &= \frac{1}{2\pi i}\oint_{|z-\lambda_j|=\delta}i\varepsilon R(z,\varepsilon)V_2R_1(z)dz\\ &= \frac{\varepsilon}{2\pi}\oint_{|z-\lambda_j|=\delta}R(z,\varepsilon)V_2R_1(z)dz\end{aligned}$$
Let $\Pi_{ac}=1-\sum_{j=1}^m\Pi_j$ be the projection of the absolutely continuous spectrum of H_1, $S_j(\varepsilon)\triangleq\varepsilon^{-1}(\Pi_j(\varepsilon)-\Pi_j)$ and $S(\varepsilon)\triangleq\sum_{j=1}^m S_j(\varepsilon)$. Then $S(\varepsilon)=-\varepsilon^{-1}(\Pi'(\varepsilon)-\Pi_{ac})\in\mathcal{L}(L^2)$ is uniformly bounded and
$$\|S_j(\varepsilon)\|\leq\frac{1}{2\pi}\oint_{|z-\lambda_j|=\delta}\|R(z,\varepsilon)\|\|V_2\|_{L^\infty}\|R_1(z)\||dz|\leq C\delta^{-1},\ s>1$$

Since H_1 is a self-adjoint operator and the spectral of H_1 on $\Pi_{ac}L^2(\mathbb{R}^n)$ is $[0,+\infty[$, then

$$\|R_1(\lambda)\Pi_{ac}\| \leq \langle\lambda\rangle^{-1},\ \lambda \leq -c_0.$$

Due to the identity

$$R(\lambda,\varepsilon)\Pi'(\varepsilon) = -\varepsilon R(\lambda,\varepsilon)S(\varepsilon) + (1 + i\varepsilon R(\lambda,\varepsilon)V_2)R_1(\lambda)\Pi_{ac}$$

then for $\lambda \leq -c_0$, we have

$$\begin{aligned} \|\langle x\rangle^{-s}\Pi'(\varepsilon)R(\lambda,\varepsilon)\Pi'(\varepsilon)\langle x\rangle^{-s}\| &\leq \|\langle x\rangle^{-s}\|_{L^\infty}\|\varepsilon R(\lambda,\varepsilon)\|\|S(\varepsilon)\|\|\langle x\rangle^{-s}\|_{L^\infty} \\ &+ \|\langle x\rangle^{-s}\|_{L^\infty}(1 + \|\varepsilon R(\lambda,\varepsilon)\|\|V_2\|_{L^\infty})\|R_1(\lambda)\Pi_{ac}\|\|\langle x\rangle^{-s}\|_{L^\infty} \\ &\leq C_\delta\langle\lambda\rangle^{-1}. \end{aligned}$$

for $s > 1$. □

For our main theorem, we need a Kato's smoothness estimate(cf.[40]) for the semigroup of contractions about non-selfadjoint operators. Let $\mathcal{H}$ be a Hilbert space, and H is a maximally dissipative operator on $\mathcal{H}$. $-iH$ is generator of a semigroup of contraction $S(t) = e^{-itH}$, $t > 0$. According to the theory of Foiaş-Sz.Nagy(cf. [22]), there is a Hilbert space $\mathcal{G} \supset \mathcal{H}$ and a unitary group $U(t) = e^{-itG}$ on $\mathcal{G}$ such that

$$\Pi U(t)|_{\mathcal{H}} = S(t), t \geq 0,$$

where Π is the projection from $\mathcal{G}$ to $\mathcal{H}$. Then G is called a selfadjoint dilation of H.

Lemma 2.3.3. *Assume that there exits $A \in \mathcal{L}(\mathcal{H})$ continuous such that*

$$\sup_{\lambda\in\mathbb{R},\delta\in]0,1]} \|A(H-(\lambda+i\delta))^{-1}A^*\| \leq \gamma,$$

then

$$\int_0^\infty (\|AS(t)f\|^2 + \|AS(t)^*f\|^2)dt \leq C_\gamma\|f\|^2,\ \forall f \in \mathcal{H}.$$

Démonstration. Let $(G,\mathcal{G})$ be a selfadjoint dilation of $(H,\mathcal{H})$, then

$$(H-z)^{-1} = \int_0^\infty e^{itz}S(t)dt = \int_0^\infty e^{itz}\Pi U(t)|_{\mathcal{H}}dt = \Pi(G-z)^{-1}|_{\mathcal{H}},$$

for $\Im z > 0$. By duality, we also have

$$(H^* - \bar{z})^{-1} = \Pi(G-\bar{z})^{-1}|_{\mathcal{H}}$$

Therefore

$$\|(A\Pi)(G-z)^{-1}(A\Pi)^*\| \leq \gamma,$$

for $z \in \{\lambda : 0 < |\Im z| \leq 1\}$. By classical Kato's smoothness estimate for the selfadjoint operators(cf. Lemma 3.6 and Theorem 5.1 in [40]),

$$\int_0^\infty \|(A\Pi)U(t)g\|^2dt \leq C\|g\|^2,\ \forall g \in \mathcal{G}.$$

with

$$C \triangleq \sup_{0<\Im z\leq 1} \|(A\Pi)((G-z)^{-1} - (G-\bar{z})^{-1})(A\Pi)^*\| \leq 2\gamma$$

For $g = f \in \mathcal{H}$, we have

$$\int_0^\infty \|AS(t)f\|^2dt \leq 2\gamma\|f\|^2.$$

Using the same method, one can consider $-H^*$ and then will obtain

$$\int_0^\infty \|AS(t)^*f\|^2dt \leq 2\gamma\|f\|^2.$$

□

In fact, using the high energy estimate in (2.2.21) and Proposition 2.2 of [78], one can obtain a slightly better smoothness estimate : $\forall s > 1$, $\exists\, C_s$ such that

$$\int_0^\infty \|\langle x\rangle^{-s}\langle D_x\rangle^{1/2}\Pi'(\epsilon)e^{-itH(\epsilon)}f\|^2\,dt \le C_s\|f\|^2, \quad \forall f \in L^2, \tag{2.21}$$

uniformly in $0 < \epsilon \le \epsilon_0$. Since 0 is a regular point of H_1, an estimate similar to (2.3.26) also holds for H_1 :

$$\int_{-\infty}^\infty \|\langle x\rangle^{-s}\Pi_{ac}e^{-itH_1}f\|^2\,dt \le C_s\|f\|^2, \quad \forall f \in L^2, s > 1. \tag{2.22}$$

1

Theorem 2.3.4. *Assume that 0 is neither an eigenvalue nor a resonance of H_1, $\rho > 2$ and $n \ge 3$. Then for $\varepsilon > 0$ small enough,*

$$\text{Ran } W_-(H(\varepsilon), H_0) = \text{Ran } \Pi'(\varepsilon),$$

Furthermore, Ran $W_-(H(\varepsilon), H_0)$ *is closed and then by Theorem 2.2.14, the dissipative scattering operator $S(H(\varepsilon), H_0)$ is bijective.*

Démonstration. Firstly, we claim that Ran $W_-(H(\varepsilon), H_0) \subset$ Ran $\Pi'(\varepsilon)$. Assume that λ is an eigenvalue of $H(\varepsilon)$ with $\Im\lambda < 0$ and $\Pi_\lambda(\varepsilon)$ is the Riesz projection associated to λ. Then there exists $k \in \mathbb{N}$ such that $(H(\varepsilon) - \lambda)^k\Pi_\lambda(\varepsilon) = 0$. Thus for $\varphi, \phi \in \mathcal{D}(H_0)$, one has

$$\langle \Pi_\lambda(\varepsilon)e^{-itH(\varepsilon)}\varphi, \phi\rangle = e^{-it\lambda}\sum_{j=0}^{k-1}\frac{(-it)^j}{j!}\langle \Pi_\lambda(\varepsilon)(H(\varepsilon)-\lambda)^j\varphi, \phi\rangle.$$

Hence,

$$\begin{aligned}
|\langle \Pi_\lambda(\varepsilon)W_-(H, H_0)\varphi, \phi\rangle| &= \lim_{t\to+\infty}|\langle \Pi_\lambda(\varepsilon)e^{-itH}e^{itH_0}\varphi, \phi\rangle| \\
&\le \lim_{t\to+\infty} e^{t\Im\lambda_j}\sum_{j=0}^{k-1}\frac{t^j}{j!}\|\Pi_\lambda(\varepsilon)(H(\varepsilon)-\lambda)^j\varphi\|\|\phi\| \\
&= 0
\end{aligned}$$

This means Ran $W_-(H(\varepsilon), H_0) \subset$ Ran $\Pi'(\varepsilon)$, and then

$$W_-(H(\varepsilon), H_0) = \Pi'(\varepsilon)W_-(H(\varepsilon), H_0) = \Pi'(\varepsilon)W_-(H(\varepsilon), H_1)W_-(H_1, H_0).$$

By Theorem XIII in [58], it is known that $W_-(H_1, H_0)$ is complete, i.e.

$$\text{Ran } W_-(H_1, H_0) = \text{Ran } \Pi_{ac}.$$

So

$$W_-(H(\varepsilon), H_0) = \Pi'(\varepsilon)W_-(H(\varepsilon), H_1)\Pi_{ac}W_-(H_1, H_0),$$

where Π_{ac} is the eigenprojection of the absolutely continuous spectrum of H_1. On the other hand,

$$\begin{aligned}
\frac{d}{dt}\langle \Pi'(\varepsilon)e^{-itH(\varepsilon)}e^{itH_1}\Pi_{ac}u, v\rangle &= \frac{d}{dt}\langle e^{itH_1}\Pi_{ac}u, e^{itH^*(\varepsilon)}\Pi'(\varepsilon)^*v\rangle \\
&= \langle iH_1e^{itH_1}\Pi_{ac}u, e^{itH^*(\varepsilon)}\Pi'(\varepsilon)^*v\rangle \\
&\quad +\langle e^{itH_1}\Pi_{ac}u, iH^*(\varepsilon)e^{itH^*(\varepsilon)}\Pi'(\varepsilon)^*v\rangle \\
&= i\langle \Pi'(\varepsilon)e^{-itH(\varepsilon)}(H_1 - H(\varepsilon))e^{itH_1}\Pi_{ac}u, v\rangle \\
&= -\varepsilon\langle \Pi'(\varepsilon)e^{-itH(\varepsilon)}V_2e^{itH_1}\Pi_{ac}u, v\rangle
\end{aligned}$$

for $\forall u, v \in \mathcal{S}$. Therefore

$$\langle \Pi'(\varepsilon)e^{-itH(\varepsilon)}e^{itH_1}\Pi_{ac}u, v\rangle = \langle \Pi'(\varepsilon)\Pi_{ac}u, v\rangle - \varepsilon \int_0^t \langle \Pi'(\varepsilon)e^{-isH}V_2 e^{isH_1}\Pi_{ac}u, v\rangle ds.$$

Since for $\frac{\rho_0}{2} \geq s > 1$ and by taking $A_1 = \langle x\rangle^{-s}\Pi_{ac}$ and $A_\varepsilon = \langle x\rangle^{-s}\Pi'(\varepsilon)$ in Lemma 2.3.3,

$$\begin{aligned}
|\int_0^t \langle \Pi'(\varepsilon)e^{-isH}V_2 e^{isH_1}\Pi_{ac}u, v\rangle ds| &= |\int_0^t \langle e^{-itH(\varepsilon)}\Pi'(\varepsilon)V_2\Pi_{ac}e^{itH_1}u, v\rangle dt| \\
&= |\int_0^t \langle \sqrt{V_2}\Pi_{ac}e^{itH_1}u, \sqrt{V_2}\Pi'(\varepsilon)e^{itH^*(\varepsilon)}v\rangle dt| \\
&\leq C\{\int_0^\infty \|\langle x\rangle^{-s}\Pi_{ac}e^{itH_1}u\|^2 dt\}^{\frac{1}{2}} \\
&\quad \cdot\{\int_0^\infty \|\langle x\rangle^{-s}\Pi'(\varepsilon)e^{itH^*(\varepsilon)}v\|^2 dt\}^{\frac{1}{2}} \\
&\leq C\|u\|\|v\|,
\end{aligned}$$

then by taking $t \to \infty$, one has

$$\Pi'(\varepsilon)W_-(H(\varepsilon), H_1)\Pi_{ac} = \Pi'(\varepsilon)\Pi_{ac} - \varepsilon K(\varepsilon),$$

where

$$K(\varepsilon) \triangleq \int_0^\infty e^{-itH(\varepsilon)}\Pi'(\varepsilon)V_2 e^{itH_1}\Pi_{ac}dt = \int_0^\infty e^{-itH(\varepsilon)}\Pi'(\varepsilon)V_2\Pi_{ac}e^{itH_1}dt \tag{2.23}$$

satisfying

$$|\langle K(\varepsilon)u, v\rangle| \leq C\|u\|\|v\|$$

uniformly in $\varepsilon > 0$ small enough. This means that $K(\varepsilon)$ is uniformly bounded in $\mathcal{L}(L^2(\mathbb{R}^n))$. Then one can check that

$$\Pi'(\varepsilon)\Pi_{ac} - \varepsilon K(\varepsilon) : \text{Ran } \Pi_{ac} \to \text{Ran } \Pi'(\varepsilon)$$

is bijective for $\varepsilon > 0$ small enough. In fact, we have

$$\Pi'(\varepsilon)\Pi_{ac}(H_1) - \varepsilon K(\varepsilon) = \Pi'(\varepsilon)\{1 + \varepsilon(S(\varepsilon) - K(\varepsilon))\} = \{1 - \varepsilon(S(\varepsilon) + K(\varepsilon))\}\Pi_{ac}(H_1).$$

For $\varepsilon > 0$ small enough, $1 + \varepsilon(S(\varepsilon) - K(\varepsilon))$ and $1 - \varepsilon(S(\varepsilon) + K(\varepsilon))$ are invertible in $L^2(\mathbb{R}^n)$. For $g \in$ Ran Π_{ac} such that $(\Pi'(\varepsilon)\Pi_{ac} - \varepsilon K(\varepsilon))g = 0$, then $\{1 - \varepsilon(S(\varepsilon) + K(\varepsilon))\}g = 0$. Thus $g = 0$. So $\Pi'(\varepsilon)\Pi_{ac} - \varepsilon K(\varepsilon)$ is an injection for $\varepsilon > 0$ small enough. On the other hand, for $f \in$ Ran $\Pi'(\varepsilon)$, set

$$g = f + \sum_{k=1}^\infty (-\varepsilon\Pi'(\varepsilon)(S(\varepsilon) - K(\varepsilon)))^k f,$$

where $\Pi_d(H_1) = 1 - \Pi_{ac}$. For $\varepsilon > 0$ small enough the series is convergent and one has $(\Pi'(\varepsilon)\Pi_{ac} - \varepsilon K(\varepsilon))g = f$. Thus $\Pi'(\varepsilon)\Pi_{ac} - \varepsilon K(\varepsilon)$ is a surjection. So it is a bijection from Ran Π_{ac} to Ran $\Pi'(\varepsilon)$. Then because of Ran $W_-(H_1, H_0) =$ Ran Π_{ac}, one obtain that Ran $\Pi'(\varepsilon) \subset$ Ran $W_-(H(\varepsilon), H_0)$. So we can deduce that Ran $W_-(H(\varepsilon), H_0) =$ Ran $\Pi'(\varepsilon)$.

By Lemma 2.2.14, $S(H(\varepsilon), H_0)$ is bijective if and only if Ran $W_-(H(\varepsilon), H_0)$ is closed. In our case, Ran $\Pi(\varepsilon)$ is of finite dimension and $\Pi(\varepsilon)$ is bounded on $L^2(\mathbb{R}^n)$, so there exists a set of functions $\{\varphi_j\}_{j=1}^{N_1}$ such that $\Pi(\varepsilon)f = \sum_{j=1}^{N_1} c_j(f)\varphi_j$, for $f \in L^2(\mathbb{R}^n)$, where c_j is a bounded operator on $L^2(\mathbb{R}^n)$. So we can find a dual basis $\{\phi_j\}_{j=1}^{N_1}$, such that $\Pi(\varepsilon) = \sum_{j=1}^{N_1}\langle\cdot, \phi_j\rangle\varphi_j$. It follows that $\Pi'(\varepsilon) = 1 - \sum_{j=1}^{N_1}\langle\cdot, \phi_j\rangle\varphi_j$ and then Ran $\Pi'(\varepsilon)$ is closed. So $S(H(\varepsilon), H_0)$ is bijective on $L^2(\mathbb{R}^n)$ for $\varepsilon > 0$ small enough.

□

3

Asymptotic Expansion in Time of the Solutions to Dissipative Schrödinger Equations

3.1 Main results

In this chapter, we consider the solution to the following Cauchy problem of the dissipative Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) = H(\varepsilon)u(t,x),\ t\geq 0,\ x\in\mathbb{R}^n,\ n\geq 3, \\ u(0,x)=u_0(x). \end{cases} \tag{3.1}$$

By the assumption (1.2), we know that $H(\varepsilon)$ is maximally dissipative with domain $D(H(\varepsilon)) = H^2(\mathbb{R}^n)$. In this case, $\mathbb{C}_+ = \{z\in\mathbb{C} : \Im z > 0\}$ is included in the resolvent set $\rho(H(\varepsilon))$ and $H(\varepsilon)$ generates a contraction semigroup $U(t,\varepsilon) = e^{-itH(\varepsilon)}$ on L^2. Thus the solution of (3.1) can be expressed by $u(t,x) = U(t,\varepsilon)u_0(x)$. The main task in this chapter is to get the asymptotic expansion of $U(t,\varepsilon)$ in $vL(0,s;0,-s)$, $s>1$ large enough as t tends to infinity, i.e. Theorem 1.4.1, Theorem 1.4.5 and Theorem 1.4.7.

This chapter is organized as follows. In Section 3.2, we will recall some known results of the free resolvent and the distribution of the eigenvalues of $H(\varepsilon)$ which can be found in [77]. We first state the low spectral analysis of $H(\varepsilon)$ for the 3-dimensional case in Section 3.3 and then discuss the large-time expansion of the semigroup in Section 3.4. In particular, we will discuss some properties of the Riesz projection of $H(\varepsilon)$ associated with the eigenvalues near 0 in Section 3.3. At last, we will discuss the case that 0 is only a resonance but not an eigenvalue of H_1 for the dimension $n=4$ in Section 3.5 and the case that 0 is both a resonance and an eigenvalue of H_1 for the dimension $n=4$ in Section 3.6.

3.2 Preliminaries

In this section, we first recall some properties about the free resolvent $R_0(z) = (-\Delta - z)^{-1}$ which will be used later. It is well-known that $R_0(z)$ is a convolution operator from $H^{-1,s}(\mathbb{R}^n)$ to $H^{1,-s}(\mathbb{R}^n)$, $s>1$. Let $z^\gamma = |z|^\gamma e^{i\gamma \arg z}$ and $\ln z = \ln|z| + i\arg z$ with $\arg z \in]0,2\pi[$ for $\gamma\in]0,\infty[$. The convolution kernel is

$$K_3(x;z) = \frac{1}{4\pi}\frac{e^{iz^{\frac{1}{2}}|x|}}{|x|}$$

if $n = 3$ and

$$K_4(x;z) = \frac{iz^{\frac{1}{2}}}{8\pi|x|} H_1^{(1)}(z^{\frac{1}{2}}|x|)$$

if $n = 4$, where $H_1^{(1)}(\xi)$ is the first Hankel function. Then without proof, we present the following two lemmas about the expansions of the 3-dimensional and the 4-dimensional free resolvents near zero. Let $B(z_0, \delta) = \{z \in \mathbb{C} : |z - z_0| < \delta\}$. If no confusion is possible, we denote $\|\cdot\|$ the norm of functions on L^2 or the matrix norm on l^2 or the operator norm on L^2.

Lemma 3.2.1. *Let $s > N + \frac{1}{2}$ for $N \in \mathbb{N}$ and $\alpha \in]0, \min\{1, s - N - \frac{1}{2}\}[$. Then there exists $\delta > 0$ such that for $z \in B(0, \delta) \setminus \mathbb{R}_+$, we have the expansion of the 3-dimensional free resolvent in $\mathcal{L}(-1, s; 1, -s)$*

$$R_0(z) = \sum_{j=0}^{N} z^{\frac{j}{2}} G_j + G_{N+\alpha}(z),$$

where each G_j is a Hilbert-Schmidt convolution operator in $\mathcal{L}(-1, s_j; 1, -s_j)$, $s_j > \max\{1, j + \frac{1}{2}\}$ with kernel $\frac{i^j|x|^{j-1}}{4\pi j!}$ and $G_{N+\alpha}(z)$ is a C^N operator-value function of z from $B(0, \delta)\backslash\mathbb{R}_+$ to $\mathcal{L}(-1, s; 1, -s)$. More precisely, G_j is a finite-rank operator for j odd. Moreover, we have the estimates for $\alpha =]0, \min\{1, s - N - \frac{1}{2}\}[$

$$\|\langle x\rangle^{-s} \frac{d^k}{dz^k} G_{N+\alpha}(z) \langle x\rangle^{-s}\| \leq C|z|^{\frac{N+\alpha}{2} - k}, \; k = 0, 1, \ldots, N.$$

Lemma 3.2.2. *Let $s > 2N + 1$ for $N \in \mathbb{N}$ and $\alpha \in]0, \min\{1, \frac{s}{2} - N - \frac{1}{2}\}[$. Then there exists $\delta > 0$ such that for $z \in B(0, \delta) \setminus \mathbb{R}_+$, we have the asymptotic expansion of the 4-dimensional free resolvent in $\mathcal{L}(-1, s; 1, -s)$*

$$R_0(z) = G_0 + \sum_{k=0}^{1} \ln^k z \sum_{j=1}^{N} z^j G_j^k + G_{N+\alpha}(z),$$

where $G_0 \in \mathcal{L}(-1, s_0; 1, -s_0)$ and all $G_j^k \in \mathcal{L}(-1, s_j; 1, -s_j)$ are Hilbert-Schmidt convolution operators for $s_j > 2j + 1$ and $G_{N+\alpha}(z)$ is a C^{2N} operator-value function of z from $B(0, \delta) \setminus \mathbb{R}_+$ to $\mathcal{L}(-1, s; 1, -s)$. In particular, each G_j^1 is of finite rank for $j = 1, \ldots, N$. Moreover, one has the estimates

$$\|\langle x\rangle^{-s} \frac{d^k}{dz^k} G_{N+\alpha}(z) \langle x\rangle^{-s}\| \leq C|z|^{N+\alpha-k}, \; k = 0, 1, \ldots, 2N.$$

These results can be found in lots of works (see [31],[33],[75],[76],[84]). From these two lemmas, it is easy to check that

$$\mathcal{M} = \{\phi \in H^{1,-s} : (1 + G_0 V_1)\phi = 0, \text{ for any } s > \frac{1}{2}\} \tag{3.2}$$

both for the 3-dimensional case and the 4-dimensional case.

Definition 3.2.3. *If $\dim \mathcal{M} = 0$, then we call that zero is a regular point of H_1. Otherwise, zero is an exceptional point of H_1. Furthermore, if $\mathcal{M}_1 \neq \emptyset$ and $\mathcal{M}_2 = \emptyset$, zero is said to be an exceptional point of the first kind. If $\mathcal{M}_1 = \emptyset$ and $\mathcal{M}_2 \neq \emptyset$, then zero is said to be an exceptional point of the second kind. And if $\mathcal{M}_1 \neq \emptyset$ and $\mathcal{M}_2 \neq \emptyset$, then zero is said to be an exceptional point of the third kind.*

And then we list some properties of the functions in $\mathcal{M}$.

Lemma 3.2.4. *(a). If n=3, then for any $\phi \in \mathcal{M}$ and ϕ_1, $\phi_2 \in \mathcal{M} \cap L^2$, we have*

$$G_1 V_1 \phi = \frac{i}{4\pi} \langle V_1 \phi, 1\rangle \begin{cases} = 0, & \text{if } \phi \in L^2, \\ \neq 0, & \text{if } \phi \notin L^2. \end{cases} \tag{3.3}$$

$$\langle G_2 V_1 \phi_1, V_1 \phi_2\rangle = \langle \phi_1, \phi_2\rangle. \tag{3.4}$$

(b). If n=4, then for any $\phi \in \mathcal{M}$, we have

$$G_1^1 V_1 \phi = -\frac{1}{(4\pi)^2}\langle V_1\phi, 1\rangle \begin{cases} = 0, & \text{if } \phi \in L^2, \\ \neq 0, & \text{if } \phi \notin L^2, \end{cases} \tag{3.5}$$

Then we apply the Grushin method to analyze the discrete spectrum of $H(\varepsilon)$ near 0 which is valid for all dimension $n \geq 3$ (see [77]). Because $G_0 V_1$ is compact in $\mathcal{L}(1,-s;1,-s)$ for $\rho_0 > 2$ and $s \in]1, \rho_0 - 1[$, $\mathcal{M}$ is a finite dimensional space and we denote $m = \dim \mathcal{M} < \infty$. Moreover, it can be check that the form

$$\mathcal{M} \times \mathcal{M} \ni (\phi, \varphi) \to \langle \phi, -V_1 \varphi \rangle,$$

is positive definite. Then by the Gram-Schmidt process, one can choose a basis $\{\phi_j\}_{j=1}^m$ of $\mathcal{M}$ such that

$$\langle \phi_j, -V_1\phi_k\rangle = \delta_{jk} = \begin{cases} 1 & , j = k \\ 0 & , j \neq k \end{cases} . \tag{3.6}$$

Let Q be the projection from $H^{1,-s}$ to $\mathcal{M}$ such that for $\phi \in H^{1,-s}$,

$$Q\phi = \sum_{j=1}^{m} \langle \phi, -V_1\phi_j\rangle \phi_j$$

and let $Q' = 1 - Q$. In [76], the author gave a proposition of the projection Q as follows.

Proposition 3.2.5. *(1). For $s > 1$, one has the decomposition*

$$H^{1,-s} = \mathcal{M} \oplus \text{Ran }(1 + G_0 V_1).$$

Q is the projection from $H^{1,-s}$ onto $\mathcal{M}$ with Ker Q = Ran $(1 + G_0V_1)$.

(2). $Q'(1 + G_0V_1)Q'$ is invertible on the range of Q' and $(Q'(1 + G_0V_1)Q')^{-1}Q' \in \mathcal{L}(1,-s;1,-s)$ for $s > 1$.

Let $R(z,\varepsilon) = (H(\varepsilon) - z)^{-1}$, for $z \notin \sigma(H(\varepsilon))$. Because of

$$R(z,\varepsilon) = (1 + R_0(z)(V_1 - i\varepsilon V_2))^{-1}R_0(z), \tag{3.7}$$

we have that the eigenvalues of $H(\varepsilon)$ coincide with the poles of $z \to W(z,\varepsilon)^{-1} = (1 + R_0(z)(V_1 - i\varepsilon V_2))^{-1}$ in $\mathcal{L}(1,-s;1,-s)$, $s > 1$. By an argument of perturbation and Proposition 3.2.5, one can prove that for δ and ε sufficiently small, $(Q'W(z,\varepsilon)Q')^{-1}Q'$ exists on $H^{1,-s}$ for $z \in B(0,\delta) \setminus \mathbb{R}_+$. One can construct the Grushin problem as follows.

For $s > 1$, let

$$\mathcal{W}(z,\varepsilon) = \begin{pmatrix} W(z,\varepsilon) & T \\ S & 0 \end{pmatrix} : H^{1,-s} \times \mathbb{C}^m \to H^{1,-s} \times \mathbb{C}^m,$$

where $T : \mathbb{C}^m \to \mathcal{M}$ and $S : H^{1,-s} \to \mathbb{C}^m$ are defined as

$$Tc = \sum_{j=1}^{m} c_j\phi_j,\ c = (c_1, \ldots, c_m) \in \mathbb{C}^m,$$

$$S\phi = (\langle \phi, -V_1\phi_1\rangle, \ldots, \langle \phi, -V_1\phi_m\rangle),\ \phi \in H^{1,-s}.$$

It is easy to check that

$$TS = Q,\ ST = \text{Id}_{\mathbb{C}^m}.$$

Moreover the inverse of $\mathcal{W}(z,\varepsilon)$ is given by

$$\mathcal{E}(z,\varepsilon) = \mathcal{W}^{-1}(z,\varepsilon) = \begin{pmatrix} E(z,\varepsilon) & E_+(z,\varepsilon) \\ E_-(z,\varepsilon) & E_{-+}(z,\varepsilon) \end{pmatrix},$$

where

$$\begin{aligned}
E(z,\varepsilon) &= (Q'W(z,\varepsilon)Q')^{-1}Q', \\
E_+(z,\varepsilon) &= T - E(z,\varepsilon)W(z,\varepsilon)T, \qquad (3.8)\\
E_-(z,\varepsilon) &= S - SW(z,\varepsilon)E(z,\varepsilon), \qquad (3.9)\\
E_{-+}(z,\varepsilon) &= -SW(z,\varepsilon)T + SW(z,\varepsilon)E(z,\varepsilon)W(z,\varepsilon)T.
\end{aligned}$$

Thus it is easy to verify that

$$W(z,\varepsilon)^{-1} = E(z,\varepsilon) - E_+(z,\varepsilon)E_{-+}(z,\varepsilon)^{-1}E_-(z,\varepsilon) \qquad (3.10)$$

on $H^{1,-s}$. Moreover, $E(z,\varepsilon)$ and $E_\beta(z,\varepsilon)$ are holomorphic and uniformly bounded for $z \in B(0,\delta) \setminus \mathbb{R}_+$ and $\varepsilon > 0$ small enough, where β is one of $-$, $+$ and $-+$. In particular, $E_{-+}(z,\varepsilon)$ is an $m \times m$ matrix with the representation

$$(E_{-+}(z,\varepsilon))_{kj} = \langle (-W(z,\varepsilon) + W(z,\varepsilon)E(z,\varepsilon)W(z,\varepsilon))\phi_j, -V_1\phi_k\rangle. \qquad (3.11)$$

Thus by (3.10), to get the expansion of the resolvent near zero, it is sufficient to discuss the inverse of $E_{-+}(z,\varepsilon)$. Let $F(z,\varepsilon) = \det E_{-+}(z,\varepsilon)$. Then z_0 is a pole of $W(z,\varepsilon)^{-1}$ if and only if $F(z_0,\varepsilon) = 0$.

In [77], the distribution of the eigenvalues of $H(\varepsilon)$ under assumption (1.2) has been proved for $\varepsilon > 0$ sufficiently small. First since $\rho_0 > 2$, there are only finite number of eigenvalues of H_1 on $]-\infty, 0[$ denoted by $\mu_1 < \ldots < \mu_l < 0$ (see Theorem XIII.6 in [58]). Let μ_j be of the multiplicity n_j for $j = 1, \ldots, l$. It was proved in [77] that for $\varepsilon > 0$ small enough, $H(\varepsilon)$ has n_j eigenvalues located in the domain $\{z \in \mathbb{C} : |z - \mu_j| < C\varepsilon, -C\varepsilon < \Im z < -c\varepsilon\}$ for some $0 < c < C$. On the other hand, if 0 is an exceptional point which means that 0 is an eigenvalue or a resonance of H_1, the distribution of the eigenvalues near 0 of $H(\varepsilon)$ was provided by the following proposition.

Proposition 3.2.6 (Theorem 3.2. in [77]). *Suppose* $\rho_0 > 4$.
(a). If zero is an eigenvalue of multiplicity m*, but not a resonance of* H_1*, then there exist* δ*,* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$*,* $H(\varepsilon)$ *has* m *eigenvalues in* $B_-(0,\delta) \triangleq B(0,\delta) \cup \mathbb{C}_-$.
(b). If $n = 4$ *and zero is a resonance, but not an eigenvalue of* H_1*, then there exist* δ*,* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$*,* $H(\varepsilon)$ *has one eigenvalue in* $B_-(0,\delta)$.

This theorem covers the first two cases we consider and for the third case we will give a proof of the distribution of the eigenvalues of $H(\varepsilon)$ in Section 6.

It is permitted in [77] that the real part of potential function has a critical decay rate $O(|x|^{-2})$ for $|x|$ sufficiently large and it includes the cases we consider here. It is proved that each eigenvalues of $H(\varepsilon)$ has negative imaginary part $-c\varepsilon + o(\varepsilon)$ for some $c > 0$ if 0 is only an eigenvalue but not a resonance of H_1. But if zero is a resonance of H_1, it can only include the 4-dimensional case but it is invalid for 3-dimensional case.

Due to (3.10), we divide $R(z,\varepsilon)$ into two parts as follows

$$\begin{aligned}
R_I(z,\varepsilon) &= E(z,\varepsilon)R_0(z), \\
R_{II}(z,\varepsilon) &= \widetilde{E}(z,\varepsilon)R_0(z),
\end{aligned}$$

where $\widetilde{E}(z,\varepsilon) = -E_+(z,\varepsilon)E_{-+}(z,\varepsilon)^{-1}E_-(z,\varepsilon)$. As we presented below, for $\delta > 0$ and $\varepsilon \in]0,\varepsilon_0]$ small enough, $R_I(z,\varepsilon)$ is uniformly bounded analytic operator in $\Omega = B(0,2\delta) \setminus \mathbb{R}_+$ and $R_{II}(z,\varepsilon)$ is of finite rank in $\mathcal{L}(-1,s;1,-s)$ for $s > 1$ and any fixed $z \in \Omega \cap \rho(H(\varepsilon))$.

3.3 Analysis of the resolvent in dimension three

In this section, we will discuss the asymptotic behavior of the resolvent near $z = 0$ under the assumption that $n = 3$ and that 0 is only an eigenvalue but not a resonance of H_1.

Firstly, we consider the expansion of $R_I(z, \varepsilon)$ for $z \in \Omega$. Under the assumption of Theorem 1.4.1, the expansion of $W(z, \varepsilon)$ in $\mathcal{L}(1, -s; 1, -s)$ for $s \in]N + \frac{1}{2}, \frac{\rho_0}{2}]$ has the form

$$W(z,\varepsilon) = (1 + G_0V_1) - i\varepsilon G_0V_2 + \sum_{j=1}^{N} z^{\frac{j}{2}} G_j(V_1 - i\varepsilon V_2) + G_{N+\alpha}(z)(V_1 - i\varepsilon V_2), \tag{3.12}$$

for $z \in \Omega$. On the other hand,

$$\begin{aligned} E(z,\varepsilon) &= (Q'W(z,\varepsilon)Q')^{-1}Q' \\ &= (Q'(1 + G_0V_1 - i\varepsilon G_0V_2 + N(z,\varepsilon))Q')^{-1}Q' \\ &= (1 - i\varepsilon E(0)G_0V_2Q' + E(0)N(z,\varepsilon)Q')^{-1}E(0) \\ &= E(0) + \sum_{l=1}^{\infty}(-1)^{l+1}(-i\varepsilon E(0)G_0V_2 + E(0)N(z,\varepsilon))^l E(0) \\ &= E(0) + \varepsilon N_1(\varepsilon) + N_2(z,\varepsilon) \end{aligned} \tag{3.13}$$

where $E(0) = (Q'(1 + G_0V_1)Q')^{-1}Q'$, $N_1(\varepsilon) = \sum_{l=1}^{\infty} \varepsilon^{l-1}(E(0)G_0V_2)^l E(0)$ and

$$N(z,\varepsilon) = O(|z|^{\frac{1}{2}}), N_2(z,\varepsilon) = z^{\frac{1}{2}}E^1(\varepsilon) + z^1E^2(\varepsilon) + z^{\frac{3}{2}}E^3(\varepsilon) + O(|z|^{\alpha})$$

are analytic in $D(0, \delta) \setminus \mathbb{R}_+$. Thusl with the help of an argument of perturbation, for δ, ε_0 small enough and $s \in]N + \frac{1}{2}, \frac{\rho_0}{2}]$, we have that

$$E(z,\varepsilon) = \sum_{j=0}^{N} z^{\frac{j}{2}} E_j(\varepsilon) + E_{N+\alpha}(z,\varepsilon),\ z \in \Omega,\ \varepsilon \in]0, \varepsilon_0], \tag{3.14}$$

where

$$\begin{aligned} E_0(\varepsilon) &= (Q'(1 + G_0V_1 - i\varepsilon G_0V_2)Q')^{-1}Q' = (Q'(1 + G_0V_1)Q')^{-1}Q' + O(\varepsilon), \\ E_1(\varepsilon) &= -E_0(\varepsilon)G_1(V_1 - i\varepsilon V_2)E_0(\varepsilon), \end{aligned}$$

and other terms can be also computed explicitly in $\mathcal{L}(1, -s; 1, -s)$. In fact, $E_j(\varepsilon)$ is a uniformly bounded operator on ε in $\mathcal{L}(1, -s_j; 1, -s_j)$ for $s_0 > 1$ and $s_j > j + \frac{1}{2}$, $j = 1, \ldots, N$. Furthermore, the remainder $E_{N+\alpha}(z, \varepsilon)$ is a uniformly bounded operator on $\varepsilon \in]0, \varepsilon_0]$ and $z \in \Omega$ in $\mathcal{L}(1, -s; 1, -s)$ satisfying that

$$\|\langle x\rangle^{-s} \frac{d^k}{dz^k} E_{N+\alpha}(z,\varepsilon)\langle x\rangle^{s}\| \le C|z|^{\frac{N+\alpha}{2}-k},\ k = 0, 1, \ldots, N.$$

In particular, one can see that $E_j(\varepsilon)$ is of finite rank for j odd.

Lemma 3.3.1. *Under the assumption of Theorem 1.4.1, for $z \in \Omega$, then we have the following expansion*

$$R_I(z,\varepsilon) = \sum_{j=0}^{N} z^{\frac{j}{2}} R_{1,j}(\varepsilon) + R_{1,N+\alpha}(z,\varepsilon),$$

where $R_{1,j} = \sum_{k=0}^{j} E_k(\varepsilon)G_{j-k} \in \mathcal{L}(-1, s_j; 1, -s_j)$ for $s_0 > 1$ and $s_j > j + \frac{1}{2}$, $j = 1, \ldots, N$. The remainder $R_{1,N+\alpha} \in \mathcal{L}(-1, s; 1, -s)$ satisfies that

$$\|\langle x\rangle^{-s} \frac{d^k}{dz^k} R_{1,N+\alpha}(z,\varepsilon)\langle x\rangle^{-s}\| \le C|z|^{\frac{N+\alpha}{2}-k},\ k = 0, 1, \ldots, N.$$

Furthermore $R_{1,j}(\varepsilon)$ is of finite rank for odd j.

Suppose that ε_0 and δ are small enough such that the expansion of free resolvent in Lemma 3.2.1 and the assumptions of Proposition 3.2.6 (a) are valid. Then we call $\Omega_1 \triangleq B(0,2\delta) \setminus (B(0,c_1\varepsilon) \cup \mathbb{R}_+)$ the intermediate energy part and $\tilde{\Omega}_2 \triangleq B(0,2c_1\varepsilon) \setminus \mathbb{R}_+$ the small energy part. Here c_1 defined below is a constant such that all the eigenvalues near 0 are located in $\tilde{\Omega}_2$. In the next two parts of this section, we will discuss the expansion of $R_{II}(z,\varepsilon)$ in the intermediate and the small energy parts.

3.3.1 Intermediate energy part

This part would not determine the expansion of the semigroup, and it yields a term with any decay rate in the expansion of the semigroup for s large enough. Throughout this subsection, we suppose that $\varepsilon \in]0,\varepsilon_0]$, $z \in \Omega_1$ and that the assumption of Theorem 1.4.1 is valid. By (3.12) and Lemma 3.2.4, we have that for $z \in \Omega_1$

$$\begin{aligned} & \langle W(z,\varepsilon)\phi_j, V_1\phi_k\rangle \\ = \; & i\varepsilon\langle V_2\phi_j,\phi_k\rangle + z\langle \phi_j,\phi_k\rangle - i\varepsilon z\langle G_2V_2\phi_j, V_1\phi_k\rangle \\ & + \sum_{j=3}^{N} z^{\frac{j}{2}}\langle G_j(V_1-i\varepsilon V_2)\phi_j, V_1\phi_k\rangle + \langle G_{N+\alpha}(z)(V_1-i\varepsilon V_2)\phi_j, V_1\phi_k\rangle. \end{aligned}$$

Here we use the relations (3.2), (3.3) and (3.4). Then by (3.11) and (3.14) we have

$$\begin{aligned} E_{-+}(z,\varepsilon) &= E_{-+,0}(z,\varepsilon) + \varepsilon^2\hat{E}_{-+,0} + \varepsilon^2 z^{\frac{1}{2}}E_{-+,1}(\varepsilon) + \varepsilon z E_{-+,2}(\varepsilon) \\ &\quad + \sum_{j=3}^{2N} z^{\frac{j}{2}}E_{-+,j}(\varepsilon) + E_{-+,N+\alpha}(z,\varepsilon), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} (E_{-+,0})_{kj}(z,\varepsilon) &= i\varepsilon\langle V_2\phi_j,\phi_k\rangle + z\langle\phi_j,\phi_k\rangle, \\ (\hat{E}_{-+,0})_{kj}(\varepsilon) &= -\langle V_2E_0(\varepsilon)G_0V_2\phi_j,\phi_k\rangle, \\ (E_{-+,1})_{kj}(\varepsilon) &= -\langle (V_2E_1(\varepsilon)G_0V_2 + V_2E_0(\varepsilon)G_1V_2)\phi_j,\phi_k\rangle, \\ (E_{-+,2})_{kj}(\varepsilon) &= -i\langle G_2V_2\phi_j, V_1\phi_k\rangle + \langle (iG_2(V_1-i\varepsilon V_2)E_0(\varepsilon)G_0V_2 + \varepsilon G_0V_2E_2(\varepsilon)G_0V_2 \\ &\quad + iG_0V_2E_0(\varepsilon)G_2(V_1-i\varepsilon V_2) + \varepsilon G_0V_2E_1(\varepsilon)G_1V_2)\phi_j, V_1\phi_k\rangle, \\ (E_{-+,3})_{kj}(\varepsilon) &= \langle G_3(V_1-i\varepsilon V_2)\phi_j, V_1\phi_k\rangle + \varepsilon\langle (iG_3(V_1-i\varepsilon V_2)E_0(\varepsilon)G_0V_2 \\ &\quad + \varepsilon G_0V_2E_3(\varepsilon)G_0V_2 + iG_0V_2E_0(\varepsilon)G_3(V_1-i\varepsilon V_2) \\ &\quad + iG_2(V_1-i\varepsilon V_2)E_1(\varepsilon)G_0V_2 + iG_0V_2E_1(\varepsilon)G_2(V_1-i\varepsilon V_2) \\ &\quad + \varepsilon G_0V_2E_2(\varepsilon)G_1V_2 + iG_2(V_1-i\varepsilon V_2)E_0(\varepsilon)G_1V_2)\phi_j, V_1\phi_k\rangle, \end{aligned}$$

and other terms can be calculated directly. In particular $\hat{E}_{-+,0}(\varepsilon)$ and $E_{-+,j}(\varepsilon)$, $j=0,\ldots,N$ are uniformly bounded matrices on ε and $E_{-+,N+\alpha}(z,\varepsilon)$ satisfies that

$$\|\frac{d^k}{dz^k}E_{-+,N+\alpha}(z,\varepsilon)\| \le C|z|^{\frac{N+\alpha}{2}-k},\; k=0,1,\ldots,N.$$

It is obvious that $\mathcal{U} = (\langle\phi_j,\phi_k\rangle)_{1\le j,k\le m}$ and $\mathcal{V} = (\langle V_2\phi_j,\phi_k\rangle)_{1\le j,k\le m}$ are positive definite, because of the assumption on $V_2(x)$. It follows that there exist m zeros $\{-i\varepsilon\lambda_j\}_{j=1}^m$ of $F_0(z,\varepsilon) = \det E_{-+,0}(z,\varepsilon)$, where $0<\lambda_1\le\ldots\le\lambda_m$. Moreover, we have

$$E_{-+,0}(z,\varepsilon) = i\varepsilon\mathcal{A}^*((\mathcal{A}^*)^{-1}\mathcal{V}\mathcal{A}^{-1} + \frac{z}{i\varepsilon})\mathcal{A},$$

where $\mathcal{U} = \mathcal{A}^*\mathcal{A}$ and $\mathcal{A}$ is an invertible matrix. Let P_j be the eigenprojection of $(\mathcal{A}^*)^{-1}\mathcal{V}\mathcal{A}^{-1}$ corresponding to λ_j. Then one has

$$E_{-+,0}(z,\varepsilon)^{-1} = \sum_{j=1}^{m} \frac{\mathcal{A}^{-1}P_j(\mathcal{A}^*)^{-1}}{z + i\varepsilon\lambda_j}.$$

In [77], using the Rouché's Theorem, the author proved that there are m zeros $\{z_j(\varepsilon)\}_{j=1}^m$ of $F(z,\varepsilon) = \det E_{-+}(z,\varepsilon)$ satisfying that

$$|z_j(\varepsilon) + i\varepsilon\lambda_j| \leq c\varepsilon^{\frac{3}{2}},$$

for some $c > 0$. Set $c_1 = 2\lambda_m$ and then for $z \in \Omega_1$ we have

$$|z + i\varepsilon\lambda_j| \geq |z| - \varepsilon\lambda_j \geq \frac{1}{2}|z|,\ \varepsilon \leq \frac{1}{c_1}|z|.$$

It follows that $E_{-+,0}(z,\varepsilon)^{-1} = O(|z|^{-1})$. By these observations, we can prove the following lemma.

Lemma 3.3.2. *For $\rho_0 > 2N+1$, $s \in]N + \frac{1}{2}, \frac{\rho_0}{2}]$, $z \in \Omega_1$ and $\varepsilon \in]0, \varepsilon_0]$, we have the expansions of $E_{-+}(z,\varepsilon)^{-1}$ and its derivatives as follows*

$$\frac{d^k}{dz^k}E_{-+}(z,\varepsilon)^{-1} = \frac{d^k}{dz^k}E_{-+,0}(z,\varepsilon)^{-1} + A_k(z,\varepsilon) = (-1)^k k! \sum_{l=1}^{m} \frac{\mathcal{A}^{-1}P_l(\mathcal{A}^*)^{-1}}{(z + i\varepsilon\lambda_l)^{k+1}} + A_k(z,\varepsilon), \tag{3.16}$$

where $A_k(z,\varepsilon)$ is a matrix with $\|A_k(z,\varepsilon)\| = O(|z|^{-k-\frac{1}{2}})$, $k = 0, \ldots, N$.

Démonstration. Let $E'(z,\varepsilon) = E_{-+}(z,\varepsilon) - E_{-+,0}(z,\varepsilon)$. For $z \in \Omega_1$, we have

$$E_{-+}(z,\varepsilon) = E_{-+,0}(z,\varepsilon)(1 + E_{-+,0}(z,\varepsilon)^{-1}E'(z,\varepsilon)) = E_{-+,0}(z,\varepsilon)(1 + O(|z|^{\frac{1}{2}})),$$

and by Neumann's series one can check that for δ and ε_0 small enough $E_{-+}(z,\varepsilon)^{-1}$ exists with

$$\|E_{-+}(z,\varepsilon)^{-1}\| \leq C_{\delta,\varepsilon_0}\|E_{-+,0}(z,\varepsilon)^{-1}\| \leq O(|z|^{-1}).$$

Then we can obtain that

$$\begin{aligned} E_{-+}(z,\varepsilon)^{-1} &= (1 + E_{-+,0}(z,\varepsilon)^{-1}E'(z,\varepsilon))^{-1}E_{-+,0}(z,\varepsilon)^{-1} \\ &= E_{-+,0}(z,\varepsilon)^{-1} + A_0(z,\varepsilon), \end{aligned}$$

where

$$A_0(z,\varepsilon) = -E_{-+,0}(z,\varepsilon)^{-1}E'(z,\varepsilon)E_{-+}(z,\varepsilon)^{-1}.$$

It is easy to check that $\|A_0(z,\varepsilon)\| = O(|z|^{-\frac{1}{2}})$. On the other hand, it can be also checked that

$$\frac{d^j}{dz^j}E_{-+,0}(z,\varepsilon)^{-1} = (-1)^j j! \sum_{l=1}^{m} \frac{\mathcal{A}^{-1}P_l(\mathcal{A}^*)^{-1}}{(z + i\varepsilon\lambda_l)^{j+1}},$$

$$\frac{d^j}{dz^j}(E_{-+,0}(z,\varepsilon)^{-1}E'(z,\varepsilon)) = O(|z|^{-j+\frac{1}{2}}),$$

for $j \geq 1$. Therefore noting that

$$\begin{aligned} &\frac{d^k}{dz^k}E_{-+}(z,\varepsilon)^{-1} = (1 + E_{-+,0}(z,\varepsilon)^{-1}E'(z,\varepsilon))^{-1}\frac{d^k}{dz^k}E_{-+,0}(z,\varepsilon)^{-1} \\ &+ \sum_{j=1}^{k} \sum_{\substack{j_1+\ldots+j_p=j \\ j_q \geq 1, q=1,\ldots,p}} (c_{j_1,\ldots,j_p} \prod_{q=1}^{p}(1 + E_{-+,0}(z,\varepsilon)^{-1}E'(z,\varepsilon))^{-1}\frac{d^{j_q}}{dz^{j_q}}(E_{-+,0}(z,\varepsilon)^{-1}E'(z,\varepsilon))) \\ &\cdot(1 + E_{-+,0}(z,\varepsilon)^{-1}E'(z,\varepsilon))^{-1}\frac{d^{k-j}}{dz^{k-j}}E_{-+,0}(z,\varepsilon)^{-1}, \end{aligned}$$

we can obtain (3.16). □

Lemma 3.3.3. *For $\rho_0 > 2N+1$, $s \in]N+\frac{1}{2}, \frac{\rho_0}{2}]$, $z \in \Omega_1$ and $\varepsilon \in]0, \varepsilon_0]$, we have the expansions of $\widetilde{E}(z,\varepsilon)$ and its derivatives as follows*

$$\frac{d^k}{dz^k}\widetilde{E}(z,\varepsilon) = (-1)^{k+1}k!\sum_{l=1}^{m}\frac{T\mathcal{A}^{-1}P_l(\mathcal{A}^*)^{-1}S}{(z+i\varepsilon\lambda_l)^{k+1}} + \widetilde{E}_k(z,\varepsilon),$$

where $\widetilde{E}_k(z,\varepsilon) \in \mathcal{L}(1,-s;1,-s)$ with $\|\langle x\rangle^{-s}\widetilde{E}_k(z,\varepsilon)\langle x\rangle^{s}\| = O(|z|^{-k-\frac{1}{2}})$.

Démonstration. In light of (3.8) and (3.9), it is easy to check for $z \in \Omega$ that,

$$E_+(z,\varepsilon) = (1+\varepsilon E_{+,0}(\varepsilon)+\varepsilon z^{\frac{1}{2}}E_{+,1}(\varepsilon)+\sum_{j=2}^{N}z^{\frac{j}{2}}E_{+,j}(\varepsilon)+E_{+,N+\alpha}(z,\varepsilon))T, \tag{3.17}$$

$$E_-(z,\varepsilon) = S(1+\varepsilon E_{-,0}(\varepsilon)+\varepsilon z^{\frac{1}{2}}E_{-,1}(\varepsilon)+\sum_{j=2}^{N}z^{\frac{j}{2}}E_{-,j}(\varepsilon)+E_{-,N+\alpha}(z,\varepsilon)), \tag{3.18}$$

where

$$\begin{aligned}
E_{+,0}(\varepsilon) &= iE_0(\varepsilon)G_0V_2,\\
E_{+,1}(\varepsilon) &= iE_0(\varepsilon)G_1V_2 + iE_1(\varepsilon)G_0V_2,\\
E_{+,2}(\varepsilon) &= -E_0(\varepsilon)G_2(V_1-i\varepsilon V_2)+i\varepsilon E_2(\varepsilon)G_0V_2+i\varepsilon E_1(\varepsilon)G_1V_2,\\
E_{+,3}(\varepsilon) &= -E_0(\varepsilon)G_3(V_1-i\varepsilon V_2)+i\varepsilon E_3(\varepsilon)G_0V_2-E_1(\varepsilon)G_2(V_1-i\varepsilon V_2)\\
&\quad +i\varepsilon E_2(\varepsilon)G_1V_2,
\end{aligned}$$

and

$$\begin{aligned}
E_{-,0}(\varepsilon) &= iG_0V_2E_0(\varepsilon),\\
E_{-,1}(\varepsilon) &= iG_0V_2E_1(\varepsilon),\\
E_{-,2}(\varepsilon) &= i\varepsilon G_0V_2E_2(\varepsilon)-G_2(V_1-i\varepsilon V_2)E_0(\varepsilon),\\
E_{-,3}(\varepsilon) &= i\varepsilon G_0V_2E_3(\varepsilon)-G_3(V_1-i\varepsilon V_2)E_0(\varepsilon)-G_2(V_1-i\varepsilon V_2)E_1(\varepsilon).
\end{aligned}$$

By the same way, $E_{+,j}(\varepsilon)$ and $E_{-,j}(\varepsilon)$, $j=4,\dots,N$ can be calculated directly and the remainders satisfy

$$\|\langle x\rangle^{-s}\frac{d^k}{dz^k}E_{\pm,N+\alpha}(z,\varepsilon)\langle x\rangle^{s}\| \le C|z|^{\frac{N+\alpha}{2}-k},\ k=0,1,\dots,N.$$

Actually, one can check that $E_{+,j}(\varepsilon), E_{-,j}(\varepsilon) \in \mathcal{L}(1,-s_j;1,-s_j)$ are uniformly bounded operators for $s_0 > 1$ and $s_j > j+\frac{1}{2}$, $j=1,\dots,N$, and the remainders $E_{+,N+\alpha}(z,\varepsilon)$ and $E_{-,N+\alpha}(z,\varepsilon)$ are uniformly bounded in $\mathcal{L}(1,-s;1,-s)$. Since $\varepsilon \le O(|z|)$ for $z \in \Omega_1$, one has

$$\begin{aligned}
E_+(z,\varepsilon) &= (1+O(|z|))T : \mathbb{C}^m \to H^{1,-s},\\
E_-(z,\varepsilon) &= S(1+O(|z|)) : H^{1,-s} \to \mathbb{C}^m,
\end{aligned}$$

and the properties of their derivatives

$$\begin{aligned}
\|\langle x\rangle^{-s}\frac{d^k}{dz^k}E_+(z,\varepsilon)\|_{l^2\to L^2} &\le O(|z|^{-k+\frac{1}{2}}),\\
\|\frac{d^k}{dz^k}E_-(z,\varepsilon)\langle x\rangle^{s}\|_{L^2\to l^2} &\le O(|z|^{-k+\frac{1}{2}}),
\end{aligned}$$

for $k=1,\dots,N$. Consequently, noting that $\widetilde{E}(z,\varepsilon) = -E_+(z,\varepsilon)E_{-+}(z,\varepsilon)E_-(z,\varepsilon)$, we can complete the proof of lemma. □

By $R_{II}(z,\varepsilon)=\widetilde{E}(z,\varepsilon)R_0(z)$, we have the following lemma.

Lemma 3.3.4. *For $\rho_0>2N+1$, $s\in]N+\frac{1}{2},\frac{\rho_0}{2}]$, $z\in\Omega_1$ and $\varepsilon\in]0,\varepsilon_0]$, we have the expansions of $R_{II}(z,\varepsilon)$ and its derivatives as follows*

$$\frac{d^k}{dz^k}R_{II}(z,\varepsilon) \;=\; (-1)^{k+1}k!\sum_{l=1}^{m}\frac{T\mathcal{A}^{-1}P_l(\mathcal{A}^*)^{-1}SG_0}{(z+i\varepsilon\lambda_l)^{k+1}}+O(|z|^{-k-\frac{1}{2}}),$$

in $\mathcal{L}(-1,s;1,-s)$.

3.3.2 Small energy part

Since the zeros of $F_0(z,\varepsilon)$ are $\{-i\varepsilon\lambda_j\}_{j=1}^m$, one can choose a constant $c_2>0$ such that $z_j(\varepsilon)\in B(-i\varepsilon\lambda_j,c_2\varepsilon)\subset\mathbb{C}_-$ and $B(-i\varepsilon\lambda_j,c_2\varepsilon)\cap B(-i\varepsilon\lambda_k,c_2\varepsilon)=\emptyset$ for $\lambda_j\neq\lambda_k$, $j,k=1,\dots,m$. In this part, we want to discuss the expansion of $R_{II}(z,\varepsilon)$ in $\Omega_2\triangleq\tilde{\Omega}_2\setminus(\cup_{j=1}^m B(-i\varepsilon\lambda_j,c_2\varepsilon))$. Throughout this subsection, we always assume that $z\in\Omega_2$ and $\varepsilon\in]0,\varepsilon_0]$. Note that the expansions (3.14),(3.17),(3.18), (3.12), (3.15) of $E(z,\varepsilon)$, $E_\pm(z,\varepsilon)$, $W(z,\varepsilon)$ and $E_{-+}(z,\varepsilon)$ respectively are valid for $z\in\Omega_2\subset\Omega$. The object of this subsection is to prove the following lemma. Furthermore, some expressions of the terms will be given in the proof (see (3.23) and (3.23)).

Lemma 3.3.5. *Suppose that $3\le N\in\mathbb{N}$, $\rho_0>2N+1$ and $s\in]N+\frac{1}{2},\frac{\rho_0}{2}]$, $\alpha\in]0,\min\{1,s-N-\frac{1}{2}\}[$, $z\in\Omega_2$, $\varepsilon\in]0,\varepsilon_0]$. Then we have the following expansion*

$$\begin{aligned}\widetilde{E}(z,\varepsilon) \;=\;& \frac{1}{\varepsilon}W_0(\varepsilon)+zW_1(\varepsilon)+\sum_{j=2}^{N}\frac{z^{\frac{j}{2}}}{\varepsilon^{[\frac{j}{2}]+1}}W_j(\varepsilon)\\ &+\frac{1}{\varepsilon^{\frac{N+\alpha}{2}+\frac{1}{2}}}W_{1,N+\alpha}(z,\varepsilon)+\frac{1}{\varepsilon^{\frac{N+\alpha}{2}+1}}W_{2,N+\alpha}(z,\varepsilon),\end{aligned} \tag{3.19}$$

where $W_0(\varepsilon)=iT\mathcal{V}^{-1}S+O(\varepsilon)\in\mathcal{L}(1,-s_0;1,-s_0)$, $W_j(\varepsilon)\in\mathcal{L}(1,-s_j;1,-s_j)$ are uniformly bounded on ε for $s_0>1$ and $s_j>j+\frac{1}{2}$, $j=1,\dots,N$, and $W_{l,N+\alpha}(z,\varepsilon)$, $l=1,2$ are uniformly bounded operators on ε, z in $\mathcal{L}(1,-s;1,-s)$. In particular $W_j(\varepsilon)$, $W_{l,N+\alpha}(z,\varepsilon)$ are of finite rank for any fixed $z\in\Omega_2$ and the remainders satisfy

$$\|\langle x\rangle^{-s}\frac{d^k}{dz^k}W_{N+\alpha,l}(z,\varepsilon)\langle x\rangle^{s}\|\le C|z|^{\frac{N+\alpha}{2}-k},\; k=0,1,\dots,N,\; l=1,2,$$

and $W_{N+\alpha,2}(\lambda+i0,\varepsilon)=W_{N+\alpha,2}(\lambda-i0,\varepsilon)$ for $\lambda\in[0,2c_1\varepsilon[$. Furthermore, it follows that for $z\in\Omega_2$

$$\begin{aligned}R_{II}(z,\varepsilon) \;=\;& \frac{1}{\varepsilon}R_{2,0}(\varepsilon)+z^{\frac{1}{2}}R_{2,1}(\varepsilon)+\sum_{j=2}^{N}\frac{z^{\frac{j}{2}}}{\varepsilon^{[\frac{j}{2}]+1}}R_{2,j}(\varepsilon)\\ &+\frac{1}{\varepsilon^{\frac{N+\alpha}{2}+\frac{1}{2}}}R_{2,1,N+\alpha}(z,\varepsilon)+\frac{1}{\varepsilon^{\frac{N+\alpha}{2}+1}}R_{2,2,N+\alpha}(z,\varepsilon),\end{aligned} \tag{3.20}$$

where $R_{2,0}(\varepsilon)=iT\mathcal{V}^{-1}SG_0+O(\varepsilon)\in\mathcal{L}(-1,s_0;1,-s_0)$, $R_{2,j}(\varepsilon)\in\mathcal{L}(-1,s_j;1,-s_j)$ are uniformly bounded on ε for $s_0>1$ and $s_j>j+\frac{1}{2}$, $j=1,\dots,N$, and $R_{2,l,N+\alpha}(z,\varepsilon)$, $l=1,2$ are uniformly bounded operators on ε, z in $\mathcal{L}(-1,s;1,-s)$. In particular $R_{2,j}(\varepsilon)$, $R_{2,l,N+\alpha}(z,\varepsilon)$ are of finite rank for any fixed $z\in\Omega_2$ and the remainders satisfy that

$$\|\langle x\rangle^{-s}\frac{d^k}{dz^k}R_{2,l,N+\alpha}(z,\varepsilon)\langle x\rangle^{-s}\|\le C|z|^{\frac{N+\alpha}{2}-k},\; k=0,1,\dots,N,\; l=1,2,$$

and $R_{2,2,N+\alpha}(\lambda+i0,\varepsilon)=R_{2,2,N+\alpha}(\lambda-i0,\varepsilon)$ for $\lambda\in[0,2c_1\varepsilon[$.

Let $\widetilde{E}_{-+,0}(z,\varepsilon) = E_{-+,0}(z,\varepsilon) + \varepsilon^2\hat{E}_{-+,0}(\varepsilon)$. To get the expansion of $\widetilde{E}(z,\varepsilon)$, we divide the proof of Lemma 3.3.5 into some steps.

Lemma 3.3.6. *Under the assumptions of Lemma 3.3.5, one has that*

$$\widetilde{E}_{-+,0}(z,\varepsilon)^{-1} = \frac{1}{\varepsilon}B_0(\varepsilon) + \sum_{j=1}^{[\frac{N}{2}]}\frac{z^j}{\varepsilon^{j+1}}B_j(\varepsilon) + \frac{z^{[\frac{N}{2}]+1}}{\varepsilon^{[\frac{N}{2}]+2}}B_{[\frac{N}{2}]+1}(z,\varepsilon),$$

where

$$B_0(\varepsilon) = (1 - i\varepsilon\mathcal{V}^{-1}\hat{E}_{-+,0}(\varepsilon))^{-1}(i\mathcal{V})^{-1} = -i\mathcal{V}^{-1} + O(\varepsilon),$$

$B_j(\varepsilon)$, $j = 1,\dots,[\frac{N}{2}]$ *are uniformly bounded on* ε *and* $B_{[\frac{N}{2}]+1}(z,\varepsilon)$ *is a uniformly bounded matrix on* z, ε *satisfying that for* $k \in \mathbb{N}$

$$\|\frac{d^k}{dz^k}B_{[\frac{N}{2}]+1}(z,\varepsilon)\| \le O(\varepsilon^{-k}).$$

Démonstration. Since

$$(i\varepsilon\mathcal{V} + z\mathcal{U})^{-1} = \sum_{l=1}^{m}\frac{\mathcal{A}^{-1}P_l(\mathcal{A}^*)^{-1}}{z + i\varepsilon\lambda_l} = O(\varepsilon^{-1}),$$

for $z \in \Omega_2$, we have that

$$\widetilde{E}_{-+,0}(z,\varepsilon) = (i\varepsilon\mathcal{V} + z\mathcal{U})(1 + O(\varepsilon)).$$

Thus by Neumann's series, one obtains that for $z \in \Omega_2$ and δ, ε small enough, $\widetilde{E}_{-+,0}(z,\varepsilon)^{-1}$ exists with

$$\widetilde{E}_{-+,0}(z,\varepsilon)^{-1} = (1 + \varepsilon^2(i\varepsilon\mathcal{V} + z\mathcal{U})^{-1}\hat{E}_{-+,0}(\varepsilon))^{-1}(i\varepsilon\mathcal{V} + z\mathcal{U})^{-1}. \tag{3.21}$$

Thus we have

$$\begin{aligned}(i\varepsilon\mathcal{V} + z\mathcal{U})^{-1} &= \sum_{j=0}^{[\frac{N}{2}]}\frac{z^j}{\varepsilon^{j+1}}\Big(\sum_{l=1}^{m}\frac{(-1)^j\mathcal{A}^{-1}P_l(\mathcal{A}^*)^{-1}}{(i\lambda_l)^{j+1}}\Big)\\ &\quad + \frac{z^{[\frac{N}{2}]+1}}{\varepsilon^{[\frac{N}{2}]+2}}\sum_{l=1}^{m}\frac{(-1)^{[\frac{N}{2}]+1}\mathcal{A}^{-1}P_l(\mathcal{A}^*)^{-1}}{(i\lambda_l)^{[\frac{N}{2}]+1}(\frac{z}{\varepsilon} + i\lambda_l)}.\end{aligned}$$

Consequently, taking this into (3.21), we can obtain the conclusion. Furthermore,

$$\begin{aligned}B_0(\varepsilon) &= (1 - i\varepsilon\mathcal{V}^{-1}\hat{E}_{-+,0}(\varepsilon))^{-1}(i\mathcal{V})^{-1}\\ &= -i\mathcal{V}^{-1} + O(\varepsilon).\end{aligned}$$

□

Lemma 3.3.7. *Under the assumption of Lemma 3.3.5, one can obtain the following expansion*

$$\begin{aligned}E_{-+}(z,\varepsilon)^{-1} &= \frac{1}{\varepsilon}B_0(\varepsilon) - z^{\frac{1}{2}}B_0(\varepsilon)E_{-+,1}(\varepsilon)B_0(\varepsilon) + \sum_{j=2}^{N}\frac{z^{\frac{j}{2}}}{\varepsilon^{[\frac{j}{2}]+1}}C_j(\varepsilon)\\ &\quad + \frac{1}{\varepsilon^{\frac{N+\alpha}{2}+\frac{1}{2}}}C_{1,N+\alpha}(z,\varepsilon) + \frac{1}{\varepsilon^{\frac{N+\alpha}{2}+1}}C_{2,N+\alpha}(z,\varepsilon),\end{aligned} \tag{3.22}$$

where $C_j(\varepsilon)$, $j = 2,\dots,N$ *are uniformly bounded matrices and* $C_{l,N+\alpha}(z,\varepsilon)$, $l = 1,2$ *satisfy that*

$$\|\frac{d^k}{dz^k}C_{l,N+\alpha}(z,\varepsilon)\| \le C|z|^{\frac{N+\alpha}{2}-k},\ k = 0,1,\dots,N,\ l = 1,2.$$

Furthermore, $C_{2,N+\alpha}(\lambda + i0,\varepsilon) = C_{2,N+\alpha}(\lambda - i0,\varepsilon)$ *for* $\lambda \in [0, 2c_1\varepsilon[$.

Démonstration. Note that

$$\begin{aligned} E_{-+}(z,\varepsilon) &= \widetilde{E}_{-+,0}(z,\varepsilon)(1+\widetilde{E}_{-+,0}(z,\varepsilon)^{-1}(\varepsilon^2 z^{\frac{1}{2}}E_{-+,1}(\varepsilon)+\varepsilon z E_{-+,2}(\varepsilon) \\ &\quad +\sum_{j=3}^{N} z^{\frac{j}{2}}E_{-+,j}(\varepsilon)+E_{-+,N+\alpha}(z,\varepsilon))). \end{aligned}$$

It is easy to check that

$$E_{-+}(z,\varepsilon)=\widetilde{E}_{-+,0}(z,\varepsilon)(1+O(\varepsilon^{\frac{1}{2}})),$$

for $z\in\Omega_2$. So $E_{-+}(z,\varepsilon)^{-1}$ exists for $\varepsilon_0>0$ small enough. Then one can deduce (3.22). Actually, noting that

$$\begin{aligned} E_{-+}(\lambda,\varepsilon) &= (1+(\varepsilon^2\lambda E^1_{-+}(\varepsilon)+\varepsilon\lambda^2E^2_{-+}(\varepsilon)+\sum_{j=3}^{N}\lambda^jE^j_{-+}(\varepsilon) \\ &\quad +E^{N+\alpha}_{-+}(\lambda,\varepsilon))E^0_{-+}(\lambda,\varepsilon)^{-1})E^0_{-+}(\lambda,\varepsilon), \end{aligned}$$

we have that

$$\begin{aligned} E_{-+}(\lambda,\varepsilon)^{-1} &= E^0_{-+}(\lambda,\varepsilon)^{-1}(1+(\varepsilon^2\lambda E^1_{-+}(\varepsilon)+\varepsilon\lambda^2E^2_{-+}(\varepsilon)+\sum_{j=3}^{N}\lambda^jE^j_{-+}(\varepsilon) \\ &\quad +E^{N+\alpha}_{-+}(\lambda,\varepsilon))E^0_{-+}(\lambda,\varepsilon)^{-1})^{-1} \\ &= E^0_{-+}(\lambda,\varepsilon)^{-1}\{\sum_{l=0}^{N}(-(\varepsilon^2\lambda E^1_{-+}(\varepsilon)+\varepsilon\lambda^2E^2_{-+}(\varepsilon)+\sum_{j=3}^{N}\lambda^jE^j_{-+}(\varepsilon) \\ &\quad +E^{N+\alpha}_{-+}(\lambda,\varepsilon))E^0_{-+}(\lambda,\varepsilon)^{-1})^l+(-1)^N((\varepsilon^2\lambda E^1_{-+}(\varepsilon)+\varepsilon\lambda^2E^2_{-+}(\varepsilon) \\ &\quad +\sum_{j=3}^{N}\lambda^jE^j_{-+}(\varepsilon)+E^{N+\alpha}_{-+}(\lambda,\varepsilon))E^0_{-+}(\lambda,\varepsilon)^{-1})^{N+1}(1+(\varepsilon^2\lambda E^1_{-+}(\varepsilon) \\ &\quad +\varepsilon\lambda^2E^2_{-+}(\varepsilon)+\sum_{j=3}^{N}\lambda^jE^j_{-+}(\varepsilon)+E^{N+\alpha}_{-+}(\lambda,\varepsilon))E^0_{-+}(\lambda,\varepsilon)^{-1})^{-1}\}. \end{aligned}$$

Firstly, due to Lemma 3.21, we check that for N even

$$\begin{aligned} &(\varepsilon^2\lambda E^1_{-+}(\varepsilon)+\varepsilon\lambda^2E^2_{-+}(\varepsilon)+\sum_{j=3}^{N}\lambda^jE^j_{-+}(\varepsilon)+E^{N+\alpha}_{-+}(\lambda,\varepsilon))E^0_{-+}(\lambda,\varepsilon)^{-1} \\ =\ &(\varepsilon^2\lambda E^1_{-+}(\varepsilon)+\varepsilon\lambda^2E^2_{-+}(\varepsilon)+\sum_{j=3}^{N}\lambda^jE^j_{-+}(\varepsilon)+E^{N+\alpha}_{-+}(\lambda,\varepsilon))(\frac{1}{\varepsilon}B^0(\varepsilon)+\sum_{j=1}^{\frac{N}{2}}\frac{\lambda^{2j}}{\varepsilon^{j+1}}B^j(\varepsilon) \\ &+\frac{\lambda^{N+2}}{\varepsilon^{\frac{N}{2}+2}}B^{\frac{N}{2}+1}(\lambda,\varepsilon)) \\ =\ &\varepsilon\lambda E^1_{-+}(\varepsilon)B^0(\varepsilon)+\lambda^2E^2_{-+}(\varepsilon)B^0(\varepsilon) \\ &+\sum_{l=2}^{\frac{N}{2}}\{\lambda^{2l-1}(\frac{1}{\varepsilon^{l-2}}E^1_{-+}(\varepsilon)B^{l-1}(\varepsilon)+\sum_{j=2}^{l}\frac{1}{\varepsilon^{l-j+1}}E^{2j-1}_{-+}(\varepsilon)B^{l-j}(\varepsilon)) \\ &+\lambda^{2l}(\frac{1}{\varepsilon^{l-1}}E^2_{-+}(\varepsilon)B^{l-1}(\varepsilon)+\sum_{j=2}^{l}\frac{1}{\varepsilon^{l-j+1}}E^{2j}_{-+}(\varepsilon)B^{l-j}(\varepsilon))\} \end{aligned}$$

$$+\{\lambda^{N+1}(\frac{1}{\varepsilon^{\frac{N}{2}-1}}E^1_{-+}(\varepsilon)B^{\frac{N}{2}}(\varepsilon)+\sum_{j=2}^{\frac{N}{2}}\frac{1}{\varepsilon^{\frac{N}{2}-j+2}}E^{2j-1}_{-+}(\varepsilon)B^{\frac{N}{2}+1-j}(\varepsilon))$$
$$+\lambda^{N+2}(\frac{1}{\varepsilon^{\frac{N}{2}}}E^2_{-+}(\varepsilon)B^{\frac{N}{2}}(\varepsilon)+\sum_{j=2}^{\frac{N}{2}}\frac{1}{\varepsilon^{\frac{N}{2}-j+2}}E^{2j}_{-+}(\varepsilon)B^{\frac{N}{2}+1-j}(\varepsilon))\}$$
$$+\sum_{l=\frac{N}{2}+2}^{N}\{\lambda^{2l-1}\sum_{j=l-\frac{N}{2}}^{\frac{N}{2}}\frac{1}{\varepsilon^{l-j+1}}E^{2j-1}_{-+}(\varepsilon)B^{l-j}+\lambda^{2l}\sum_{j=l-\frac{N}{2}}^{\frac{N}{2}}\frac{1}{\varepsilon^{l-j+1}}E^{2j}_{-+}(\varepsilon)B^{l-j}(\varepsilon)\}$$
$$+E^{N+\alpha}_{-+}(\lambda,\varepsilon)(\frac{1}{\varepsilon}B^0(\varepsilon)+\sum_{j=1}^{\frac{N}{2}}\frac{\lambda^{2j}}{\varepsilon^{j+1}}B^j(\varepsilon)+\frac{\lambda^{N+2}}{\varepsilon^{\frac{N}{2}+2}}B^{\frac{N}{2}+1}(\lambda,\varepsilon))$$
$$+\frac{\lambda^{N+2}}{\varepsilon^{\frac{N}{2}+2}}(\varepsilon^2\lambda E^1_{-+}(\varepsilon)+\varepsilon\lambda^2E^2_{-+}(\varepsilon)+\sum_{j=3}^{N}\lambda^jE^j_{-+}(\varepsilon)+E^{N+\alpha}_{-+}(\lambda,\varepsilon))B^{\frac{N}{2}+1}(\lambda,\varepsilon)$$

and for N odd

$$\begin{aligned}
&(\varepsilon^2\lambda E^1_{-+}(\varepsilon)+\varepsilon\lambda^2E^2_{-+}(\varepsilon)+\sum_{j=3}^{N}\lambda^jE^j_{-+}(\varepsilon)+E^{N+\alpha}_{-+}(\lambda,\varepsilon))E^0_{-+}(\lambda,\varepsilon)^{-1}\\
=\;&(\varepsilon^2\lambda E^1_{-+}(\varepsilon)+\varepsilon\lambda^2E^2_{-+}(\varepsilon)+\sum_{j=3}^{N}\lambda^jE^j_{-+}(\varepsilon)+E^{N+\alpha}_{-+}(\lambda,\varepsilon))(\frac{1}{\varepsilon}B^0(\varepsilon)+\sum_{j=1}^{\frac{N-1}{2}}\frac{\lambda^{2j}}{\varepsilon^{j+1}}B^j(\varepsilon)\\
&+\frac{\lambda^{N+1}}{\varepsilon^{\frac{N-1}{2}+2}}B^{\frac{N-1}{2}+1}(\lambda,\varepsilon))\\
=\;&\varepsilon\lambda E^1_{-+}(\varepsilon)B^0(\varepsilon)+\lambda^2E^2_{-+}(\varepsilon)B^0(\varepsilon)\\
&+\sum_{l=2}^{\frac{N-1}{2}}\{\lambda^{2l-1}(\frac{1}{\varepsilon^{l-2}}E^1_{-+}(\varepsilon)B^{l-1}(\varepsilon)+\sum_{j=2}^{l}\frac{1}{\varepsilon^{l-j+1}}E^{2j-1}_{-+}(\varepsilon)B^{l-j}(\varepsilon))\\
&+\lambda^{2l}(\frac{1}{\varepsilon^{l-1}}E^2_{-+}(\varepsilon)B^{l-1}(\varepsilon)+\sum_{j=2}^{l}\frac{1}{\varepsilon^{l-j+1}}E^{2j}_{-+}(\varepsilon)B^{l-j}(\varepsilon))\}\\
&+\{\lambda^{N}(\frac{1}{\varepsilon^{\frac{N-3}{2}}}E^1_{-+}(\varepsilon)B^{\frac{N-1}{2}}(\varepsilon)+\sum_{j=2}^{\frac{N+1}{2}}\frac{1}{\varepsilon^{\frac{N+3}{2}-j}}E^{2j-1}_{-+}(\varepsilon)B^{\frac{N+1}{2}-j}(\varepsilon))\\
&+\lambda^{N+1}(\frac{1}{\varepsilon^{\frac{N-1}{2}}}E^2_{-+}(\varepsilon)B^{\frac{N-1}{2}}(\varepsilon)+\sum_{j=2}^{\frac{N-1}{2}}\frac{1}{\varepsilon^{\frac{N+3}{2}-j}}E^{2j}_{-+}(\varepsilon)B^{\frac{N+1}{2}-j}(\varepsilon))\}\\
&+\sum_{l=\frac{N-1}{2}+2}^{N}\{\lambda^{2l-1}\sum_{j=l-\frac{N-1}{2}}^{\frac{N+1}{2}}\frac{1}{\varepsilon^{l-j+1}}E^{2j-1}_{-+}(\varepsilon)B^{l-j}(\varepsilon)+\lambda^{2l}\sum_{j=l-\frac{N-1}{2}}^{\frac{N-1}{2}}\frac{1}{\varepsilon^{l-j+1}}E^{2j}_{-+}(\varepsilon)B^{l-j}\}\\
&+E^{N+\alpha}_{-+}(\lambda,\varepsilon)(\frac{1}{\varepsilon}B^0(\varepsilon)+\sum_{j=1}^{\frac{N-1}{2}}\frac{\lambda^{2j}}{\varepsilon^{j+1}}B^j(\varepsilon)+\frac{\lambda^{N+1}}{\varepsilon^{\frac{N-1}{2}+2}}B^{\frac{N-1}{2}+1}(\lambda,\varepsilon))\\
&+\frac{\lambda^{N+1}}{\varepsilon^{\frac{N-1}{2}+2}}(\varepsilon^2\lambda E^1_{-+}(\varepsilon)+\varepsilon\lambda^2E^2_{-+}(\varepsilon)+\sum_{j=3}^{N}\lambda^jE^j_{-+}(\varepsilon)+E^{N+\alpha}_{-+}(\lambda,\varepsilon))B^{\frac{N-1}{2}+1}(\lambda,\varepsilon)
\end{aligned}$$

In the expansion (3.22), the singularities of the terms for $j\geq 3$ odd are determined by

$-z^{\frac{3}{2}}\widetilde{E}_{-+,0}(z,\varepsilon)^{-1}E_{-+,3}(\varepsilon)\widetilde{E}_{-+,0}(z,\varepsilon)^{-1}$ and the ones for $j \geq 2$ even are dependent on $\widetilde{E}_{-+,0}(z,\varepsilon)^{-1}$. Therefore, the singularities on ε for j odd and even are different. On the other hand, also due to this observation, there would appear two kinds of remainders. The first kind $C_{1,N+\alpha}(z,\varepsilon)$ is dependent on the odd power of $z^{\frac{1}{2}}$ and $E_{-+,N+\alpha}(z,\varepsilon)$, and the second one $C_{2,N+\alpha}(z,\varepsilon)$ is dependent on the even power of $z^{\frac{1}{2}}$. Furthermore, $C_{2,N+\alpha}(z,\varepsilon)$ is analytic on $z \in \Omega_2$. □

Proof of Lemma 3.3.5. From (3.10), (3.17), (3.18) and (3.22), one can get the expansion (3.19) of $\widetilde{E}(z,\varepsilon)$. In particular,

$$\begin{aligned}
W_0(\varepsilon) &= -TB_0(\varepsilon)S - \varepsilon(E_{+,0}(\varepsilon)TB_0(\varepsilon)S + TB_0(\varepsilon)SE_{-,0}(\varepsilon)) \\
&\quad -\varepsilon^2 E_{+,0}(\varepsilon)TB_0(\varepsilon)SE_{-,0}(\varepsilon) \\
&= iT\mathcal{V}^{-1}S + O(\varepsilon), \\
W_1(\varepsilon) &= -E_{+,1}(\varepsilon)TB_0(\varepsilon)S(1+\varepsilon E_{-,0}(\varepsilon)) - (1+\varepsilon E_{+,0}(\varepsilon))TB_0(\varepsilon)SE_{-,1}(\varepsilon) \\
&\quad +(1+\varepsilon E_{+,0}(\varepsilon))TB_0(\varepsilon)E_{-+,1}(\varepsilon)B_0(\varepsilon)S(1+\varepsilon E_{-,0}(\varepsilon)),
\end{aligned}$$

and the other terms can be also computed directly. Consequently, by (3.7) and Lemma 3.2.1, it is easy to get (3.20) and

$$\begin{aligned}
R_{2,0}(\varepsilon) &= W_0(\varepsilon)G_0 \\
&= iT\mathcal{V}^{-1}SG_0 + O(\varepsilon), \\
R_{2,1}(\varepsilon) &= \frac{1}{\varepsilon}W_0(\varepsilon)G_1 + W_1(\varepsilon)G_0 \\
&= -TB_0(\varepsilon)SE_{-,0}(\varepsilon)G_1 \\
&\quad -(E_{+,1}(\varepsilon)TB_0(\varepsilon)S + TB_0(\varepsilon)SE_{-,1}(\varepsilon) - TB_0(\varepsilon)E_{-+,1}(\varepsilon)B_0(\varepsilon)S)G_0 + O(\varepsilon), \\
R_{2,j}(\varepsilon) &= \varepsilon^{[\frac{j}{2}]-1}W_0(\varepsilon)G_j + \varepsilon^{[\frac{j}{2}]}W_1(\varepsilon)G_{j-1} + \sum_{k=2}^{j}\varepsilon^{[\frac{j}{2}]-[\frac{k}{2}]}W_k(\varepsilon)G_{j-k},\ j = 2,\dots,N.
\end{aligned}$$

Here we use the relation $SG_1 = 0$ for $R_{2,1}(\varepsilon)$, and the properties of the remainders in (3.19) and (3.20) can be easily checked. We omit the details here.

□

Together with Lemma 3.3.1, 3.3.4 and 3.3.5, we can get the following theorem about the resolvent in Ω.

Theorem 3.3.8. *Suppose that $N > 3$, $\rho_0 > 2N+1$, $\varepsilon \in]0,\varepsilon_0]$ and $s \in]N+\frac{1}{2}, \frac{\rho_0}{2}]$. Then for $z \in \Omega_1$, one has the expansions of $R(z,\varepsilon)$ and its derivatives as follows*

$$\frac{d^k}{dz^k}R(z,\varepsilon) = (-1)^{k+1}k!\sum_{l=1}^{m}\frac{T\mathcal{A}^{-1}P_l(\mathcal{A}^*)^{-1}SG_0}{(z+i\varepsilon\lambda_l)^{k+1}} + O(|z|^{-k-\frac{1}{2}}), \tag{3.23}$$

in $\mathcal{L}(-1,s;1,-s)$. For $z \in \Omega_2$, we have the following expansion of $R(z,\varepsilon)$

$$\begin{aligned}
R(z,\varepsilon) &= R_{1,0}(\varepsilon) + \frac{1}{\varepsilon}R_0(\varepsilon) + z^{\frac{1}{2}}R_1(\varepsilon) \\
&\quad + \sum_{j=2}^{N}\frac{z^{\frac{j}{2}}}{\varepsilon^{[\frac{j}{2}]+1}}R_j(\varepsilon) + \frac{1}{\varepsilon^{\frac{N+\alpha+1}{2}}}R^{(1)}_{N+\alpha}(z,\varepsilon) + \frac{1}{\varepsilon^{\frac{N+\alpha+2}{2}}}R^{(2)}_{N+\alpha}(z,\varepsilon),
\end{aligned}$$

for some $\alpha \in]0, \min\{1, s-N-\frac{1}{2}\}[$. Here $R_1(\varepsilon) = R_{1,1}(\varepsilon) + R_{2,1}(\varepsilon)$ and $R_j(\varepsilon) = \varepsilon^{[\frac{j}{2}]+1}R_{1,j}(\varepsilon) + R_{2,j}(\varepsilon)$, $j = 2,\dots,N$ and $R^{(1)}_{N+\alpha}(z,\varepsilon) = \varepsilon^{\frac{N+\alpha+1}{2}}R_{1,N+\alpha}(z,\varepsilon) + R_{2,1,N+\alpha}(z,\varepsilon)$, $R^{(2)}_{N+\alpha}(z,\varepsilon) = R_{2,2,N+\alpha}(z,\varepsilon)$.

Remark 3.3.9. *It can be calculated directly that*

$$\begin{aligned} R_1(\varepsilon) &= R_{1,1}(\varepsilon) + R_{2,1}(\varepsilon) \\ &= (1 - i(1+\varepsilon E_{+,0}(\varepsilon))TB_0(\varepsilon)SG_0V_2)(E_0(\varepsilon)G_1 + E_1(\varepsilon)G_0) \\ &\quad (1 - iV_2TB_0(\varepsilon)S(1+\varepsilon E_{-,0}(\varepsilon))G_0). \end{aligned}$$

Furthermore, noting that $(E_0(\varepsilon)G_1 + E_1(\varepsilon)G_0) = E_0(\varepsilon)G_1(1 - (V_1 - i\varepsilon V_2)E_0(\varepsilon)G_0)$ *is of rank one,* $R_1(\varepsilon)$ *is a uniformly bounded operator of rank one at most in* $\mathcal{L}(-1, s_1; 1, -s_1)$ *for* $s_1 > \frac{3}{2}$. *On the other hand, we can compute the limit*

$$\begin{aligned} \lim_{\varepsilon \to 0_+} R_1(\varepsilon) &= (1 - T\mathcal{V}^{-1}SG_0V_2)(E_0(0)G_1 + E_1(0)G_0)(1 - V_2T\mathcal{V}^{-1}SG_0) \\ &= (1 - T\mathcal{V}^{-1}SG_0V_2)E_0(0)G_1(1 - V_1E_0(0)G_0)(1 - V_2T\mathcal{V}^{-1}SG_0) \end{aligned}$$

is a nontrival bounded operator of rank one in $\mathcal{L}(-1, s_1; 1, -s_1)$ *for* $s_1 > \frac{3}{2}$. *So* $R_1(\varepsilon)$ *is of rank one for* $\varepsilon \in]0, \varepsilon_0]$.

3.3.3 Properties of the Riesz projection

At the end of this section, we shall analyze the Riesz projection associated with the eigenvalues near 0. Without loss of generality, we suppose that $-i\varepsilon\lambda_j$, $j = 1, \ldots, m$ are all simple roots of $F_0(z, \varepsilon) = 0$. For $z \in \Omega_2$, we have

$$|z + i\varepsilon\lambda_j| \geq c_2\varepsilon, \; |z| \leq 2c_1\varepsilon.$$

Since $E(z, \varepsilon)$ is uniformly bounded on ε for $z \in \Omega_2$, one has for $z \in \Omega_2$

$$W(z, \varepsilon)^{-1} = -TE_{-+,0}(z, \varepsilon)^{-1}S + W_1(z, \varepsilon)$$

in $\mathcal{L}(1, -s; 1, -s)$, where $\|\langle x\rangle^{-s}W_1(z, \varepsilon)\langle x\rangle^{s}\| = O(\varepsilon^{-\frac{1}{2}})$. Consider the Riesz projection associated with $z_j(\varepsilon)$

$$\Pi_j(\varepsilon) = -\frac{1}{2\pi i}\oint_{\partial B(-i\varepsilon\lambda_j, c_2\varepsilon)} R(z, \varepsilon)dz.$$

Therefore, by (3.7), one has

$$\begin{aligned} \Pi_j(\varepsilon) &= -\frac{1}{2\pi i}\oint_{\partial B(-i\varepsilon\lambda_j, c_2\varepsilon)} -TE_{-+,0}(z, \varepsilon)^{-1}SG_0 + O(\varepsilon^{-\frac{1}{2}})dz \\ &= \frac{1}{2\pi i}\sum_{j=1}^{m}\oint_{\partial B(-i\varepsilon\lambda_j, c_2\varepsilon)} \frac{T\mathcal{A}^{-1}P_j(\mathcal{A}^*)^{-1}SG_0}{z + i\varepsilon\lambda_j}dz + O(\varepsilon^{\frac{1}{2}}) \\ &= T\mathcal{A}^{-1}P_j(\mathcal{A}^*)^{-1}SG_0 + O(\varepsilon^{\frac{1}{2}}). \end{aligned} \tag{3.24}$$

One can see that $\Pi_j = T\mathcal{A}^{-1}P_j(\mathcal{A}^*)^{-1}SG_0$ is a mapping from $L^{2,s}$ to $\mathcal{M} \subset L^2$. Furthermore, it can be extended to a projection from L^2 to $\mathcal{M}$. Formally, we have for $\phi \in L^2$

$$\Pi_j\phi = T\mathcal{A}^{-1}P_j(\mathcal{A}^*)^{-1}(\{\langle\phi, \phi_k\rangle\}_{k=1}^m).$$

So let $\widetilde{S} = \{\langle\cdot, \phi_k\rangle\}_{k=1}^m$ be a mapping from L^2 to $\mathbb{C}^m$. It is easy to see that $\widetilde{S} = T^*$. We can verify that $\widetilde{\Pi}_j = T\mathcal{A}^{-1}P_j(\mathcal{A}^*)^{-1}T^*$ is an orthogonal projection from L^2 to $\mathcal{M}$. In fact,

$$\begin{aligned} \widetilde{\Pi}_j\widetilde{\Pi}_j &= T\mathcal{A}^{-1}P_j(\mathcal{A}^*)^{-1}T^*T\mathcal{A}^{-1}P_j(\mathcal{A}^*)^{-1}T^* \\ &= T\mathcal{A}^{-1}P_j(\mathcal{A}^*)^{-1}\mathcal{U}\mathcal{A}^{-1}P_j(\mathcal{A}^*)^{-1}T^* \\ &= T\mathcal{A}^{-1}P_jP_j(\mathcal{A}^*)^{-1}T^* = \widetilde{\Pi}_j, \end{aligned}$$

and

$$\widetilde{\Pi}_j^* = (T\mathcal{A}^{-1}P_j(\mathcal{A}^*)^{-1}T^*)^* = \widetilde{\Pi}_j.$$

If no confusion is possible, we also denote Π_j the eigen-projection of $H_1 = -\Delta + V_1(x)$ on L^2. Then one has that

$$\sum_{j=1}^{m} \Pi_j = T\mathcal{A}^{-1}(\mathcal{A}^*)^{-1}T^* = T\mathcal{U}^{-1}T^*$$

is the orthogonal eigenprojection P_0 of H_1 associated with 0. On the other hand, by (3.24) we have the estimate for $s > \frac{7}{2}$

$$\|\Pi_j(\varepsilon) - \Pi_j\|_{\mathcal{L}(-1,s;1,-s)} \geq O(\varepsilon^{\frac{1}{2}}),$$

which implies

$$\lim_{\varepsilon\to 0_+} \Pi_j(\varepsilon) = \Pi_j, \text{ in } \mathcal{L}(-1,s;1,-s).$$

More precisely, we can also get the estimate of the projection as an operator on L^2.

Proposition 3.3.10. *Suppose $\rho_0 > 7$ and $\varepsilon \in]0, \varepsilon_0]$ small enough. Then it holds that*

$$\|\Pi_j(\varepsilon) - \Pi_j\|_{L^2\to L^2} \leq O(\varepsilon^{\frac{1}{2}}),\ j = 1, \dots, m. \tag{3.25}$$

Démonstration. Let $\Pi^{(0)}(\varepsilon) = \sum_{j=1}^m \Pi_j(\varepsilon)$ be the Riesz projection associated with the eigenvalues of $H(\varepsilon)$ near zero and then $P_0 = \Pi^{(0)}(0)$ be the orthogonal projection onto the eigenfunction space of H_1 associated with 0. Denote $P_0' = 1 - P_0$. It is known that

$$\begin{aligned} \|R_1(z)\| &\leq \frac{1}{(|\Im z|^2 + |(\Re z)_-|^2)^{\frac{1}{2}}}, \\ \|\langle x\rangle^{-s} P_0' R_1(z) P_0' \langle x\rangle^{-s}\| &\leq \frac{C}{|z|^{\frac{1}{2}}}, \end{aligned}$$

for $z \in \mathbb{C} \setminus \mathbb{R}_+$ and $s > \frac{7}{2}$, where $(\Re z)_- = \min\{0, \Re z\}$. First, we consider the inverse of $P_0'(H(\varepsilon) - z)P_0'$ on L_2. Denote $E_0(z) = (P_0'(H_1 - z)P_0')^{-1}P_0'$ and we have

$$\|E_0(z)\| \leq \frac{1}{(|\Im z|^2 + |(\Re z)_-|^2)^{\frac{1}{2}}},$$

and

$$\|\langle x\rangle^{-s} E_0(z) \langle x\rangle^{-s}\| \leq \frac{C}{|z|^{\frac{1}{2}}}.$$

Let $\{\varphi_j\}_{j=1}^m \subset L^2$ be an orthogonal basis of the eigenfunction space $\mathcal{M}$. It is known that the eigenvalues of $H(\varepsilon)$ are all in $B(-i\varepsilon\lambda_j, c_2\varepsilon)$ for $j = 1, \dots, m$, where λ_j, $j = 1, \dots, m$ are the zeros of

$$F_0(\lambda) = \det(\lambda I_m - (\langle V_2\varphi_j, \varphi_k\rangle)_{1\leq j,k,\leq m}) = \det(\lambda\langle\phi_j, \phi_k\rangle - (\langle V_2\phi_j, \phi_k\rangle)_{1\leq j,k,\leq m}),$$

where $\{\phi_j\}_{j=1}^m$ is the basis satisfying (3.6). Let $\tilde{\Omega} = B(0, C_1\varepsilon) \setminus \{B(0, C_2\varepsilon) \cup \{z \in \mathbb{C} : |\Im z| \leq c'\varepsilon\}\}$ for some $c' > 0$ and $C_1 > C_2 > 0$ satisfying that $\cup_{j=1}^m B(-i\varepsilon\lambda_j, c_2\varepsilon) \subset B(0, C_2\varepsilon)$. Formally, We note that

$$(P_0'(H(\varepsilon) - z)P_0')^{-1}P_0' = E_0(z) + \sum_{j=0}^{\infty} (i\varepsilon)^{j+1} E_0(z)\sqrt{V_2}(\sqrt{V_2}E_0(z)\sqrt{V_2})^j \sqrt{V_2}E_0(z).$$

and for $z \in \tilde{\Omega}$,

$$\|\sqrt{V_2}E_0(z,\varepsilon)\sqrt{V_2}\| \leq O(\varepsilon^{-\frac{1}{2}}).$$

So $E(z,\varepsilon) = (P_0'(H(\varepsilon)-z)P_0')^{-1}P_0'$ exists and we have the estimates

$$\begin{aligned} \|E(z,\varepsilon)\| &\leq O(\varepsilon^{-1}), \\ \|\langle x\rangle^{-s}E(z,\varepsilon)\langle x\rangle^{-s}\| &\leq O(\varepsilon^{-\frac{1}{2}}), \end{aligned}$$

for $z \in \tilde{\Omega}$ and $s > 7$. We define the mapping $R_- : L^2 \to \mathbb{C}^m$ and $R_+ : \mathbb{C}^m \to \text{Ran } P_0$ by

$$\begin{aligned} R_-\varphi &= \{\langle \varphi, \varphi_j\rangle\}_{j=1}^m, \text{ for } \varphi \in L^2; \\ R_+a &= \sum_{j=1}^m a_j\phi_j, \text{ for } a = \{a_k\}_{j=1}^m \in \mathbb{C}^m. \end{aligned}$$

Then they satisfy that $R_-R_+ = Id_{\mathbb{C}^m}$ and $R_+R_- = P_0$. Following a linear transformation from $\{\varphi_j\}_{j=1}^m$ to $\{\phi_j\}_{j=1}^m$, we can obtain that $\Pi_j = R_+P_jR_-$. By the Grushin method, one can deduce that

$$R(z,\varepsilon) = E(z,\varepsilon) - E_+(z,\varepsilon)E_{-+}(z,\varepsilon)^{-1}E_-(z,\varepsilon),$$

where

$$\begin{aligned} E_+(z,\varepsilon) &= (1 - E(z,\varepsilon)(H(\varepsilon)-z))R_+, \\ E_-(z,\varepsilon) &= R_-(1-(H(\varepsilon)-z)E(z,\varepsilon)), \\ E_{-+}(z,\varepsilon) &= R_-((H(\varepsilon)-z)E(z,\varepsilon)(H(\varepsilon)-z) - (H(\varepsilon)-z))R_+. \end{aligned}$$

Therefore, it can be checked that for $z \in \tilde{\Omega}$,

$$\begin{aligned} \|E_+(z,\varepsilon)\|_{l^2\to L^2} &= O(1), \\ \|E_-(z,\varepsilon)\|_{L^2\to l^2} &= O(1). \end{aligned}$$

On the other hand, it can be calculated that

$$(E_{-+})_{jk}(z,\varepsilon) = z\delta_{jk} + i\varepsilon\langle V_2\varphi_k, \varphi_j\rangle - \varepsilon^2\langle V_2E(z,\varepsilon)V_2\varphi_k, \varphi_j\rangle.$$

Let $\bar{E}_{-+}(z,\varepsilon) = zI_m + i\varepsilon\mathcal{V}^0$ where $\mathcal{V}^0_{jk} = \langle V_2\varphi_k, \varphi_j\rangle$. So we have that

$$E_{-+}(z,\varepsilon)^{-1} = E^0_{-+}(z,\varepsilon)^{-1} + O(\varepsilon^{-\frac{1}{2}}).$$

Here $E^0_{-+}(z,\varepsilon)^{-1} = \sum_{j=1}^m \frac{P_j}{z+i\varepsilon\lambda_j}$, and P_j is the eigenprojection of $\mathcal{V}^0$ associated with the eigenvalue λ_j, which is the same projection on $\mathbb{C}^m$ defined before. Thus we have that

$$R(z,\varepsilon) = E(z,\varepsilon) - \sum_{j=1}^m \frac{R_+P_jR_-}{z+i\varepsilon\lambda_j} + O(\varepsilon^{-\frac{1}{2}}).$$

Consequently, we have the following expansion of the Riesz Projection $\Pi_j(\varepsilon)$

$$\begin{aligned} \Pi_j(\varepsilon) &= -\frac{1}{2\pi i}\oint_{\partial B(-i\varepsilon\lambda_j, c_2\varepsilon)} R(z,\varepsilon)dz \\ &= \frac{1}{2\pi i}\sum_{j=1}^m \oint_{\partial B(-i\varepsilon\lambda_j, c_2\varepsilon)} \frac{R_+P_jR_-}{z+i\varepsilon\lambda_j}dz + O(\varepsilon^{\frac{1}{2}}) \\ &= \Pi_j + O(\varepsilon^{\frac{1}{2}}). \end{aligned}$$

Here we use the analyticity of $E(z,\varepsilon)$ in $\tilde{\Omega}$. So we have the estimate (3.25). □

Second, similar to the case considered in [79] where it is supposed that 0 is neither a resonance nor an eigenvalue of H_1, we can deduce the global estimate in the following proposition.

Proposition 3.3.11. *For $\rho_0 > 7$, $s > \frac{7}{2}$ and $\varepsilon > 0$ small enough, then the global estimate*

$$\|\langle x\rangle^{-s}\Pi'(\varepsilon)R(\lambda \pm i0,\varepsilon)\Pi'(\varepsilon)\langle x\rangle^{-s}\| \le C_0\varepsilon^{-\frac{1}{2}},\ \lambda \in \mathbb{R} \tag{3.26}$$

holds for some $C_0 > 0$.

Démonstration. Let δ be the constant defined as before. Then for $|\lambda| > \delta$, it is the same to the regular case in [79]. On the other hand, we note that for $\kappa > 0$ small enough such that $\lambda \pm i\kappa \in \Omega_2$ and dist $(\lambda \pm i\kappa, \partial B(-i\varepsilon\lambda_j, c_2\varepsilon)) > d\varepsilon$ for some $d > 0$ and $j = 1,\dots,m$,

$$\begin{aligned} R(\lambda \pm i\kappa,\varepsilon)\Pi_j(\varepsilon) &= -\frac{1}{2\pi i}\oint_{\partial B(-i\varepsilon\lambda_j,c_2\varepsilon)} R(\lambda \pm i\kappa,\varepsilon)R(z,\varepsilon)dz \\ &= -\frac{1}{2\pi i}\oint_{\partial B(-i\varepsilon\lambda_j,c_2\varepsilon)} \frac{R(z,\varepsilon) - R(\lambda \pm i\kappa,\varepsilon)}{z-(\lambda \pm i\kappa)}dz \\ &= -\frac{1}{2\pi i}\oint_{\partial B(-i\varepsilon\lambda_j,c_2\varepsilon)} \frac{R(z,\varepsilon)}{z-(\lambda \pm i\kappa)}dz \\ &= -\frac{T\mathcal{A}^{-1}P_j(\mathcal{A}^*)^{-1}SG_0}{\lambda \pm i\kappa + i\varepsilon\lambda_j} + O(\varepsilon^{-\frac{1}{2}}), \end{aligned}$$

in $\mathcal{L}(-1,s;1,-s)$. Consequently, we have in $\mathcal{L}(-1,s;1,-s)$

$$R(\lambda \pm i\kappa,\varepsilon)(1-\Pi^{(0)}(\varepsilon)) = R(\lambda \pm i\kappa,\varepsilon) + \sum_{j=1}^{m}\frac{T\mathcal{A}^{-1}P_j(\mathcal{A}^*)^{-1}SG_0}{\lambda \pm i\kappa + i\varepsilon\lambda_j} + O(\varepsilon^{-\frac{1}{2}}) = O(\varepsilon^{-\frac{1}{2}}).$$

Let $\Pi^{(d)}(\varepsilon) = \Pi(\varepsilon) - \Pi^{(0)}(\varepsilon)$ be the Riesz projection associated with the eigenvalues of $H(\varepsilon)$ which are near the negative eigenvalues of H_1 (see [77]). As shown in [79], one can see that

$$\|R(\lambda \pm i\kappa,\varepsilon)\Pi^{(d)}(\varepsilon)\| \le C_\delta.$$

So let κ tends to 0 and (3.26) can be obtained for $|\lambda| \le \delta$. □

Remark 3.3.12. *For the selfadjoint case satisfying that 0 is only an eigenvalue of H_1 and $\rho_0 > 7$, $s > \frac{7}{2}$, one has the estimate*

$$\|\langle x\rangle^{-s}\Pi_{ac}R_1(\lambda \pm i0)\Pi_{ac}\langle x\rangle^{-s}\| \le C_0|\lambda|^{-\frac{1}{2}},$$

for $\lambda \in \mathbb{R}\setminus\{0\}$, where Π_{ac} is the eigenprojection onto the absolutely continuous space of H_1. Furthermore, in [31], it was indicated that the singularity is due to $P_0V_1G_3V_1P_0$. If this term can be canceled, then one can also deduce the global estimate

$$\|\langle x\rangle^{-s}\Pi_{ac}R_1(\lambda \pm i0)\Pi_{ac}\langle x\rangle^{-s}\| \le C_0.$$

Thus applying the selfadjoint dilation (see [60], [22], [78]), one can establish Kato's smoothness estimate (see [40]) both for H_1 and $H(\varepsilon)$. Then by the same method of perturbation as in [79], the asymptotic completeness of the scattering operator for the pair $(H(\varepsilon), H_0)$ can be proved. Actually in [31], the authors gave an example in which $V_1(x)$ is the spherical square well potential defined as follows

$$V_1(x) = \begin{cases} -V_0, & |x| < r_0, \\ 0, & |x| \ge r_0 \end{cases}$$

for some $V_0 > 0$ and $r_0 > 0$. And then one can choose some suitable V_0 and r_0 such that 0 is only an eigenvalue but not a resonance of H_1 and $P_0V_1G_3V_1P_0 = 0$.

3.4 Proof of Theorem 1.4.1

First, we state the existence of $R(\lambda \pm i0, \varepsilon)$ and their derivatives in some weighted L^2 space for $\lambda \geq \delta$.

Lemma 3.4.1. *Under assumption (1.2) for $\rho_0 > J+1$ and $s > J + \frac{1}{2}$, one has that $\frac{d^j}{d\lambda^j} R(\lambda \pm i0, \varepsilon)$, $\lambda \in [\delta, \infty[$, $j = 0, \ldots, J$ exist in $\mathcal{L}(0, s; 0, -s)$ for any fixed $\delta > 0$ and $\varepsilon \in]0, \varepsilon_0]$ where $\varepsilon_0 = \varepsilon_0(\delta)$ small enough. Moreover, the estimates*

$$\|\langle x\rangle^{-s} \frac{d^j}{d\lambda^j} R(\lambda \pm i0, \varepsilon)\langle x\rangle^{-s}\| \leq C_{s,j,\delta_0}\langle\lambda\rangle^{-\frac{j+1}{2}},\ \lambda \in [\delta, \infty[,\ j = 0, \ldots, J \tag{3.27}$$

hold.

Démonstration. Under assumption (1.2) for $\rho_0 > j+1$ on V_1, and $s > j + \frac{1}{2}$, $\frac{d^j}{d\lambda^j} R_1(\lambda \pm i0)$ exists in $\mathcal{L}(0, s; 0, -s)$ satisfying the estimate

$$\|\langle x\rangle^{-s} \frac{d^j}{d\lambda^j} R_1(\lambda \pm i0)\langle x\rangle^{-s}\| \leq C_{s,j,\delta}\langle\lambda\rangle^{-\frac{j+1}{2}}, \quad \lambda \geq \delta,$$

for any $\delta > 0$. One can see this from Theorem 9.2 in [31]. With help of Neumann's series, one can obtain that

$$\|\langle x\rangle^{-s}(1 - i\varepsilon R_1(\lambda \pm i0)V_2)^{-1}\langle x\rangle^{s}\| \leq C_s,$$

for $\lambda \geq \delta$ and $\varepsilon \in]0, \varepsilon_0]$ where $\varepsilon_0 = \varepsilon_0(\delta) > 0$ small enough. Noting that

$$R(\lambda \pm i0, \varepsilon) = (1 - i\varepsilon R_1(\lambda \pm i0)V_2)^{-1} R_1(\lambda \pm i0),$$

and

$$\begin{aligned}&\frac{d}{d\lambda} R(\lambda \pm i0, \varepsilon) = (1 - i\varepsilon R_1(\lambda \pm i0)V_2)^{-1} \frac{d}{d\lambda} R_1(\lambda \pm i0)\\ &+ i\varepsilon(1 - i\varepsilon R_1(\lambda \pm i0)V_2)^{-1} \frac{d}{d\lambda} R_1(\lambda \pm i0)V_2(1 - i\varepsilon R_1(\lambda \pm i0)V_2)^{-1} R_1(\lambda \pm i0),\end{aligned}$$

by induction we have that for $s > j + \frac{1}{2}$, $R(\lambda \pm i0, \varepsilon)$ exists in $\mathcal{L}(0, s; 0, -s)$ for $\lambda \in [\delta, \infty[$, $j = 0, \ldots, J$ with the estimate (3.27). □

Before we prove Theorem 1.4.1, we check the formula (1.10) in the following lemma.

Lemma 3.4.2. *Under assumption (1.2) for $\rho_0 > 3$ and $s > \frac{5}{2}$, the formula (1.10) holds for $t > 0$ in $\mathcal{L}(0, s; 0, -s)$.*

Démonstration. Since $\rho_0 > 3$ and $s > \frac{5}{2}$, Lemma 3.4.1 holds for $J = 2$. Then from Theorem 2.1 in [78], we have that for $\varepsilon \in]0, \varepsilon_0]$ small enough

$$U(t, \varepsilon) = \frac{1}{2\pi i} \int_{\mathbb{R}} R(\lambda + i0, \varepsilon) e^{-it\lambda} d\lambda \tag{3.28}$$

holds for $t > 0$ in $\mathcal{L}(0, s; 0, -s)$. Choose $L > 0$ sufficiently large such that $\sigma_{disc}(H(\varepsilon)) \subset \mathcal{F} \triangleq \{z \in \mathbb{C} : |\Re z| < L, -L^{\frac{1}{3}} < \Im z < 0\}$. Then applying Cauchy's integral formula, we have that

$$\begin{aligned} U(t, \varepsilon)\Pi'(\varepsilon) &= \frac{1}{2\pi i}\{\int_{-\infty}^{-L} R(\lambda, \varepsilon) e^{-it\lambda} d\lambda - \int_0^{L^{\frac{1}{3}}} R(-L - i\mu, \varepsilon) e^{-it(-L-i\mu)} d\mu \\ &+ \int_{-L}^{L} R(\mu - iL, \varepsilon) e^{-it(\mu - iL)} d\mu + \int_0^{L^{\frac{1}{3}}} R(L - i\mu, \varepsilon) e^{-it(L - i\mu)} d\mu \\ &- \int_0^{L} R(\lambda - i0, \varepsilon) e^{-it\lambda} d\lambda + \int_0^{\infty} R(\lambda + i0, \varepsilon) e^{-it\lambda} d\lambda\}. \end{aligned} \tag{3.29}$$

Because the integration in (3.28) is convergent, the first term in (3.29) will tends to 0 as $L \to \infty$. Noting that for L large enough,

$$\|\langle x\rangle^{-s} R_1(\pm L - i\mu)\langle x\rangle^{-s}\| = O(L^{-\frac{1}{2}})$$

and $R(\pm L - i\mu, \varepsilon) = (1 - i\varepsilon R_1(\pm L - i\mu)V_2)^{-1} R_1(\pm L - i\mu)$, one has that

$$\|\langle x\rangle^{-s} R(\pm L - i\mu, \varepsilon)\langle x\rangle^{-s}\| = O(L^{-\frac{1}{2}}).$$

Therefore, we have that the second and fourth terms equal to $O(L^{-\frac{1}{6}})$. On the other hand, it is easy to see that the third term in (3.29) has exponential decay on L. Consequently, let L tend to the infinity, we can get (1.10).

□

To get the expansion of the semigroup for $t > 0$ large, we need to divide the integration term in (1.10) into three parts : the small energy part, the intermediate energy part and the high energy part. Let $\chi_j(\lambda)$, $j = 1, 2, 3$ be $C^\infty([0, \infty[, [0, 1])$ cutoff functions satisfying that

- $\chi_1(\lambda) + \chi_2(\lambda) + \chi_3(\lambda) = 1$, for $\lambda \in [0, \infty[$,
- supp $\chi_1 \subset [0, 2c_1\varepsilon[$, supp $\chi_2 \subset]c_1\varepsilon, 2\delta[$ and supp $\chi_3 \subset]\delta, \infty[$,
- $\chi_1(\lambda) = 1$, $\lambda \in [0, c_1\varepsilon]$; $\chi_2(\lambda)$, $\lambda \in [2c_1\varepsilon, \delta]$; $\chi_3(\lambda) = 1$, $\lambda \in [2\delta, \infty[$,
- For $k \in \mathbb{N}$, $|\frac{d^k}{d\lambda^k}\chi_1(\lambda)| \le C_k\varepsilon^{-k}$; $|\frac{d^k}{d\lambda^k}\chi_2(\lambda)| \le C_k\varepsilon^{-k}$, for $\lambda \in [c_1\varepsilon, 2c_1\varepsilon]$ and $|\frac{d^k}{d\lambda^k}\chi_2(\lambda)| \le C_k\delta^{-k}$, for $\lambda \in [\delta, 2\delta]$; $|\frac{d^k}{d\lambda^k}\chi_3(\lambda)| \le C_k\delta^{-k}$.

Denote the integration in (1.10) by $I(t)$. Let

$$I_j(t) \;=\; \int_0^{+\infty} e^{-it\lambda}(R(\lambda + i0, \varepsilon) - R(\lambda - i0, \varepsilon))\chi_j(\lambda)d\lambda,$$

and thus $I(t) = I_1(t) + I_2(t) + I_3(t)$.

Proof of Theorem 1.4.1. Applying the stationary method and interpolation, we obtain that

$$\begin{aligned}
\|\langle x\rangle^{-s} I_2(t)\langle x\rangle^{-s}\| &\le O(\varepsilon^{-\frac{N+\alpha}{2}-\frac{1}{2}} t^{-\frac{N+\alpha}{2}-1}),\\
\|\langle x\rangle^{-s} I_3(t)\langle x\rangle^{-s}\| &\le O(t^{-N+\alpha}).
\end{aligned}$$

for $\rho_0 > 2N + 1$, $s \in]N + \frac{1}{2}, \frac{\rho_0}{2}]$ and $\alpha =]0, \min\{1, s - N - \frac{1}{2}\}[$. Here we use the estimates (3.23), (3.27) and the properties of the cutoff functions.

In light of Lemma 10.2 in [31], one has that for $s \in]N + \frac{1}{2}, \frac{\rho_0}{2}]$ and $\alpha =]0, \min\{1, s - N - \frac{1}{2}\}[$,

$$\|\langle x\rangle^{-s}\int_0^\infty \chi_1(\lambda) R^{(1)}_{N+\alpha}(\lambda, \varepsilon) e^{-it\lambda} d\lambda \langle x\rangle^{-s}\| = O(t^{-\frac{N+\alpha}{2}-1}).$$

On the other hand, note that

$$\begin{aligned}
&\int_0^\infty \chi_1(\lambda)(\lambda + i0)^{\frac{k}{2}} e^{-it\lambda} d\lambda\\
=\;& \int_0^\infty (\lambda + i0)^{\frac{k}{2}} e^{-it\lambda} d\lambda + \int_0^\infty (\chi_1(\lambda) - 1)(\lambda + i0)^{\frac{k}{2}} e^{-it\lambda} d\lambda\\
\triangleq\;& I + II.
\end{aligned}$$

Due to the Fourier transform of the homogeneous distribution $\chi_+^s(\lambda)$ for $s \notin \mathbb{N}$ (See [24]), we have that

$$I = -2\sin\frac{k\pi}{4}\Gamma(\frac{k}{2} + 1)t^{-\frac{k}{2}-1}.$$

For the second term, with help of integration by parts and the properties of χ_1, it follows that

$$II = O(\varepsilon^{\frac{k}{2}+1-l}t^{-l}),$$

for any $l \in \mathbb{N}$. Since $(\lambda + i0)^{\frac{k}{2}} = (-1)^k(\lambda - i0)^{\frac{k}{2}}$ and $R^{(2)}_{N+\alpha}(\lambda + i0, \varepsilon) = R^{(2)}_{N+\alpha}(\lambda - i0, \varepsilon)$, we have that

$$I_1(t) = c_1 t^{-\frac{3}{2}} R_1(\varepsilon) + \sum_{j=2}^{[\frac{N+1}{2}]} c_{2j-1} \frac{t^{-\frac{2j+1}{2}}}{\varepsilon^j} R_{2j-1}(\varepsilon) + O(\varepsilon^{-\frac{N+\alpha}{2}-\frac{1}{2}} t^{-\frac{N+\alpha}{2}-1}),$$

where $c_k = -4\sin\frac{k\pi}{4}\Gamma(\frac{k}{2}+1)$. Then let

$$T_j(\varepsilon) = \frac{c_{2j-1}}{2\pi i} R_{2j-1}(\varepsilon),\ j = 1, \ldots, [\frac{N+1}{2}],$$

and then the expansion (1.11) in Theorem 1.4.1 can be obtained. By Lemma 3.3.1 and 3.3.5, $T_j(\varepsilon)$, $j = 1, \ldots, [\frac{N+1}{2}]$ are finite-rank operators. In particular, by Remark 3.3.9, one can obtain that $T_1(\varepsilon)$ is of rank one. □

3.5 The four-dimensional case I

In this section, we will prove Theorem 1.4.5. We consider the case that $n = 4$ and that 0 is only a resonance but not an eigenvalue of H_1.

3.5.1 Resolvent analysis

As in the 3-dimensional eigenvalue case, we will first discuss the behavior of the resolvent near zero. It is known that $\dim \mathcal{M} = 1$ (See [33]). Let $0 \neq \phi \in \mathcal{M}$ be a resonant state of H_1 at 0 satisfying $\langle \phi, -V_1\phi\rangle = 1$ and $\langle V_1\phi, 1\rangle \neq 0$ by (3.5).

By (3.2.2), it is easy to check that

$$\begin{aligned} W(z,\varepsilon) &= 1 + G_0V_1 - i\varepsilon G_0V_2 + \ln z \sum_{j=1}^{N} z^j G_j^1(V_1 - i\varepsilon V_2) \\ &\quad + \sum_{j=1}^{N} z^j G_j^0(V_1 - i\varepsilon V_2) + O(z^{N+\alpha}). \end{aligned} \tag{3.30}$$

Thus, by Neumann's series for δ, $\varepsilon_0 > 0$ sufficiently small, we have the expansion

$$E(z,\varepsilon) = \sum_{j=0}^{N}\sum_{k=0}^{j} z^j \ln^k z E_j^k(\varepsilon) + O(z^{N+\alpha}), \tag{3.31}$$

for $z \in \Omega$ and $\varepsilon \in]0, \varepsilon_0]$, where each $E_j^k(\varepsilon)$ is uniformly bounded in $\mathcal{L}(1, -s_j; 1, -s_j)$ for $s_j > 2j+1$ and for $k \geq 1$, $E_j^k(\varepsilon)$ is of finite rank. More precisely, it can be computed that

$$\begin{aligned} E_0^0(\varepsilon) &= (Q'(1 + G_0V_1 - i\varepsilon G_0V_2)Q')^{-1}Q' \\ &= (Q'(1 + G_0V_1)Q')^{-1}Q' + O(\varepsilon), \\ E_1^0(\varepsilon) &= -E_0^0(\varepsilon)G_1^0(V_1 - i\varepsilon V_2)E_0^0(\varepsilon), \\ E_1^1(\varepsilon) &= -E_0^0(\varepsilon)G_1^1(V_1 - i\varepsilon V_2)E_0^0(\varepsilon). \end{aligned}$$

Thus we have the following lemma about $R_I(z,\varepsilon)$ in $\Omega = B(0, 2\delta) \setminus \mathbb{R}_+$.

Lemma 3.5.1. *Under assumption of Theorem 1.4.5, we have the asymptotic expansion in* $\mathcal{L}(-1,s;1,-s)$

$$R_I(z,\varepsilon)=\sum_{j=0}^{N}\sum_{k=0}^{j} z^j \ln^k z R_{1,j}^k(\varepsilon)+R_{1,N+\alpha}(z,\varepsilon), \tag{3.32}$$

for $z\in\Omega$*, where*

$$\begin{aligned}
R_{1,0}^0(\varepsilon) &= E_0^0(\varepsilon)G_0,\\
R_{1,1}^1(\varepsilon) &= E_1^1(\varepsilon)G_0+E_0^0(\varepsilon)G_1^1,\\
R_{1,1}^0(\varepsilon) &= E_1^0(\varepsilon)G_0+E_0^0(\varepsilon)G_1^0,\\
R_{1,j}^k(\varepsilon) &= E_j^k(\varepsilon)G_0+\sum_{l=1}^{j-k}E_{j-l}^k(\varepsilon)G_l^0+\sum_{l=0}^{j-k+1}E_{j-l}^{k-1}(\varepsilon)G_l^1\in\mathcal{L}(-1,s_j;1,-s_j),
\end{aligned}$$

for $k\le j$ *and* $s_j>2j+1$*. Furthermore,* $R_{1,j}^k$ *for* $k\ge 1$ *is of finite rank and the r-th derivative of the remainder* $R_{1,N+\alpha}(z,\varepsilon)\in\mathcal{L}(-1,s;1,-s)$ *has order* $O(z^{N+\alpha-r})$ *for* $r=1,2,\dots,2N$.

Below, we will state the expansion of $R_{II}(z,\varepsilon)$ in $\mathcal{L}(-1,s;1,-s)$, $s>4N+2$ for $z\in\Omega$. By a direct calculation, we can compute that

$$\begin{aligned}
E_+(z,\varepsilon) &= (1+\varepsilon E_{+,0}^0(\varepsilon)+\sum_{j=1}^{N}\sum_{k=0}^{j}z^j\ln^k zE_{+,j}^k(\varepsilon)+O(z^{N+\alpha}))T, &(3.33)\\
E_-(z,\varepsilon) &= S(1+\varepsilon E_{-,0}^0(\varepsilon)+\sum_{j=1}^{N}\sum_{k=0}^{j}z^j\ln^k zE_{-,j}^k(\varepsilon)+O(z^{N+\alpha})), &(3.34)
\end{aligned}$$

where

$$\begin{aligned}
E_{+,0}^0(\varepsilon) &= iE_0^0(\varepsilon)G_0V_2,\\
E_{+,1}^0(\varepsilon) &= i\varepsilon E_1^0(\varepsilon)G_0V_2-E_0^0(\varepsilon)G_1^0(V_1-i\varepsilon V_2)=-E_0^0(\varepsilon)G_1^0(V_1-i\varepsilon V_2)(1+\varepsilon E_{+,0}^0(\varepsilon)),\\
E_{+,1}^1(\varepsilon) &= i\varepsilon E_1^1(\varepsilon)G_0V_2-E_0^0(\varepsilon)G_1^1(V_1-i\varepsilon V_2)=-E_0^0(\varepsilon)G_1^1(V_1-i\varepsilon V_2)(1+\varepsilon E_{+,0}^0(\varepsilon)),
\end{aligned}$$

and

$$\begin{aligned}
E_{-,0}^0(\varepsilon) &= iG_0V_2E_0^0(\varepsilon),\\
E_{-,1}^0(\varepsilon) &= i\varepsilon G_0V_2E_1^0(\varepsilon)-G_1^0(V_1-i\varepsilon V_2)E_0^0(\varepsilon)=-(1+\varepsilon E_{-,0}^0(\varepsilon))G_1^0(V_1-i\varepsilon V_2)E_0^0(\varepsilon),\\
E_{-,1}^1(\varepsilon) &= i\varepsilon G_0V_2E_1^1(\varepsilon)-G_1^1(V_1-i\varepsilon V_2)E_0^0(\varepsilon)=-(1+\varepsilon E_{-,0}^0(\varepsilon))G_1^1(V_1-i\varepsilon V_2)E_0^0(\varepsilon),
\end{aligned}$$

and other terms can be computed explicitly. Then by (3.30) and (3.31), we have the expansion of the scalar function $E_{-+}(z,\varepsilon)$ that

$$E_{-+}(z,\varepsilon) = \varepsilon E_{-+,0}^0(\varepsilon)+\sum_{j=1}^{N}\sum_{k=0}^{j}z^j\ln^k zE_{-+,j}^k(\varepsilon)+O(z^{N+\alpha}),$$

where

$$\begin{aligned}
E_{-+,0}^0 &= i\langle V_2\phi,\phi\rangle-\varepsilon\langle V_2E_0^0(\varepsilon)G_0V_2\phi,\phi\rangle=i\langle V_2\phi,\phi\rangle+O(\varepsilon),\\
E_{-+,1}^1 &= -S(G_1^1(V_1-i\varepsilon V_2)+i\varepsilon G_0V_2E_0^0(\varepsilon)G_1^1(V_1-i\varepsilon V_2)\\
&\quad +i\varepsilon G_1^1(V_1-i\varepsilon V_2)E_0^0(\varepsilon)G_0V_2+\varepsilon^2G_0V_2E_1^1(\varepsilon)G_0V_2)T\\
&= -S(1+\varepsilon E_{-,0}^0(\varepsilon))G_1^1(V_1-i\varepsilon V_2)(1+\varepsilon E_{+,0}^0(\varepsilon))T\\
&= -\frac{|\langle V_1\phi,1\rangle|^2}{(4\pi)^2}+O(\varepsilon),\\
E_{-+,1}^0 &= \langle G_1^0(V_1-i\varepsilon V_2)\phi,V_1\phi\rangle+i\varepsilon\langle G_1^0(V_1-i\varepsilon V_2)E_0^0(\varepsilon)G_0V_2\phi,V_1\phi\rangle\\
&\quad +i\varepsilon\langle G_0^0V_2E_0^0(\varepsilon)G_1^0(V_1-i\varepsilon V_2)\phi,V_1\phi\rangle-\varepsilon^2\langle V_2E_1^0(\varepsilon)G_0V_2\phi,\phi\rangle
\end{aligned}$$

and other terms can be also calculated.

Denote $F_0(z,\varepsilon) = \frac{|\langle V_1\phi,1\rangle|^2}{(4\pi)^2}(i\varepsilon a - z\ln z)$ where $a = \frac{(4\pi)^2\langle V_2\phi,\phi\rangle}{|\langle V_1\phi,1\rangle|^2} > 0$. Consider the equation $F_0(z,\varepsilon) = 0$. One can check that there exists a unique solution $z_0(\varepsilon) = r_0(\varepsilon)e^{i\theta_0(\varepsilon)}$ in $B_-(0,\delta)$. Furthermore, $r_0(\varepsilon) \to 0_+$ and $\theta_0(\varepsilon) \to \frac{3\pi}{2}_+$ as $\varepsilon \to 0_+$. On the other hand, we have that $M^{-1}\varepsilon \le r_0(\varepsilon)|\ln r_0(\varepsilon)| \le M\varepsilon$ for some positive constant $M > 0$. Thus $C^{-1}\varepsilon|\ln\varepsilon|^{-1} \le r_0(\varepsilon) \le C\varepsilon|\ln\varepsilon|^{-1}$ for some positive constant C. Then we can choose some constant $c > 0$ such that for $z \in \partial B(z_0(\varepsilon), c\varepsilon|\ln\varepsilon|^{-1})$,

$$|F_0(z,\varepsilon)| \ge a_1\varepsilon,\ |E_{-+}(z,\varepsilon) - F_0(z,\varepsilon)| = O(\varepsilon^2 + \varepsilon z\ln z + z) \le a_2\varepsilon|\ln\varepsilon|^{-1},$$

for some positive constants a_1, $a_2 > 0$. By the analyticity on z of $F_0(z,\varepsilon)$ and $E_{-+}(z,\varepsilon)$ and using the Rouché's Theorem as in [78], it can be prove that there exist $\varepsilon_0 > 0$ small enough and $c > 0$ such that $E_{-+}(z,\varepsilon)$ has a zero $z_1(\varepsilon)$ in disc $B_0 \triangleq B(z_0(\varepsilon), c\varepsilon|\ln\varepsilon|^{-1})$ and dist $(B_0, \mathbb{R}_+) > c_1\varepsilon|\ln\varepsilon|^{-1}$ for some $c_1 > 0$ and $\varepsilon \in]0,\varepsilon_0]$. Furthermore, we can compare this with Theorem 1.2(b) in [77]. There, it needs some additional condition (1.8). Actually, this condition is only needed in the case $\nu_1 = \frac{1}{2}$. And for $\nu_1 \in]\frac{1}{2}, 1]$, in (3.30) in the proof of Theorem 1.2 in [77], the term of order γz_{ν_1} can be treated as a high-order term as we discuss here.

Denote $\Omega_1 = B(0,2\delta) \setminus \{B(0, c_2\varepsilon|\ln\varepsilon|^{-1}) \cup \mathbb{R}_+\}$ and $\Omega_2 = B(0, 2c_2\varepsilon|\ln\varepsilon|^{-1}) \setminus \{B_0 \cup \mathbb{R}_+\}$. Here c_2 is chosen such that $|z\ln z| \ge 2|z_0(\varepsilon)\ln z_0(\varepsilon)|$ for $z \in \Omega_1$.

Suppose $z \in \Omega_1$ and then we have

$$C_1^{-1}|z||\ln z| \le |F_0(z,\varepsilon)| \le C_1|z||\ln z|,$$

for some $C_1 > 0$. Thus by

$$E_{-+}(z,\varepsilon) = F_0(z,\varepsilon)(1 + O(|\ln z|^{-1})),$$

we have

$$E_{-+}(z,\varepsilon)^{-1} = O(|z\ln z|^{-1}).$$

Noting that

$$\frac{d^j}{dz^j}E_{-+}(z,\varepsilon) = \begin{cases} O(\ln z), & j = 1, \\ O(z^{-j+1}), & j \ge 2, \end{cases}$$

and

$$\frac{d^j}{dz^j}E_{-+}(z,\varepsilon)^{-1} = \sum_{\substack{j_1+\cdots+j_l=j \\ j_k\ge 1, k=1,\ldots,l}} C_{j_1,\ldots,j_l}E_{-+}(z,\varepsilon)^{-l-1}\prod_{k=1}^{l}\left(\frac{d^{j_k}}{dz^{j_k}}E_{-+}(z,\varepsilon)\right),$$

one can obtain that

$$\frac{d^j}{dz^j}E_{-+}(z,\varepsilon)^{-1} = O(z^{-j-1}\ln^{-1} z). \tag{3.35}$$

By Lemma 3.2.2 the following estimates

$$\|\langle x\rangle^{-s}\frac{d^j}{dz^j}E_+(z,\varepsilon)\|_{l^2(\mathbb{C})\to L^2(\mathbb{R}^4)} = \begin{cases} O(1), & j = 0, \\ O(\ln z), & j = 1, \\ O(z^{-j+1}), & 2 \le j \le 2N, \end{cases}$$

$$\|\frac{d^j}{dz^j}E_-(z,\varepsilon)\langle x\rangle^{s}\|_{L^2(\mathbb{R}^4)\to l^2(\mathbb{C})} = \begin{cases} O(1), & j = 0, \\ O(\ln z), & j = 1, \\ O(z^{-j+1}), & 2 \le j \le 2N, \end{cases}$$

hold. Consequently, we can get the following lemma.

Lemma 3.5.2. *Under assumption of Thorem 1.4.5 and for $z \in \Omega_1$, we have the estimate*

$$\|\langle x\rangle^{-s}\frac{d^j}{dz^j}R_{II}(z,\varepsilon)\langle x\rangle^{-s}\| = O(z^{-j-1}|\ln z|^{-1}),\ j = 0,\ldots,2N.$$

On the other hand, suppose that $z \in \Omega_2$ and we can deduce that

$$C_2^{-1}\varepsilon|\ln\varepsilon|^{-1} \le |F_0(z,\varepsilon)| \le C_2\varepsilon|\ln\varepsilon|^{-1},$$
$$|z| \le c_1\varepsilon|\ln\varepsilon|^{-1}, |z\ln z| \le c_2'\varepsilon,$$

for some $C_2, c_2' > 0$. Let $\widetilde{F}_0(z,\varepsilon) = E^0_{-+,0}(z,\varepsilon) + z\ln z E^1_{-+,1} = F_0(z,\varepsilon) + O(\varepsilon^2 + \varepsilon z\ln z)$. Thus one can prove that $\widetilde{F}_0(z,\varepsilon)$ is invertible in Ω_2 and the inverse has the expansion

$$\widetilde{F}_0(z,\varepsilon)^{-1} = \sum_{j=0}^{N} \frac{A_j(\varepsilon)}{\varepsilon^{j+1}}(z\ln z)^j + \frac{A_{N+1}(z,\varepsilon)}{\varepsilon^{N+1}},$$

where

$$A_j(\varepsilon) = \frac{(-E^1_{-+,1}(\varepsilon))^j}{(i\langle V_2\phi,\phi\rangle - \varepsilon\langle V_2E_0^0(\varepsilon)G_0^0V_2\phi,\phi\rangle)^{j+1}},\ j = 0,\dots,N$$

are uniformly bounded functions of ε and

$$A_{N+1}(z,\varepsilon) = \frac{\varepsilon(-E^1_{-+,1}(\varepsilon))^{N+1}}{(i\langle V_2\phi,\phi\rangle - \varepsilon\langle V_2E_0^0(\varepsilon)G_0^0V_2\phi,\phi\rangle)^{N+1}\widetilde{F}_0(z,\varepsilon)} = O((z\ln z)^{N+1}).$$

Therefore, using Neumann's series, we can get the expansion of $E_{-+}(z,\varepsilon)$ for $z \in \Omega_2$ as follows

$$E_{-+}(z,\varepsilon)^{-1} = \frac{C_0^0(\varepsilon)}{\varepsilon} + \sum_{j=1}^{N}\sum_{k=0}^{j} \frac{C_j^k(\varepsilon)}{\varepsilon^{k+1}} z^j \ln^k z + \frac{O(z^{N+\alpha})}{\varepsilon^{N+\alpha+1}},$$

where

$$\begin{aligned} C_0^0(\varepsilon) &= A_0(\varepsilon), \\ C_1^0(\varepsilon) &= -A_0(\varepsilon)E^0_{-+,1}(\varepsilon), \\ C_1^1(\varepsilon) &= A_1(\varepsilon) = -A_0(\varepsilon)^2E^1_{-+,1}(\varepsilon), \end{aligned}$$

and the other $C_j^k(\varepsilon)$, $j = 2,\dots,N,\ k = 0,\dots,j$ are uniformly bounded on ε and can be calculated directly. Furthermore the j-th derivative of the remainder is of order $O(\frac{z^{N-j+\alpha}}{\varepsilon^{N+\alpha+1}})$ for $j = 0,1,\dots,2N$. Thus using (3.33) and (3.34), we have the expansion

$$\widetilde{E}(z,\varepsilon) = \sum_{j=0}^{N}\sum_{k=0}^{j} z^j\ln^k z \frac{W_j^k(\varepsilon)}{\varepsilon^{k+1}} + \frac{O(z^{N+\alpha})}{\varepsilon^{N+\alpha+1}},$$

in $\mathcal{L}(1,-s;1,-s)$,where

$$\begin{aligned} W_0^0(\varepsilon) &= -A_0(\varepsilon)(1+\varepsilon E^0_{+,0}(\varepsilon))Q(1+\varepsilon E^0_{-,0}(\varepsilon)), \\ W_1^0(\varepsilon) &= -A_0(\varepsilon)((1+\varepsilon E^0_{+,0}(\varepsilon))QE^0_{-,1}(\varepsilon) + E^0_{+,1}(\varepsilon)Q(1+\varepsilon E^0_{-,0}(\varepsilon))) \\ &\quad -C_1^0(1+\varepsilon E^0_{+,0}(\varepsilon))Q(1+\varepsilon E^0_{-,0}(\varepsilon)), \\ W_1^1(\varepsilon) &= -\varepsilon A_0(\varepsilon)((1+\varepsilon E^0_{+,0}(\varepsilon))QE^1_{-,1}(\varepsilon) + E^1_{+,1}(\varepsilon)Q(1+\varepsilon E^0_{-,0}(\varepsilon))) \\ &\quad -C_1^1(1+\varepsilon E^0_{+,0}(\varepsilon))Q(1+\varepsilon E^0_{-,0}(\varepsilon)) \end{aligned}$$

and we omit the expressions of other terms. Here $\alpha \in]0,\min\{1,\frac{s}{2}-N-\frac{1}{2}\}[$. Then by Lemma 3.2.2, one has the expansion for $z \in \Omega_2$.

Lemma 3.5.3. *Under assumption of Theorem 1.4.5, we have the expansion in* $\mathcal{L}(-1,s;1,-s)$

$$R_{II}(z,\varepsilon) = \sum_{j=0}^{N}\sum_{k=0}^{j} z^j \ln^k z \frac{R_{2,j}^k(\varepsilon)}{\varepsilon^{k+1}} + \frac{R_{2,N+\alpha}(z,\varepsilon)}{\varepsilon^{N+\alpha+1}},$$

for $z \in \Omega_2$*, where*

$$\begin{aligned} R_{2,0}^0(\varepsilon) &= W_0^0(\varepsilon)G_0, \\ R_{2,1}^0(\varepsilon) &= W_0^0(\varepsilon)G_1^0 + W_1^0(\varepsilon)G_0, \\ R_{2,1}^1(\varepsilon) &= \varepsilon W_0^0(\varepsilon)G_1^1 + W_1^1(\varepsilon)G_0, \end{aligned}$$

and the other terms are uniformly bounded on ε *in* $\mathcal{L}(-1,s_j;1,-s_j)$ *for* $s_j > 2j+1$*. Furthermore, the j-th derivative of the remainder* $R_{2,N+\alpha}(z,\varepsilon)$ *is of* $O(z^{N+\alpha-j})$ *in* $\mathcal{L}(-1,s;1,-s)$ *for* $j = 0,\dots,2N$*.*

By Lemma 3.5.1, 3.5.2 and 3.5.3, we have the expansion of the resolvent near $z = 0$.

Theorem 3.5.4. *Suppose that* $N > 3$*,* $\rho_0 > 4N+2$ *and* $\varepsilon \in]0,\varepsilon_0]$*. Then for* $z \in \Omega_1$*, one has the expansions of* $R(z,\varepsilon)$ *and its derivatives as follows*

$$\frac{d^j}{dz^j}R(z,\varepsilon) = O(z^{-j-1}|\ln z|^{-1}) \tag{3.36}$$

in $\mathcal{L}(-1,s_j;1,-s_j)$*,* $s_j > 2j+1$*,* $j = 0,\dots,2N$*. For* $z \in \Omega_2$*, we have the following expansion of* $R(z,\varepsilon)$ *in* $\mathcal{L}(-1,s;1,-s)$

$$R(z,\varepsilon) = \sum_{j=0}^{N}\sum_{k=0}^{j} z^j \ln^k z \frac{R_j^k(\varepsilon)}{\varepsilon^{k+1}} + \frac{R_{N+\alpha}(z,\varepsilon)}{\varepsilon^{N+\alpha+1}},$$

for $s \in]2N+1, \frac{\rho_0}{2}]$ *and* $\alpha \in]0, \min\{1, \frac{s}{2}-N-\frac{1}{2}\}[$*. Here* $R_j^k(\varepsilon) = \varepsilon^{k+1}R_{1,j}^k(\varepsilon) + R_{2,j}^k(\varepsilon)$ *and* $R_{N+\alpha}(z,\varepsilon) = \varepsilon^{N+\alpha+1}R_{1,N+\alpha}(z,\varepsilon) + R_{2,N+\alpha}(z,\varepsilon)$*.*

Remark 3.5.5. *We can compute that*

$$\begin{aligned} R_1^1(\varepsilon) &= \varepsilon^2 R_{1,1}^1(\varepsilon) + R_{2,1}^1(\varepsilon) \\ &= (\varepsilon E_0^0(\varepsilon) - A_0(\varepsilon)K(\varepsilon))G_1^1\{\varepsilon - (V_1 - i\varepsilon V_2)(\varepsilon E_0^0(\varepsilon) - A_0(\varepsilon)K(\varepsilon))G_0\}, \end{aligned}$$

where $K(\varepsilon) = (1+\varepsilon E_{+,0}^0(\varepsilon))Q(1+\varepsilon E_{-,0}^0(\varepsilon))$*. Since* G_1^1 *is of rank one, the rank of* $R_{1,1}^1(\varepsilon) + \varepsilon^{-2}R_{2,1}^1(\varepsilon)$ *is at most one. Actually, as* $\varepsilon \to 0_+$*, we have*

$$\lim_{\varepsilon\to 0_+} R_1^1(\varepsilon) = \lim_{\varepsilon\to 0_+} R_{2,1}^1(\varepsilon) = \frac{|\langle V_1\phi, 1\rangle|^2}{(4\pi)^2|\langle V_2\phi,\phi\rangle|^2}\langle\cdot,\phi\rangle\phi.$$

Therefore, $R_1^1(\varepsilon)$ *is of rank one for* ε_0 *sufficiently small.*

3.5.2 Expansion of the semigroup

Consequently, similar to the 3-dimensional case we can obtain the large-time expansion of $U(t,\varepsilon)$ following Theorem 3.5.4. First we state the following Fourier transform in the sense of distribution. For the proof, one can see Section 2.4 of Chapter II in [24].

Lemma 3.5.6. *For* $\gamma \in \mathbb{R}$ *and* $k \in \mathbb{N}$*, we have that*

$$\int_0^\infty (x+i0)^\gamma \ln^k(x+i0)e^{-itx}dx = \sum_{l=0}^{k} e^{(l+\frac{1}{2})\pi i}C_k^l \frac{d^{k-l}}{d\gamma^{k-l}}\{e^{i\frac{\pi\gamma}{2}}\Gamma(\gamma+1)\}t^{-\gamma-1}\ln^l t,$$

for $t > 0$*.*

Proof of Theorem 1.4.5 : Choose the cutoff functions $\chi_j(\lambda)$, $j = 1, 2, 3$ satisfying that

- $\chi_j(\lambda) \in C^\infty([0, \infty[\,;[0,1])$, $j = 0, 1, 2, 3$,
- $\chi_1(\lambda) + \chi_2(\lambda) + \chi_3(\lambda) = 1$, for $\lambda \in [0, \infty[$,
- supp $\chi_1 \subset [0, 2c_1\varepsilon|\ln\varepsilon|^{-1}[$, supp $\chi_2 \subset]c_1\varepsilon|\ln\varepsilon|^{-1}, 2\delta[$ and supp $\chi_3 \subset]\delta, \infty[$,
- $\chi_1(\lambda) = 1$, $\lambda \in [0, c_1\varepsilon|\ln\varepsilon|^{-1}]$; $\chi_2(\lambda)$, $\lambda \in [2c_1\varepsilon|\ln\varepsilon|^{-1}, \delta]$; $\chi_3(\lambda) = 1$, $\lambda \in [2\delta, \infty[$,
- For $k \in \mathbb{N}$, $|\frac{d^k}{d\lambda^k}\chi_1(\lambda)| \leq C_k\varepsilon^{-k}|\ln\varepsilon|^k$; $|\frac{d^k}{d\lambda^k}\chi_2(\lambda)| \leq C_k\varepsilon^{-k}|\ln\varepsilon|^k$, for $\lambda \in [c_1\varepsilon|\ln\varepsilon|^{-1}, 2c_1\varepsilon|\ln\varepsilon|^{-1}]$ and $|\frac{d^k}{d\lambda^k}\chi_2(\lambda)| \leq C_k\delta^{-k}$, for $\lambda \in [\delta, 2\delta]$; $|\frac{d^k}{d\lambda^k}\chi_3(\lambda)| \leq C_k\delta^{-k}$.

Denote the integration in (1.10) by $I(t)$. Let

$$I_j(t) = \int_0^{+\infty} e^{-it\lambda}(R(\lambda + i0, \varepsilon) - R(\lambda - i0, \varepsilon))\chi_j(\lambda)d\lambda, \ j = 1, 2, 3,$$

and thus $I(t) = I_1(t) + I_2(t) + I_3(t)$.

For $I_3(t)$, similar to the 3-dimensional case, one has that for any $s > j + \frac{1}{2}$, $j \geq 2$ and $\rho_0 > j + 1$,

$$\|\langle x\rangle^{-s}I_3(t)\langle x\rangle^{-s}\| \leq O(t^{-j}).$$

For $I_2(t)$, by using the stationary phase method and the interpolation, we can get that

$$\|\langle x\rangle^{-s}I_2(t)\langle x\rangle^{-s}\| \leq O((\frac{\varepsilon t}{|\ln\varepsilon|})^{-N-1-\alpha}),$$

following (3.36) where $\alpha \in]0, \min\{1, \frac{s-2N-1}{2}\}[$.

For $I_1(t)$, we first note that for $\lambda > 0$

$$\begin{aligned} z^j \ln^k z|_{\lambda-i0}^{\lambda+i0} &= \lambda^j \ln^k \lambda - \lambda^j(\ln\lambda + 2\pi i)^k \\ &= -\lambda^j \sum_{l=0}^{k-1} C_k^l (2\pi i)^{k-l} \ln^l \lambda. \end{aligned}$$

Thus in light of Lemma 3.5.6, it follows that the following integral holds in the sense of distribution

$$\begin{aligned} &\int_0^\infty ((\lambda + i0)^j \ln^k(\lambda + i0) - (\lambda - i0)^j \ln^k(\lambda - i0))e^{-it\lambda}d\lambda \\ = &-\sum_{l=0}^{k-1} C_k^l (2\pi i)^{k-l} \int_0^\infty \lambda^j \ln^l \lambda e^{-it\lambda}d\lambda \\ = &-\sum_{l=0}^{k-1} C_k^l (2\pi i)^{k-l} \sum_{h=0}^{l} e^{(h+\frac{1}{2})\pi i} C_l^h \frac{d^{l-h}}{d\gamma^{l-h}}\{e^{i\frac{\pi\gamma}{2}}\Gamma(\gamma + 1)\}|_{\gamma=j} t^{-j-1} \ln^h t \\ = &\ t^{-j-1} \sum_{h=0}^{k-1} c_j^{k,h} \ln^h t, \end{aligned}$$

where $c_j^{k,h} = -\sum_{l=h}^{k-1} C_k^l (2\pi i)^{k-l} e^{(h+\frac{1}{2})\pi i} C_l^h \frac{d^{l-h}}{d\gamma^{l-h}} \{e^{i\frac{\pi\gamma}{2}}\Gamma(\gamma+1)\}|_{\gamma=j}$. Therefore, together with Theorem 3.5.4, we have that

$$\begin{aligned} I_1(t) &= \sum_{j=1}^{N}\sum_{k=1}^{j}\frac{1}{\varepsilon^{k+1}}\int_0^\infty (1+\chi_1(\lambda)-1)((\lambda+i0)^j \ln^k(\lambda+i0) \\ &\quad -(\lambda-i0)^j\ln^k(\lambda-i0))e^{-it\lambda}d\lambda R_j^k(\varepsilon) + (\varepsilon|\ln\varepsilon|^{-1})^{-N-1-\alpha}O(t^{-N-1-\alpha}) \\ &= \sum_{j=1}^{N} t^{-j-1}\sum_{k=1}^{j}\frac{1}{\varepsilon^{k+1}}\sum_{l=0}^{k-1} c_j^{k,l}\ln^l t R_j^k(\varepsilon) + \sum_{j=1}^{N}\sum_{k=1}^{j}\frac{(\varepsilon|\ln\varepsilon|^{-1})^{j-N-\alpha}}{\varepsilon^{k+1}}O(t^{-N-1-\alpha}) \\ &\quad +(\varepsilon|\ln\varepsilon|^{-1})^{-N-1-\alpha}O(t^{-N-1-\alpha}) \\ &= 2\pi i\sum_{j=1}^{N}\varepsilon^{-1-j}t^{-1-j}\sum_{l=0}^{j-1}\ln^l t T_j^l(\varepsilon) + (\varepsilon|\ln\varepsilon|^{-1})^{-N-1-\alpha}O(t^{-N-1-\alpha}), \end{aligned}$$

where $T_j^l(\varepsilon) = \frac{1}{2\pi i}\sum_{k=l+1}^{j}\varepsilon^{j-k}c_j^{k,l}R_j^k(\varepsilon) \in \mathcal{L}(0,s_j;0,-s_j)$, for $s_j > 2j+1$. Furthermore, by Lemma 3.5.2 and 3.5.3, it is easy to see that each T_j^l is of finite rank. Thus (1.13) can be obtained. In particular, by Remark 3.5.5, $T_1^0(\varepsilon)$ is of rank one.

□

3.6 The four-dimensional case II

In this section, we will prove Theorem 1.4.7. Here we suppose that zero is not only a resonance of H_1 but also an eigenvalue for dimension $n = 4$. The details of the proof will be omitted and we only need to show the low-energy analysis of the resolvent $R(z,\varepsilon)$ near $z = 0$, since it is similar to the above two cases.

Without loss of generality, one can choose the basis $\{\phi_j\}_{j=1}^m$ of $\mathcal{M}$ defined in Section 2 such that ϕ_1 is a zero-resonant state of H_1 and $\{\phi_j\}_{j=2}^m$ is a collection of zero-eigenfunctions of H_1. Thus from Lemma 3.2.4 (b), it follows that $G_1^1 V_1\phi_1 \neq 0$ and $G_1^1 V_1\phi_j = 0$, for $j = 2,\dots,m$.

It is noted that the expansions (3.30) (resp. (3.31), (3.32), (3.33) and (3.34)) of $W(z,\varepsilon)$ (resp. $E(z,\varepsilon)$, $R_I(z,\varepsilon)$, $E_+(z,\varepsilon)$ and $E_-(z,\varepsilon)$) is still valid for $z\in\Omega$ and $\varepsilon\in]0,\varepsilon_0]$ small enough, although the expressions of the term in these expansions may be different from those in Section 5. On the other hand, one can compute that

$$\begin{aligned} E_{-+}(z,\varepsilon) &= i\varepsilon\mathcal{V} + \varepsilon^2 E_{-+,0}^0(\varepsilon) + z\ln z(\begin{pmatrix} -\frac{|\langle V_1\phi_1,1\rangle|^2}{(4\pi)^2} & 0_{1\times(m-1)} \\ 0_{(m-1)\times 1} & 0_{(m-1)\times(m-1)}\end{pmatrix} + \varepsilon E_{-+,1}^1(\varepsilon)) \\ &\quad +z(\mathcal{U}+\varepsilon E_{-+,1}^0(\varepsilon)) \\ &\quad +\sum_{j=2}^{N}\sum_{k=0}^{j} z^j\ln^k z E_{-+,j}^k(\varepsilon) + z^{N+\alpha}E_{-+,N+\alpha}(z,\varepsilon), \end{aligned}$$

where $0_{p\times q}$ is the zero matrix of size $p \times q$ and

$$\begin{aligned}
\mathcal{V} &= SG_0V_2T = T^*V_2T,\\
\mathcal{U} &= \begin{pmatrix} 0 & 0_{1\times(m-1)} \\ 0_{(m-1)\times 1} & \tilde{\mathcal{U}} \end{pmatrix} + \begin{pmatrix} \langle G_1^0 V_1\phi_1, V_1\phi_1\rangle & \gamma^* \\ \gamma & 0_{(m-1)\times(m-1)} \end{pmatrix},\\
\tilde{\mathcal{U}} &= (\langle \phi_k, \phi_j\rangle)_{2\le j,k\le m-1},\\
\gamma &= (\langle G_1^0V_1\phi_2, V_1\phi_1\rangle, \ldots, \langle G_1^0 V_1\phi_m, V_1\phi_1\rangle)^T,\\
E^0_{-+,0}(\varepsilon) &= -SG_0^0V_2E_0^0(\varepsilon)G_0^0V_2T = -T^*V_2E_0^0(\varepsilon)G_0^0V_2T,\\
E^1_{-+,1}(\varepsilon) &= S(iG_1^1V_2 - iG_1^1(V_1 - i\varepsilon V_2)E_0^0(\varepsilon)G_0^0V_2 - \varepsilon G_0^0V_2E_1^1(\varepsilon)G_0^0V_2\\
&\quad -iG_0^0V_2E_0^0(\varepsilon)G_1^1(V_1 - i\varepsilon V_2))T,\\
E^0_{-+,1}(\varepsilon) &= S(iG_1^0V_2 - iG_1^0(V_1 - i\varepsilon V_2)E_0^0(\varepsilon)G_0^0V_2 - \varepsilon G_0^0V_2E_1^0(\varepsilon)G_0^0V_2\\
&\quad -iG_0^0V_2E_0^0(\varepsilon)G_1^0(V_1 - i\varepsilon V_2))T,
\end{aligned}$$

and each $E^k_{-+,j}(\varepsilon)$ is a uniformly bounded $(m \times m)$-matrix on ε and $E_{-+,N+\alpha}(z,\varepsilon)$ satisfies that

$$\|\frac{d^l}{dz^l}E_{-+,N+\alpha}(z,\varepsilon)\| \le O(|z|^{N+\alpha-l}),$$

for $l = 0, 1, \ldots, 2N$ and $z \in \Omega$.

Before we state the distribution of the poles of $E_{-+}(z,\varepsilon)$ for ε small enough, we need the following lemma as an ingredient of the proof.

Lemma 3.6.1. *[Lemma 2.3 in [35]] Let A be an operator matrix on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$:*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ a_{jk} : \mathcal{H}_k \to \mathcal{H}_j,$$

where a_{11}, a_{22} are closed and a_{12}, a_{21} are bounded. Suppose a_{22} has a bounded inverse. Then A has a bounded inverse if and only if

$$a \equiv (a_{11} - a_{12}a_{22}^{-1}a_{21})^{-1}$$

exists and is bounded. Furthermore, we have

$$A^{-1} = \begin{pmatrix} a & -aa_{12}a_{22}^{-1} \\ -a_{22}^{-1}a_{21}a & a_{22}^{-1}a_{21}aa_{12}a_{22}^{-1} + a_{22}^{-1} \end{pmatrix}.$$

Let $\tilde{E}_{-+}(z,\varepsilon) = (E^{pq}_{-+}(z,\varepsilon))_{2\le p,q\le m}$ be a $(m-1)\times(m-1)$-matrix. Then we get its expansion as follows

$$\begin{aligned}
\tilde{E}_{-+}(z,\varepsilon) &= i\varepsilon\tilde{\mathcal{V}} + z\tilde{\mathcal{U}} + \varepsilon^2\tilde{E}^0_{-+,0}(\varepsilon) + \varepsilon^2 z\ln z\tilde{E}^1_{-+,1}(\varepsilon) + z\varepsilon\tilde{E}^0_{-+,1}(\varepsilon)\\
&\quad + \sum_{j=2}^{N}\sum_{k=0}^{j} z^j \ln^k z\tilde{E}^k_{-+,j}(\varepsilon) + z^{N+\alpha}\tilde{E}_{-+,N+\alpha}(z,\varepsilon),
\end{aligned} \tag{3.37}$$

where

$$\begin{aligned}
\tilde{\mathcal{V}} &= (\mathcal{V}^{pq})_{2\le p,q\le m},\\
\tilde{E}^k_{-+,j}(\varepsilon) &= ((E^k_{-+,j})^{pq}(\varepsilon))_{2\le p,q\le m},\\
\tilde{E}_{-+,N+\alpha}(z,\varepsilon) &= ((E_{-+,N+\alpha})^{pq}(z,\varepsilon))_{2\le p,q\le m},
\end{aligned}$$

are all bounded $(m-1)\times(m-1)$ matrices.

Let λ_j, $j = 2, \dots, m$ denote the roots of $\det(\tilde{\mathcal{V}} - \lambda\tilde{\mathcal{U}}) = 0$. On the other hand, we denote $\beta = (\mathcal{V}^{21}, \dots, \mathcal{V}^{m1})^T$ and $v = \mathcal{V}^{11}$. Thus $\mathcal{V}$ can be written as

$$\mathcal{V} = \begin{pmatrix} v & \beta^* \\ \beta & \tilde{\mathcal{V}} \end{pmatrix}.$$

Let $a = v - \beta\tilde{\mathcal{V}}^{-1}\beta^*$. Actually, one can check that $a = \frac{\det\mathcal{V}}{\det\tilde{\mathcal{V}}} > 0$. Similar to the case we discussed in Section 3.5.1, one can derive that there exists a unique solution denoted by $\lambda_1(\varepsilon)$ of the equation

$$i\varepsilon a - \frac{|\langle V_1\phi_1, 1\rangle|^2}{(4\pi)^2} z\ln z = 0.$$

Furthermore, $\lambda_1(\varepsilon) = r_1(\varepsilon)e^{i\theta_1(\varepsilon)}$ satisfies that $|r_1(\varepsilon)| = O(\varepsilon|\ln\varepsilon|^{-1})$, $r_1(\varepsilon) \to 0_+$ and $\theta_1(\varepsilon) \to \frac{3\pi}{2}_+$ as $\varepsilon \to 0_+$. In the following lemma, we describe the zeros of $F(z,\varepsilon) = \det E_{-+}(z,\varepsilon)$ and the inverse of $E_{-+}(z,\varepsilon)$ when the distance between z and these zeros has a positive lower bound dependent on ε, which is $O(\varepsilon|\ln\varepsilon|^{-1})$ according to $\lambda_1(\varepsilon)$ and $O(\varepsilon)$ according to $-i\varepsilon\lambda_j$, $j = 2, \dots, m$.

Lemma 3.6.2. *Suppose that $\rho_0 > 4$. There exist sufficiently small constants δ, $\varepsilon_0 > 0$ such that for $\varepsilon \in]0, \varepsilon_0]$, $F(z,\varepsilon)$ has m zeros $\{z_j(\varepsilon)\}_{j=1}^m \in B_-(0,\delta)$ which coincide with the eigenvalues according to their algebraic multiplicities. More precisely, we have that $z_1(\varepsilon) \in B(\lambda_1(\varepsilon), c\varepsilon|\ln\varepsilon|^{-1})$ and $z_j(\varepsilon) \in B(-i\varepsilon\lambda_j, c\varepsilon)$, $j = 2, \dots, m$ for some $c > 0$. Here $B(\lambda_1(\varepsilon), c\varepsilon|\ln\varepsilon|^{-1}) \cap B(-i\varepsilon\lambda_j, c\varepsilon) = \emptyset$ and $B(-i\varepsilon\lambda_j, c\varepsilon) \cap B(-i\varepsilon\lambda_k, c\varepsilon) = \emptyset$ for $\lambda_j \neq \lambda_k$ and $\operatorname{dist}(B(\lambda_1(\varepsilon), c\varepsilon|\ln\varepsilon|^{-1}), \mathbb{R}_+) > c_1\varepsilon|\ln\varepsilon|^{-1}$ and $\operatorname{dist}(B(-i\varepsilon\lambda_j, c\varepsilon), \mathbb{R}_+) > c_1\varepsilon$ for some $c_1 > 0$.*

Furthermore, for $z \in \bar{\Omega} \triangleq B(0,\delta) \setminus \{B(\lambda_1(\varepsilon), c\varepsilon|\ln\varepsilon|^{-1}) \cup (\cup_{j=2}^m B(-i\varepsilon\lambda_j, c\varepsilon)) \cup \mathbb{R}_+\}$ the inverse of $E_{-+}(z,\varepsilon)$ has the form

$$\begin{aligned} & (E_{-+}(z,\varepsilon))^{-1} \\ = & \begin{pmatrix} f(z,\varepsilon)^{-1} & -f^{-1}\xi^T\tilde{E}_{-+}(z,\varepsilon)^{-1} \\ -\tilde{E}_{-+}(z,\varepsilon)^{-1}\zeta f(z,\varepsilon)^{-1} & -\tilde{E}_{-+}(z,\varepsilon)^{-1}\zeta f(z,\varepsilon)^{-1}\xi^T\tilde{E}_{-+}(z,\varepsilon)^{-1} + \tilde{E}_{-+}(z,\varepsilon)^{-1} \end{pmatrix}, \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} \xi(z,\varepsilon) &= (E_{-+}(z,\varepsilon)^{12}, \dots, E_{-+}(z,\varepsilon)^{1m})^T, \\ \zeta(z,\varepsilon) &= (E_{-+}(z,\varepsilon)^{21}, \dots, E_{-+}(z,\varepsilon)^{m1})^T, \\ f(z,\varepsilon) &= E_{-+}(z,\varepsilon)^{11} - \xi(z,\varepsilon)^T\tilde{E}_{-+}(z,\varepsilon)^{-1}\zeta(z,\varepsilon). \end{aligned} \tag{3.39}$$

Démonstration. As the proof in Section 3 and 5, it is sufficient to find the principal part of $F(z,\varepsilon)$. For $\rho_0 > 4$, one can get the expansion of $E_{-+}(z,\varepsilon)$ as follows

$$\begin{aligned} E_{-+}(z,\varepsilon) &= \begin{pmatrix} i\varepsilon v - \frac{|\langle V_1\phi_1,1\rangle|^2}{(4\pi)^2} z\ln z & i\varepsilon\beta^* \\ i\varepsilon\beta & i\varepsilon\tilde{\mathcal{V}} + z\tilde{\mathcal{U}} \end{pmatrix} \\ &\quad + \begin{pmatrix} O(\varepsilon^2 + z + \varepsilon z\ln z) & O(\varepsilon^2 + z + \varepsilon z\ln z) \\ O(\varepsilon^2 + z + \varepsilon z\ln z) & O(\varepsilon^2 + \varepsilon z + z^2\ln^2 z) \end{pmatrix}. \end{aligned}$$

It follows that for $z \in \Omega$ and $\varepsilon \in]0, \varepsilon_0]$,

$$\begin{aligned} F(z,\varepsilon) &= (i\varepsilon a - \frac{|\langle V_1\phi_1, 1\rangle|^2}{(4\pi)^2} z\ln z)\det(i\varepsilon\tilde{\mathcal{V}} + z\tilde{\mathcal{U}}) \\ &\quad + O(\varepsilon^{m+1} + \varepsilon z^{m-1} + \varepsilon^m z\ln z + z^m\ln z). \end{aligned}$$

Let $F_0(z,\varepsilon) = (i\varepsilon a - \frac{|\langle V_1\phi_1, 1\rangle|^2}{(4\pi)^2} z\ln z)\det(i\varepsilon\tilde{\mathcal{V}} + z\tilde{\mathcal{U}})$ and then $F_0(z,\varepsilon)$ has m zeros $\{\lambda_1(\varepsilon), -i\varepsilon\lambda_2, \ldots, -i\varepsilon\lambda_m\}$ in Ω. One can choose some $c > 0$ such that

$$|F_0(z,\varepsilon)| \geq C_1\varepsilon^m,\ |F(z,\varepsilon) - F_0(z,\varepsilon)| \leq C_2\varepsilon^m |\ln\varepsilon|^{-1}$$

for $z \in \partial B(\lambda_1(\varepsilon), c\varepsilon|\ln\varepsilon|^{-1})$ and some C_1, $C_2 > 0$. So applying Rouché's Theorem, we have that there exists a unique zero $z_1(\varepsilon) \in B(\lambda_1(\varepsilon), c\varepsilon|\ln\varepsilon|^{-1})$ of $F(z,\varepsilon)$. On the other hand, the conclusion about the other zeros of $F(z,\varepsilon)$ can be obtained, provided that

$$|F_0(z,\varepsilon)| \geq C_1\varepsilon^m|\ln\varepsilon|,\ |F(z,\varepsilon) - F_0(z,\varepsilon)| \leq C_2\varepsilon^m$$

for $z \in \partial B(-i\varepsilon\lambda_j, c\varepsilon)$, $j = 2, \ldots, m$. By choosing a suitable c, we can complete the proof of the zeros of $F(z,\varepsilon)$. For the expression of the inverse of $E_{-+}(z,\varepsilon)$, it is sufficient to prove the invertibility of $\tilde{E}_{-+}(z,\varepsilon)$ and $f(z,\varepsilon)$ for $z \in \bar{\Omega}$. In fact, from (3.37), one has that

$$\tilde{E}_{-+}(z,\varepsilon) = i\varepsilon\tilde{\mathcal{V}} + z\tilde{\mathcal{U}} + O(\varepsilon^2 + \varepsilon z + z^2\ln^2 z).$$

Thus also by Rouché's Theorem and choosing a suitable c, $\tilde{E}_{-+}(z,\varepsilon)^{-1}$ exists for $z \in \bar{\Omega}$. On the other hand, noting that

$$f(z,\varepsilon) = \frac{F(z,\varepsilon)}{F_0(z,\varepsilon)},$$

it is obvious that $f(z,\varepsilon)$ is invertible for $z \in \bar{\Omega}$. Therefore by Lemma 3.6.1, Lemma 3.6.2 is proved. □

Then we want to state the expansion of inverse of $E_{-+}(z,\varepsilon)$ near $z = 0$ for $\varepsilon > 0$ small enough. We divide Ω into two parts :

$$\begin{aligned}
\Omega_1 &= B(0, 2\delta) \setminus \{B(0, c_1\varepsilon|\ln\varepsilon|^{-1}) \cup (\cup_{j=2}^m B(-i\varepsilon\lambda_j, c\varepsilon)) \cup \mathbb{R}_+\} \\
\Omega_2 &= B(0, 2c_1\varepsilon|\ln\varepsilon|^{-1}) \setminus \{B(\lambda_1(\varepsilon), c\varepsilon|\ln\varepsilon|^{-1}) \cup \mathbb{R}_+\},
\end{aligned}$$

where $c_1 > 0$ is chosen such that $|z\ln z| \geq 2|\lambda_1(\varepsilon)\ln\lambda_1(\varepsilon)|$ for $z \in \Omega_1$.

Then for $\varepsilon \in]0, \varepsilon_0]$ small enough, we can get the following lemma in which we state the estimates of $E_{-+}(z,\varepsilon)$ and its derivatives for $z \in \Omega_1$ and the expansion of $E_{-+}(z,\varepsilon)$ for $z \in \Omega_2$.

Lemma 3.6.3. *Suppose that the assumptions of Theorem 1.4.7 holds.*

(1). For $z \in \Omega_1$, we have the estimates

$$\frac{d^j}{dz^j}E_{-+}(z,\varepsilon)^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & (-1)^j j!(i\varepsilon\mathcal{V} + z\mathcal{U})^{-1}(\mathcal{U}(i\varepsilon\mathcal{V} + z\mathcal{U})^{-1})^j \end{pmatrix} + O(z^{-j-1}\ln^{-1} z) \tag{3.40}$$

for $j = 0, 1, \ldots, 2N$.

(2). For $z \in \Omega_2$, we have the expansion of $E_{-+}(z,\varepsilon)$ as follows

$$E_{-+}(z,\varepsilon)^{-1} = \sum_{j=0}^{N}\sum_{k=0}^{j} \frac{z^j\ln^k z}{\varepsilon^{j+1}} C_j^k(\varepsilon) + \frac{1}{\varepsilon^{N+\alpha+1}} C_{N+\alpha}(z,\varepsilon), \tag{3.41}$$

where $C_j^k(\varepsilon)$, $j = 0, 1, \ldots, N$, $k = 0, \ldots, j$ are some bounded matrices on ε and the estimates for the remainder term

$$\|\frac{d^l}{dz^l} C_{N+\alpha}(z,\varepsilon)\| = O(z^{N+\alpha-l}),\ l = 0, \ldots, 2N, \tag{3.42}$$

hold. Furthermore, $C_0^0(\varepsilon) = -i\mathcal{V}^{-1} + O(\varepsilon)$.

Démonstration. (1). For $z \in \Omega_1$, it can be verified that

$$|z \ln z| \geq C_1 \varepsilon, \ |i\varepsilon a - \frac{|\langle V_1\phi_1, 1\rangle|^2}{(4\pi)^2} z \ln z| \geq C_2 |z \ln z|,$$

for some $C_1, C_2 > 0$. On the other hand, let $M = 2\max_{2\leq j\leq m}\{\lambda_j\}$ and then we have that for $z \in \Omega_1^1 \triangleq \Omega_1 \setminus B(0, M\varepsilon)$, $|z + i\varepsilon\lambda_j| \geq |z| - \lambda_j\varepsilon \geq (1 - \frac{\lambda_j}{M})|z|$ and for $z \in \Omega_1^2 \triangleq \Omega_1 \cap B(0, M\varepsilon)$, $|z + i\varepsilon\lambda_j| \geq c_1\varepsilon \geq \frac{c_1}{M}|z|$. So for $z \in \Omega_1$, it always holds that $|z + i\varepsilon\lambda_j| \geq C_3|z|$ for $C_3 = \min\{\frac{c_1}{M}, \frac{1}{2}\}$. Then as the intermediate part in Section 3.5.1, one can derive that

$$\frac{d^j}{dz^j}\tilde{E}_{-+}(z,\varepsilon)^{-1} = (-1)^j j!(i\varepsilon\tilde{\mathcal{V}} + z\tilde{\mathcal{U}})^{-1}(\mathcal{U}(i\varepsilon\tilde{\mathcal{V}} + z\tilde{\mathcal{U}})^{-1})^j + O(z^{-j}\ln^2 z),$$

for $z \in \Omega_1$ and $j = 0, 1, \ldots, 2N$.

Then we calculate the behavior of $f(z,\varepsilon)^{-1}$ for $z \in \Omega_1$. For $z \in \Omega_1^1$, due to the fact that $\varepsilon \leq \frac{1}{2}|z|$, we have that

$$\begin{aligned} f(z,\varepsilon) &= (i\varepsilon v - \frac{|\langle V_1\phi_1, 1\rangle|^2}{(4\pi)^2} z\ln z + O(\varepsilon^2 + \varepsilon z\ln z + z)) - (i\varepsilon\beta^* + O(\varepsilon^2 + \varepsilon z\ln z + z)) \\ &\quad ((i\varepsilon\tilde{\mathcal{V}} + z\tilde{\mathcal{U}})^{-1} + O(\ln^2 z))(i\varepsilon\beta + O(\varepsilon^2 + \varepsilon z\ln z + z)) \\ &= -\frac{|\langle V_1\phi_1, 1\rangle|^2}{(4\pi)^2} z\ln z + O(z). \end{aligned}$$

In light of

$$f(z,\varepsilon) = -\frac{|\langle V_1\phi_1, 1\rangle|^2}{(4\pi)^2} z\ln z(1 - \frac{(4\pi)^2}{|\langle V_1\phi_1, 1\rangle|^2}(z\ln z)^{-1}(f(z,\varepsilon) + \frac{|\langle V_1\phi_1, 1\rangle|^2}{(4\pi)^2} z\ln z)),$$

one can obtain that

$$\frac{d^j}{dz^j} f(z,\varepsilon)^{-1} = O(z^{-j-1}\ln^{-1} z),$$

for $z \in \Omega_1^1$ and $j = 0, 1, \ldots, 2N$ which is the same to the proof of (3.35).

On the other hand, for $z \in \Omega_1^2$, we have $|z| \leq M\varepsilon$ and then

$$\begin{aligned} \tilde{E}_{-+}(z,\varepsilon)^{-1} &= (i\varepsilon\tilde{\mathcal{V}})^{-1} - z(i\varepsilon\tilde{\mathcal{V}})^{-1}\tilde{\mathcal{U}}(i\varepsilon\tilde{\mathcal{V}} + z\tilde{\mathcal{U}})^{-1} + O(\ln^2 z) \\ &= (i\varepsilon\tilde{\mathcal{V}})^{-1} + O(\varepsilon^{-2} z + \ln^2 z). \end{aligned}$$

Denote $f_0(z,\varepsilon) = i\varepsilon a - \frac{|\langle V_1\phi_1, 1\rangle|^2}{(4\pi)^2} z\ln z$ and then by (3.39) together with $\frac{1}{M}|z| \leq \varepsilon \leq C_1^{-1}|z\ln z|$,

$$\begin{aligned} f(z,\varepsilon) &= (i\varepsilon v - \frac{|\langle V_1\phi_1, 1\rangle|^2}{(4\pi)^2} z\ln z + O(\varepsilon^2 + \varepsilon z\ln z + z)) - (i\varepsilon\beta^* + O(\varepsilon^2 + \varepsilon z\ln z + z)) \\ &\quad ((i\varepsilon\tilde{\mathcal{V}})^{-1} + O(\varepsilon^{-2} z + \ln^2 z))(i\varepsilon\beta + O(\varepsilon^2 + \varepsilon z\ln z + z)) \\ &= f_0(z,\varepsilon) + O(z). \end{aligned}$$

From

$$f(z,\varepsilon) = f_0(z,\varepsilon)(1 + f_0(z,\varepsilon)^{-1}(f(z,\varepsilon) - f_0(z,\varepsilon))),$$

it follows that

$$\frac{d^j}{dz^j} f(z,\varepsilon)^{-1} = O(z^{-j-1}\ln^{-1} z),$$

for $z \in \Omega_1^2$ and $j = 0, 1, \ldots, 2N$ similar to the proof of (3.35). Then by (3.38), we can get (3.40).

(2). For $z \in \Omega_2$, it is clear that

$$|z \ln z| \le C_4 \varepsilon;\ C_5^{-1}\varepsilon \le |f_0(z,\varepsilon)| \le C_5\varepsilon;$$
$$|z| \le C_6 \varepsilon |\ln \varepsilon|^{-1};\ |z + i\varepsilon\lambda_j| \ge c_1\varepsilon,\ j = 2, \ldots, m$$

for some positive constants C_4, C_5 and C_6. So as the proof of (3.22) and with help of Neumann's series and (3.37), one can check that for $z \in \Omega_2$,

$$\begin{aligned} \tilde{E}_{-+}(z,\varepsilon)^{-1} &= \sum_{j=0}^{N} \frac{z^j}{\varepsilon^{j+1}} A_j^0(\varepsilon) + z \ln z A_1^1(\varepsilon) \\ &\quad + \sum_{j=2}^{N}\sum_{k=1}^{j} \frac{z^j \ln^k z}{\varepsilon^j} A_j^k(\varepsilon) + \frac{1}{\varepsilon^{N+\alpha+1}} A_{N+\alpha}(z,\varepsilon), \end{aligned}$$

where

$$\begin{aligned} A_0^0(\varepsilon) &= (i\tilde{\mathcal{V}} + \varepsilon \tilde{E}_{-+,0}^0(\varepsilon))^{-1} = -i\tilde{\mathcal{V}}^{-1} + O(\varepsilon), \\ A_1^1(\varepsilon) &= -A_0^0(\varepsilon)\tilde{E}_{-+,1}^1(\varepsilon)A_0^0(\varepsilon), \end{aligned}$$

and $A_1^0(\varepsilon)$, $A_j^k(\varepsilon)$, $j = 2, \ldots, N$, $k = 0, \ldots, j$ and $A_{N+\alpha}(z,\varepsilon)$ are bounded $(m-1)\times(m-1)$ matrices satisfying the estimates

$$\|\frac{d^j}{dz^j} A_{N+\alpha}(z,\varepsilon)\| = O(|z|^{N+\alpha-j}),\ j = 0, 1, \ldots, 2N.$$

Let

$$\xi_j^k(\varepsilon) = (E_{-+,j}^k(z,\varepsilon)^{12}, \ldots, E_{-+,j}^k(z,\varepsilon)^{1m})^T$$

and

$$\zeta_j^k(\varepsilon) = (E_{-+,j}^k(z,\varepsilon)^{21}, \ldots, E_{-+,j}^k(z,\varepsilon)^{m1})^T$$

for $j = 0, \ldots, N$ and $k = 0, \ldots, j$. It follows from (3.39) that for $z \in \Omega_2$,

$$\begin{aligned} f(z,\varepsilon) &= i\varepsilon(v - \beta^* \tilde{\mathcal{V}}^{-1}\beta) + \varepsilon^2 f_0^0(\varepsilon) - z \ln z \frac{|\langle V_1\phi_1, 1\rangle|^2}{(4\pi)^2} + \sum_{j=1}^{N} \frac{z^j}{\varepsilon^{j-1}} f_j^0(\varepsilon) \\ &\quad + \sum_{j=1}^{N}\sum_{k=1}^{j} \frac{z^j \ln^k z}{\varepsilon^{j-2}} f_j^k(\varepsilon) + \frac{1}{\varepsilon^{N+\alpha-1}} f_{N+\alpha}(z,\varepsilon), \end{aligned}$$

where

$$\begin{aligned} f_0^0(\varepsilon) &= (E_{-+,0}^0)^{11}(\varepsilon) - \varepsilon^{-1}\{(i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_0^0(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon)) - i\beta^*\mathcal{V}^{-1}\beta\} \\ &= O(1), \\ f_1^1(\varepsilon) &= (E_{-+,1}^1)^{11}(\varepsilon) - \varepsilon\{(\xi_1^1)^T(\varepsilon)A_0^0(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon)) \\ &\quad + \varepsilon(i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_1^1(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon)) + (i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_0^0(\varepsilon)(\zeta_1^1)^T(\varepsilon)\}, \\ f_1^0(\varepsilon) &= (E_{-+,1}^0)^{11}(\varepsilon) - \{(\xi_1^0)^T(\varepsilon)A_0^0(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon)) \\ &\quad + \varepsilon(i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_1^0(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon)) + (i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_0^0(\varepsilon)(\zeta_1^0)^T(\varepsilon)\}, \end{aligned}$$

and other $f_j^k(\varepsilon)$, $j = 2, \ldots, N$, $k = 0, \ldots, j$ are bounded functions on ε which can be computed directly and the remainder term $f_{N+\alpha}(z,\varepsilon)$ satisfying that

$$|\frac{d^l}{dz^l} f_{N+\alpha}(z,\varepsilon)| = O(z^{N+\alpha-l})$$

for $l = 0, 1, \dots, 2N$. Thus applying Neumann's series as before, one can obtain that for $z \in \Omega_2$,

$$f(z,\varepsilon)^{-1} = \frac{1}{\varepsilon} g_0^0(\varepsilon) + \sum_{j=1}^{N} \sum_{k=0}^{j} \frac{z^j \ln^k z}{\varepsilon^{j+1}} g_j^k(\varepsilon) + \frac{1}{\varepsilon^{N+\alpha+1}} g_{N+\alpha}(z,\varepsilon),$$

where

$$\begin{aligned} g_0^0(\varepsilon) &= (i(v - \beta^* \tilde{\mathcal{V}}^{-1}\beta) + \varepsilon f_0^0(\varepsilon))^{-1}, \\ g_1^1(\varepsilon) &= (\frac{|\langle V_1\phi_1, 1\rangle|^2}{(4\pi)^2} - \varepsilon f_1^1(\varepsilon)) g_0^0(\varepsilon)^2 \\ g_1^0(\varepsilon) &= -f_1^0(\varepsilon) g_0^0(\varepsilon)^2, \end{aligned}$$

and other $g_j^k(\varepsilon)$, $j = 2, \dots, N$, $k = 0, \dots, j$ are bounded functions on ε which can be calculated explicitly and $g_{N+\alpha}(z,\varepsilon)$ satisfies that

$$|\frac{d^l}{dz^l} g_{N+\alpha}(z,\varepsilon)| = O(z^{N+\alpha-l})$$

for $l = 0, 1, \dots, 2N$. Thus by using the expression (3.38), we can obtain the expansion (3.41). In particular, it can be verified that

$$\begin{aligned} C_0^0(\varepsilon) &= \begin{pmatrix} iv & (i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon)) \\ (i\beta + \varepsilon\zeta_0^0(\varepsilon)) & i\tilde{\mathcal{V}} + \varepsilon \tilde{E}_{-+,0}^0(\varepsilon) \end{pmatrix}^{-1} \\ &= -i\mathcal{V}^{-1} + O(\varepsilon), \\ C_1^1(\varepsilon) &= -C_0^0(\varepsilon) \begin{pmatrix} -\frac{|\langle V_1\phi_1, 1\rangle|^2}{(4\pi)^2} + \varepsilon (E_{-+,1}^1)^{11}(\varepsilon) & \varepsilon(\xi_1^1)^T(\varepsilon) \\ \varepsilon\zeta_1^1(\varepsilon) & \varepsilon^2 \tilde{E}_{-+,1}^1(\varepsilon) \end{pmatrix} C_0^0(\varepsilon) \\ &\triangleq \begin{pmatrix} (C_1^1)^{11}(\varepsilon) & (C_1^1)^{12}(\varepsilon) \\ (C_1^1)^{21}(\varepsilon) & (C_1^1)^{22}(\varepsilon) \end{pmatrix}, \\ C_1^0(\varepsilon) &= -C_0^0(\varepsilon) \left(\mathcal{U} + \varepsilon E_{-+,1}^0(\varepsilon)\right) C_0^0(\varepsilon) \\ &\triangleq \begin{pmatrix} (C_1^0)^{11}(\varepsilon) & (C_1^0)^{12}(\varepsilon) \\ (C_1^0)^{21}(\varepsilon) & (C_1^0)^{22}(\varepsilon) \end{pmatrix}, \end{aligned}$$

where $C_1^1(\varepsilon)$ and $C_1^1(\varepsilon)$ have the asymptotic expansion as follows

$$\begin{aligned} (C_1^1)^{11}(\varepsilon) &= g_1^1(\varepsilon), \\ (C_1^1)^{12}(\varepsilon) &= -g_1^1(\varepsilon)(i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_0^0(\varepsilon) - \varepsilon g_0^0(\varepsilon)(\xi_1^1)^T(\varepsilon)A_0^0(\varepsilon) \\ &\quad -\varepsilon^2 g_0^0(\varepsilon)(i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_1^1(\varepsilon), \\ (C_1^1)^{21}(\varepsilon) &= -g_1^1(\varepsilon)A_0^0(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon)) - \varepsilon g_0^0(\varepsilon)A_0^0(\varepsilon)\zeta_1^1(\varepsilon) \\ &\quad -\varepsilon^2 g_0^0(\varepsilon)A_1^1(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon)), \\ (C_1^1)^{22}(\varepsilon) &= \varepsilon^2\{g_0^0(\varepsilon)A_1^1(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon))(i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_0^0(\varepsilon) \\ &\quad + g_0^0(\varepsilon)A_0^0(\varepsilon)\zeta_1^1(\varepsilon)(i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_0^0(\varepsilon) \\ &\quad + g_1^1(\varepsilon)A_0^0(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon))(i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_0^0(\varepsilon) \\ &\quad + g_0^0(\varepsilon)A_0^0(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon))(\xi_1^1)^T(\varepsilon)A_0^0(\varepsilon) \\ &\quad + g_0^0(\varepsilon)A_0^0(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon))(i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_1^1(\varepsilon) + A_1^1(\varepsilon)\}, \end{aligned}$$

and

$$
\begin{aligned}
(C_1^0)^{11}(\varepsilon) &= g_1^0(\varepsilon), \\
(C_1^0)^{12}(\varepsilon) &= -g_1^0(\varepsilon)(i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_0^0(\varepsilon) - g_0^0(\varepsilon)(\xi_1^0)^T(\varepsilon)A_0^0(\varepsilon) \\
&\quad -g_0^0(\varepsilon)(i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_1^0(\varepsilon), \\
(C_1^0)^{21}(\varepsilon) &= -g_1^0(\varepsilon)A_0^0(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon)) - g_0^0(\varepsilon)A_1^0(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon)) \\
&\quad -g_0^0(\varepsilon)A_0^0(\varepsilon)\zeta_1^0(\varepsilon), \\
(C_1^0)^{22}(\varepsilon) &= g_0^0(\varepsilon)A_1^0(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon))(i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_0^0(\varepsilon) \\
&\quad +g_0^0(\varepsilon)A_0^0(\varepsilon)\zeta_1^0(\varepsilon)(i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_0^0(\varepsilon) \\
&\quad +g_1^0(\varepsilon)A_0^0(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon))(i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_0^0(\varepsilon) \\
&\quad +g_0^0(\varepsilon)A_0^0(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon))(\xi_1^0)^T(\varepsilon)A_0^0(\varepsilon) \\
&\quad +g_0^0(\varepsilon)A_0^0(\varepsilon)(i\beta + \varepsilon\zeta_0^0(\varepsilon))(i\beta^* + \varepsilon(\xi_0^0)^T(\varepsilon))A_1^0(\varepsilon).
\end{aligned}
$$

Meanwhile, other terms in (3.41) can be also obtained explicitly and the estimate (3.42) holds. □

Thus, by taking the expansions (3.31), (3.33), (3.34), and (3.41) into (3.10) and then by the relation $R(z,\varepsilon) = W(z,\varepsilon)R_0(z)$, we can get the following theorem about the low-energy analysis of the resolvent.

Theorem 3.6.4. *Suppose that the assumptions of Theorem 1.4.7 holds. Then we have that*
(1). For $z \in \Omega_1$, we have the estimates

$$
\frac{d^j}{dz^j}W(z,\varepsilon)^{-1} = (-1)^{j+1}j!T'(i\varepsilon\mathcal{V} + z\mathcal{U})^{-1}(\mathcal{U}(i\varepsilon\mathcal{V} + z\mathcal{U})^{-1})^j S' + O(z^{-j-1}\ln^{-1} z)
$$

in $\mathcal{L}(1,-s_j;1,-s_j)$ and

$$
\frac{d^j}{dz^j}R(z,\varepsilon) = (-1)^{j+1}j!T'(i\varepsilon\mathcal{V} + z\mathcal{U})^{-1}(\mathcal{U}(i\varepsilon\mathcal{V} + z\mathcal{U})^{-1})^j S'G_0^0 + O(z^{-j-1}\ln^{-1} z)
$$

in $\mathcal{L}(-1,s_j;1,-s_j)$ for $j = 0,1,\ldots,2N$ and $s_j > 2j+1$, where T' and S' are defined by

$$
T'c' = \sum_{j=2}^{m} c_j\phi_j, \; S'\phi = (\langle\phi, -V_1\phi_2\rangle, \ldots, \langle\phi, -V_1\phi_2\rangle)
$$

for $c' = (c_2,\ldots,c_m) \in \mathbb{C}^{m-1}$ and $\phi \in H^{1,-s}$, $s > 1$.
(2). For $z \in \Omega_2$ and $s > 2N+1$, we have the expansions of $W(z,\varepsilon)^{-1}$ in $\mathcal{L}(1,-s;1,-s)$ and $R(z,\varepsilon)$ in $\mathcal{L}(-1,s;1,-s)$ as follows

$$
\begin{aligned}
W(z,\varepsilon)^{-1} &= \sum_{j=0}^{N}\sum_{k=0}^{j}\frac{z^j\ln^k z}{\varepsilon^{j+1}}W_j^k(\varepsilon) + \frac{1}{\varepsilon^{N+\alpha+1}}W_{N+\alpha}(z,\varepsilon), \\
R(z,\varepsilon) &= \sum_{j=0}^{N}\sum_{k=0}^{j}\frac{z^j\ln^k z}{\varepsilon^{j+1}}R_j^k(\varepsilon) + \frac{1}{\varepsilon^{N+\alpha+1}}R_{N+\alpha}(z,\varepsilon),
\end{aligned}
$$

where $W_j^k(\varepsilon) \in \mathcal{L}(1,-s_j;1,-s_j)$ and $R_j^k(\varepsilon) \in \mathcal{L}(-1,s_j;1,-s_j)$, $j = 0,1,\ldots,N$, $k = 0,\ldots,j$ are some bounded operators on ε for $s_j > 2j+1$ and the remainder terms $W_{N+\alpha}(z,\varepsilon) \in \mathcal{L}(1,-s;1,-s)$, $R_{N+\alpha}(z,\varepsilon) \in \mathcal{L}(-1,s;1,-s)$ for $s > 2N+1$ are uniformly bounded on z, ε with the estimates

$$
\begin{aligned}
\|\langle x\rangle^{-s}\frac{d^l}{dz^l}W_{N+\alpha}(z,\varepsilon)\langle x\rangle^{s}\| &= O(z^{N+\alpha-l}), \\
\|\langle x\rangle^{-s}\frac{d^l}{dz^l}R_{N+\alpha}(z,\varepsilon)\langle x\rangle^{-s}\| &= O(z^{N+\alpha-l})
\end{aligned}
$$

hold for $l = 0, 1, \dots, 2N$. In particular, we can derive that

$$\begin{aligned}
W_0^0(\varepsilon) &= \varepsilon E_0^0(\varepsilon) - (1+\varepsilon E_{+,0}^0(\varepsilon))SC_0^0(\varepsilon)T(1+\varepsilon E_{-,0}^0(\varepsilon)),\\
W_1^1(\varepsilon) &= \varepsilon^2 E_1^1(\varepsilon) - (1+\varepsilon E_{+,0}^0(\varepsilon))SC_1^1(\varepsilon)T(1+\varepsilon E_{-,0}^0(\varepsilon))\\
&\quad -\varepsilon\{E_{+,1}^1(\varepsilon)SC_0^0(\varepsilon)T(1+\varepsilon E_{-,0}^0(\varepsilon)) + (1+\varepsilon E_{+,0}^0(\varepsilon))SC_0^0(\varepsilon)TE_{-,1}^1(\varepsilon)\},\\
W_1^0(\varepsilon) &= \varepsilon^2 E_1^0(\varepsilon) - (1+\varepsilon E_{+,0}^0(\varepsilon))SC_1^0(\varepsilon)T(1+\varepsilon E_{-,0}^0(\varepsilon))\\
&\quad -\varepsilon\{E_{+,1}^0(\varepsilon)SC_0^0(\varepsilon)T(1+\varepsilon E_{-,0}^0(\varepsilon)) + (1+\varepsilon E_{+,0}^0(\varepsilon))SC_0^0(\varepsilon)TE_{-,1}^0(\varepsilon)\},\\
R_0^0(\varepsilon) &= W_0^0(\varepsilon)G_0^0,\\
R_1^1(\varepsilon) &= W_1^1(\varepsilon)G_0^0 + \varepsilon W_0^0(\varepsilon)G_1^1,\\
R_1^0(\varepsilon) &= W_1^0(\varepsilon)G_0^0 + \varepsilon W_0^0(\varepsilon)G_1^0.
\end{aligned}$$

Similar to the cases we considered in Section 3.5.2, the proof of Theorem 1.4.7 can be completed following Theorem 3.6.4.

Remark 3.6.5. *It can be computed that*

$$R_1^1(\varepsilon) = K(\varepsilon)G_1^1(\varepsilon - (V_1 - i\varepsilon V_2)K(\varepsilon)G_0^0),$$

where $K(\varepsilon) = \varepsilon E_0^0(\varepsilon) - (1 + i\varepsilon E_0^0(\varepsilon)G_0^0 V_2)TC_0^0(\varepsilon)S(1 + i\varepsilon G_0^0(\varepsilon)V_2)$. Thus $R_1^1(\varepsilon)$ is of rank one at most provided that G_1^1 is of rank one. Furthermore, let ε tends to zero and then one has that

$$\begin{aligned}
\lim_{\varepsilon\to 0_+} R_1^1(\varepsilon) &= T\mathcal{V}^{-1}SG_1^1V_1T\mathcal{V}^{-1}S\\
&= \frac{|\langle V_1\phi_1, 1\rangle|^2}{(4\pi)^2}T\mathcal{V}^{-1}P_1\mathcal{V}^{-1}S,
\end{aligned}$$

is non-trivial, where P_1 is a projection from $\mathbb{C}^m$ to $\{\theta(1,0,\dots,0) : \theta \in \mathbb{C}\}$ defined by $P_1 c = (c_1, 0, \dots, 0)^T$, for $c = (c_1, c_2, \dots, c_m)^T \in \mathbb{C}^m$. Therefore, for $\varepsilon > 0$ small enough, the principal term is of rank one.

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COMMUNAUTÉ D'UNIVERSITÉS ET ÉTABLISSEMENTS

UNIVERSITÉ DE NANTES

Thèse de Doctorat

Zhu Lu

Comportement en grand temps des solutions de l'équation de Schrödinger dissipative

Large-time Behavior of the Solutions to Dissipative Schrödinger Equation

Résumé

Cette thèse est consacrée à l'étude de l'équation de Schrödinger dissipative dépendant du temps, surtout à l'évolution à long terme des solutions du problème de Cauchy. Soit $H = -\Delta + V(x)$ l'opérateur de Schrödinger dissipatif, i.e. $\Im V(x) \leq 0$. De plus, on suppose que la partie imaginaire de $V(x)$ est assez petite de sorte qu'elle peuve être considérée comme une perturbation de la partie autoadjointe de l'opérateur.

D'abord, nous étudions la complétude asymptotique de l'opérateur de la diffusion pour la paire $(-\Delta, H)$, sous condition que 0 soit un point régulier de la partie autoadjointe de H, désignée par H_1. Cela signifie que 0 n'est ni une valeur propre, ni une résonance de H_1 . La preuve est basee sur une estimation globale de la résolvante qui est uniforme par rapport a la taille de la partie imaginaire du potentiel et sur la completude asymptotique de la diffusion quantique pour la paire d'opérateurs autoadjoints $(-\Delta, H_1)$.

Ensuite, pour mieux comprendre les comportements en grands temps de la dynamique quantique, nous étudions le développement asymptotique du semigroup e^{-itH} lorsque t tend vers l'infini. Nous considérons les trois cas suivants : (1). 0 est seulement une valeur propre, mais pas une résonance de H_1 en dimension trois ; (2). 0 est seulement une résonance, mais pas une valeur propre de H_1 en dimension quatre ; (3). 0 n'est pas seulement une résonance mais aussi une valeur propre de H_1 en dimension quatre.

Abstract

This thesis is devoted to studying the large time behavior of the solutions to the Cauchy problem of the dissipative Schrödinger equations. Let $H = -\Delta + V(x)$ be the Schrödinger operator. We consider that H is dissipative, i.e. $\Im V \leq 0$. More precisely, in this thesis, we assume that the imaginary part of $V(x)$ is sufficiently small such that it can be seen as a perturbation of the real part of H. Thus the main method in this thesis is the argument of perturbation.

First, we will study the asymptotic completeness of the scattering pair $(-\Delta, H)$, under the assumption that 0 is a regular point of the real part of H, denoted by H_1. It means that 0 is neither an eigenvalue nor a resonance of H_1. The proof is based on a global resolvent estimate which is uniform to the size of the imaginary part of the potential function and on the asymptotic completeness of the quantum scattering pair of the selfadjoint operators $(-\Delta, H_1)$.

Second, we will discuss the expansion in time of e^{-itH}. Here we will consider three cases: (1). 0 is only an eigenvalue but not a resonance of H_1 in dimension three; (2). 0 is only a resonance but not an eigenvalue of H_1 in dimension four; (3). 0 is not only a resonance but also an eigenvalue of H_1 in dimension four. Main tool is the low-energy analysis.

Mots clés
Operateur de Schrödinger dissipatif, développement asymptotique de la résolvante, comportement en grand temps, valeurs propres complexes, résonance au seuil, diffusion quantique dissipatif, complétude asymptotique

Key Words
Dissipative Schrödinger operators, resolvent expansion, large time behavior, complex eigenvalues, threshold resonance, dissipative quantum scattering, asymptotic completeness.