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Trois résultats sous forme normale pour les équations de Schrödinger et le système abcd de Boussinesq

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Introduction

1.1 General introduction

1.1.1 Partial differential equations

In the past decades, partial differential equations have been intensively studied. PDEs play an important role in scientific fields, especially in physics and engineering. They arise from many physical considerations, like fluid dynamics, quantum mechanics, statistical mechanics, heat diffusion, N-body problem. These different physical phenomena give rise to various mathematical models and establishes a strong connection between Mathematics and Physics.

Without being exhausted, we mention here some interesting examples. One of the most well-known PDEs is the Schrödinger equation, which describes the probability density of the presence of a non-relativistic massive particle and named after Erwin Schrödinger, who first introduced the equation in 1926 to study electrons in the atom. The discovery of the Schrödinger equation was a significant landmark in the study of quantum mechanics. To describe relativistic massive particles, we have the Dirac equation, which was derived by British physicist Paul Dirac in 1928. In the same context, we have the Klein-Gordon equation (1926).A year after the publication of the Schrödinger equation, Hartree published his research, what is now known as the Hartree equation, considering the electron systems in a spherical potential. Considering fluid dynamics, we have the Korteweg–De Vries (KdV) equation (1895), Boussinesq equation (1872), Navier Stokes equation (1845), and also the equation of Burgers (1948), which is a mathematical model of waves on shallow water surfaces. The KdV equation was first introduced by Boussinesq in 1877 and rediscovered by Diederik Korteweg and Gustav de Vries (1895). Another example for fluid dynamics is the Boussinesq equation, which is named after Joseph Boussinesq (1872), who first derived it to study solitary waves. The Navier-Stokes equations describe the motion of fluid relating pressure, temperature, and density. The Burgers' equations appear in various areas of applied mathematics, such as gas dynamics, traffic low and nonlinear acoustics. Considering the heat diffusion phenomenon, Joseph Fourier introduced the theory of the heat equation in 1807. The heat equation, along with its variants, also appears in many fields of applied mathematics, like probability theory and financial mathematics.

All of these equations indicate the importance of PDEs. Solving these equations helps us understand physical phenomena. Studying these equations, we deal with many questions, like questions about the existence of solutions, the uniqueness of solutions, the behavior of solutions, how long the solutions exist and how we can approach the solutions,... Unfortunately, there are many different types of PDEs and there is no general method or general theory for PDEs. Different methods have been developed to deal with many of the individual equations. Fortunately, PDEs considering conservative physical phenomena are Hamiltonian, which corresponds to the total energy of the system, both kinetic and potential energy.

In this thesis, we focus on study Schrödinger equations, which describe the wave function of a quantum mechanical system, and Boussinesq system, which was derived to study the water way in a shallow water regime. Both systems can be written in Hamiltonian form.

1.1.2 Hamiltonian

In 1833, starting from Lagrangian mechanics, William Rowan Hamilton developed a reformulation of Newtonian mechanics, known as Hamiltonian mechanics, which then historically played an important role in the development of quantum physics.

In classical mechanics, the time evolution of a physical system is obtained by Hamiltonian equations

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \text{for } 1 \leq i \leq N.$$

Here the coordinate (q_i, p_i) is indexed to the frame of reference of the system, p_i is generalized momentum associated with the generalized coordinate q_i. In Newtonian mechanics, the time evolution of both position and velocity are computed by applying Newton's second law to the total force being exerted on each particle in the system. In contrast, the time evolution state in Hamiltonian mechanics is obtained by computing the Hamiltonian of the system. The system with many degrees of freedom allows the exchange of energy between different modes, which makes its time evolution complicated and may cause chaotic behavior.

Hamiltonian mechanics is equivalent to Lagrangian mechanics under Legendre transform when holding q and t fixed and defining p as the dual variable. However, Hamiltonian mechanics along with its symplectic structure has demonstrated its strength in studying physical systems. In classical mechanics, the Hamiltonian induces a symplectic structure on smooth functions and forms a symplectic manifold, which is called the phase space. The Hamiltonian then can be generalized to quantum mechanics.

The Hamiltonian of a closed mechanical system is commonly expressed as the sum of functions corresponding to the kinetic and potential energies of a system in the form

$$H = H_0 + P$$

where H_0 is the kinetic energy and corresponds to the linear part of the dynamic system while P is the potential energy and normally corresponds to nonlinear part. Generally, the Hamiltonian generates the time evolution of solutions

$$iu_t = X_H(u) = Au + f(u)$$

where X_H is the Hamiltonian vector field, A is a linear operator (Laplace operator in Schrödinger equation) diagonalized on an orthonormal basis of the phase space $(\phi_j)_{j\in J}$, $f(u)$ is a nonlinear term, and $u(x,t)$ is a complex function. The linear equation gives us solutions in an explicit form

$$u(t,x) = \sum_{j\in J} c_j e^{i\omega_j t}\phi_j(x)$$

where frequency ω_j denotes the eigenvalue of A associated with the eigenfunction ϕ_j. We see that different linear modes form different trajectories without interacting each others. These linear trajectories define finite or infinite invariant tori in the phase space.

We define the Sobolev norm and Sobolev space H^s

$$H^s := \{u(x) = \sum_j c_j\phi_j(x) \mid \|u\|_s^2 := \sum_{j\in J} c_j^2\langle j\rangle^{2s} < \infty\}.$$

Here $\langle j\rangle = \sqrt{1+j^2}$. The questions now become : once the well posedness (local or global) is proved, assume that the initial data is sufficiently small, we want to know

— Do invariant tori persist or the perturbation kill these tori ?

— Whether the solutions remain bounded for all time or if there exist initial conditions that give unbounded solutions ?

— Even when we do not know for all time, can we answer for a very long but finite time ?

Much effort has been done to answer these questions, leading to the birth of many methods. KAM theory and Birkhoff normal form theory are two of the most well-known methods.

1.1.3 Poisson bracket

The Poisson bracket, which is named in honor of French mathematician, engineer, and physicist Siméon Denis Poisson (1782-1840), plays a central role in Hamiltonian dynamical equations. In canonical coordinates $(q_i,\, p_i)(q_i,\, p_i)$, given two functions $f(p_i,\, q_i, t)$, $g(p_i,\, q_i, t)$ the Poisson bracket is defined in the form

$$\{f, g\} := \sum_{i=1}^{N}(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}).$$

The Poisson brackets of the canonical coordinates, which are called the fundamental Poisson brackets, are

$$\begin{aligned} \{q_i, q_i\} &= 0 \\ \{p_i, p_i\} &= 0 \\ \{q_i, p_j\} &= -\{p_j, q_i\} = \delta_{i,j} \end{aligned}$$

here δ_{ij} is the Kronecker symbol.

Generally, let M be a smooth manifold and ω be a closed non degenerate differential 2-form on M, then the pair (M, ω) is a symplectic manifold and ω is a symplectic form. Let f and g be two differentiable functions depending on the phase space and time, their Poisson bracket $\{f, g\}$ is given as

$$\{f, g\} = \omega(X_f, X_g) = X_g f$$

where X_f denotes the vector field generated by f and $X_g f$ denotes the vector field X_g applied to the function f as a directional derivative. The Poisson bracket of two differentiable functions is a differentiable function in the phase space. The algebra of smooth functions in the phase space together with the Poisson bracket form a Poisson algebra, which is a Lie algebra under the Poisson bracket. Every symplectic manifold is Poisson manifold.

Given three functions f, g, h in the phase space and time, one has properties of the Poisson bracket

— Anticommutativity

$$\{f, g\} = -\{g, f\},$$

— Bilinearity

$$\{af + bg, h\} = a\{f, h\} + b\{g, h\} \qquad a, b \in \mathbb{R},$$

— Leibniz's rule

$$\{fg, h\} = \{f, h\}g + f\{g, h\},$$

— Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Moreover, we have an equivalent expression of the Poisson bracket of functions to the Lie bracket of the associated Hamiltonian vector fields

$$X_{\{f,g\}} = -[X_f, X_g].$$

A Hamiltonian equation can be written in term of the Poisson bracket

$$\dot{u} = \{u, H\} \tag{1.1.1}$$

where H is the Hamiltonian and u is a complex function. The Poisson bracket in (1.1.1) is defined as

$$\{f, g\} = -\mathrm{i}(\partial_u f \partial_{\overline{u}} g - \partial_u g \partial_{\overline{u}} f)$$

A Hamiltonian dynamical system typically has constants of motion besides the energy, these constants of motion commute with the Hamiltonian under the Poisson bracket. If f, g are two constants of the motion, then their Poisson bracket $\{f, g\}$ is also a constant of the motion. A Hamiltonian system that contains a maximal set of conserved quantities, i.e., there exists a maximal set of constants of motion, is completely integrable.

The Poisson bracket is preserved under symplectic transformations (canonical transformations preserve the bi-linear form ω). To be more specific, given two functions f, g

and a symplectic transformation $\Phi_S = \Phi_S^t \mid_{t=1}$, one has

$$\{f, g\} \circ \Phi_S = \{f \circ \Phi_S, g \circ \Phi_S\}.$$

This property is especially useful in studying Hamiltonian mechanics. Let Φ_S be a symplectic transformation and H be a Hamiltonian, then

$$H \circ \Phi_S = H + \{S, H\} + \sum_{n \geq 2} \frac{1}{n!} ad_S^n(H)$$

where

$$ad_S^1(H) := \{S, H\}, \qquad ad_S^n(H) = \{S, ad_S^{n-1}(H)\} \qquad \forall n \geq 2.$$

This fact is a powerful tool in studying Hamiltonian equations. Indeed, KAM theory and Birkhoff normal form theory are based on the search for symplectic transformations in order to conjugate the original Hamiltonian to a normal form. These symplectic transformations are founded by solving homological equations.

1.1.4 KAM theory

Over the past half century, the KAM theory has played an important role in studying long time behavior of solutions of non integrable Hamiltonian system. The integrable Hamiltonian system admits many invariant tori in phase space. Different initial conditions form different invariant tori. The KAM theorem states that most of such tori persist under small Hamiltonian perturbations. This result is valid under certain conditions of suitable regularity and sufficiently irrational frequencies. The persistence of such invariant tori implies that the motion continues to be quasiperiodic. The name KAM theory comes from Kolmogorov, Arnol'd and Moser who initiated the theory.

We talk about the history of the KAM theory. In 17 century, Kepler announced his study describing the orbit of a planet around its Sun as an ellipse. However, with Newton'law of gravitation, physicians and mathematicians then realized that the disturbance due to interaction between planets makes their orbits more complicated than their in Kepler'law. Scientists want to know how much disturbance affects the trajectories of the planets. Started by Poincaré, mathematicians believed that a small disturbance in a long time could push the planets far away from Kepler's orbits and the stable trajectories are exceptional. Indeed, considering general Hamiltonian tori, it was a common belief that an integrable system can be turned into an ergodic one on each energy surface under an arbitrarily small perturbation. This means that for a long time the system forgets its initial state and the trajectories eventually visit almost all points in any subset of the phase space.

However, in 1954, Kolmogorov [Kol54] in his talk at the International Congress of Mathematicians in Amsterdam, announced that the majority of tori survive and instability is possible but very rare. Arnold [Arn63] and Moser [Mos62] then completed his proof. The general result is known as the KAM theorem.

Mathematicians have made significant advances in the KAM theory since then. The theory was originally applied to perturbed Hamiltonian PDEs in one dimension. In [Kuk87 ; Kuk93], Kuksin used KAM methods to prove the existence of quasi periodic solutions of nearly integrable Hamiltonian equation in infinite dimension. In [Way90], Wayne proved the existence of periodic and quasi periodic solutions for nonlinear wave equations in dimension one with Dirichlet boundary condition. The theory was then applied to the nonlinear wave equation with periodic boundary conditions [CY00], Klein-Gordon equation [BK95] and nonlinear Schrödinger equation [KP96]. While others need external parameters to verify the non resonant condition, the result in [KP96] is remarkable since it is the first result for a Hamiltonian without an external parameter. All of these results are for Hamiltonian equations in one dimensional context.

The study of KAM theory in multidimensional space has just started recently (see [Bou98 ; Bou03 ; EK09 ; EK10]). In these just mentioned papers, the authors considered Hamiltonian PDEs with external parameters in the linear part, by which a non-resonant condition is achieved. Usually, the parameter enters the equation through the potential term $V(x)u(t,x)$ or $V(x) * u(x)$, where the potential V depends on the parameter. The techniques developed in [EK09 ; EK10] then has been extended in [EGK16] to a KAM result without external parameters (see also [Procesi15]). The approach in [EK09 ; EK10] allows to analyse the linear stability of the KAM tori. We remind that a solution of a nonlinear equation is called linear stable if the linearization of the equation at this solution has linear operator whose spectrum contains only pure imaginary eigenvalues. In [Procesi15] (see also [Procesi15 ; PPV13 ; Wang16]), applying a KAM algorithm, the authors proved the existence of large families of stable and unstable quasi periodic solutions for the NLS in any number of independent frequencies. The considered quasi periodic solutions base on non-degenerate sets $\mathcal{A}$ of linear modes.

Proving the KAM theory involves verifying the nonresonant condition of the frequencies. The nonresonant condition becomes increasingly difficult to verify for systems with more degrees of freedom. In many cases, the frequencies by themselves are resonant and we must use external parameters to achieve the nonresonant condition.

We are then interested in the stability and instability of the KAM tori. Although, using KAM theory, the stability is frequently observed, the instability also occurs. Such invariant tori exhibit hyperbolic directions which induce instability of the tori (see [EGK16]).

We consider the Hamiltonian perturbation $H = h_0 + f$, with the small perturbation f. Let us denote $(r, \theta) \in \mathbb{R}^n \times \mathbb{T}^n$ the action-angle variables associated with invariant tori, and $z = (\zeta, \eta)$, $\eta = \overline{\zeta}$ the external modes. For any perturbation f, we define its jet function f^T, the effective part of f, as a quadratic truncation of Taylor expansion of f around the origin $r = 0$, $z = 0$:

$$f^T = f(0,\theta,0) + \partial_r f(0,\theta,0) \cdot r + \partial_z f(0,\theta,0) \cdot z + \frac{1}{2}\langle \partial_z^2 f(0,\theta,0)z, z\rangle.$$

In KAM theory, we are looking for an an analytic, near-identity, symplectic transformation Ψ, which puts the original Hamiltonian $H = h_0 + f$ to a desirable form

$$(h_0 + f) \circ \Psi = \tilde{h} + g$$

where $\tilde{h}$ is a Hamiltonian on normal form and the perturbation g has its effective part $\tilde{g}^T$ vanishing.

The idea of KAM theory is to do an iterative procedure. More precisely, considering the original Hamiltonian $H = h_0 + f$, we search for an analytic, near-identity, symplectic transformation $\Phi_S = \Phi_S^t \mid_{t=1}$ that puts H into a new form

$$H_+ := H \circ \Phi_S = h_+ + f_+$$

where

— h_+ is a Hamiltonian, close to h_0, i.e, $|h_+ - h_0| \sim O(\varepsilon)$. In addition, h_+ commutes with the linear Hamiltonian h_0, i.e.,

$$\{h_0, h_+\} = 0.$$

— f_+ is a new perturbation, whose jet function f_+^T smaller than f^T. Indeed, assuming that the initial data is sufficiently small, we are finding S such that $f_+^T \sim (f^T)^\alpha$ with $\alpha > 1$ ($\alpha = \frac{3}{2}$ for example).

We say that the Hamiltonian h_+ is in normal form. Since it commutes with h_0, the normal Hamiltonian does not affect the trajectory of the linear Hamiltonian equation. Once S is achieved, we iterate this procedure to obtain a sequence of symplectic transformations

Φ_{S_j} such that

$$H_{j+1} := H_j \circ \Phi_{S_j} = h_j + f_j$$

Of course, we want this procedure to converge. The convergence problem relates to the appearance of small divisors. Indeed, in each step of the KAM procedure, we have to solve homological equations

$$f_j^T + \{h_j, S_j\} = h_+ + O(\varepsilon^\alpha).$$

The Hamiltonian h_+ is in normal form. Putting this equation in Fourier formula leads us to estimate divisors $\omega \cdot k,\ \omega \cdot k \pm \Lambda_s,\ \omega \cdot k \pm \Lambda_s \pm \Lambda_t$, with $0 \neq k \in \mathbb{Z}^n$, where the eigenvalue Λ_s corresponds to external modes. Our procedure would be false if these divisors are too small. In the KAM procedure, one usually wants to find a control of these small divisors, likely the Diophantine inequality

$$\begin{aligned} |\ \langle \omega \cdot k \rangle\ | &\geq \frac{\alpha}{|k|^\tau}, \\ |\ \langle \omega \cdot k \pm \Lambda_s \rangle\ | &\geq \frac{\alpha}{|k|^\tau}, \\ |\ \langle \omega \cdot k \pm \Lambda_s \pm \Lambda_t \rangle\ | &\geq \frac{\alpha}{|k|^\tau}, \end{aligned}$$

for all $0 \neq k \in \mathbb{Z}^n$ and some fixed constants α and τ. The major difficulty is that the frequency ω moves during the KAM iteration. The nonlinearity f_j^T decreases super exponentially of size $\varepsilon^{j\alpha}(\alpha > 1)$ allow to the convergence of symplectic transformations $\Phi_{S_1} \circ \Phi_{S_2} \circ \cdots \to \Phi_\infty$ and the normal form $h_j \to h_\infty$. Since Φ_∞ is close to identity, one can talk about the existence of periodic or quasi periodic solutions of the original equation.

1.1.5 Birkhoff normal form

KAM theory is no doubt a very useful tool in studying Hamiltonian PDEs. However, in order to prove a KAM result, we usually need to assume some undesirable hypotheses, which are not always satisfied. Mathematicians want to know more about the behavior of the solutions lying outside KAM tori. The idea of Birkhoff normal form theory came up and soon played a crucial role in studying Hamiltonian system. Unlike the KAM procedure just works for a set of finite, nondegenerate invariant tori for every time, the Birkhoff normal form works for not any special tori but for all tori in a finite but sufficiently long time. Birkhoff normal form was first derived by Birkhoff and Lewis [BL34] in 1934 (see also [Lew34 ; Mos77]) where they consider the neighborhood of elliptic, non-constant, periodic orbits of Hamiltonian systems. In their paper, they put the Hamiltonian systems in fourth order normal form, namely

$$H = H_0 + G_4 + R_5$$

where G_4 is a homogeneous polynomial of degree 4 depending only on the actions and R_5 is a remainder term having a zero of at least fifth order at the origin.

The idea of Birkhoff normal form theory was then intensively studied in [Bam03 ; BG06 ; BDGS07 ; GIP09 ; Del12 ; FGL13 ; FI20 ; BG20]. Birkhoff normal form theory describes the behavior of solutions in neighborhoods of elliptic equilibrium points. Typically, it ensures some properties of stability for a finite but very long time.

In the finite dimensional context, we consider the Hamiltonian

$$H = H_0 + P$$

where P is a smooth function having a zero of order at least 3 at the origin, H_0 is the linear Hamiltonian

$$H_0 = \sum_{j \in J} \omega_j |u_j|^2.$$

Provided nonresonant hypothesis, the Birkhoff normal form theory states that, for each $r \geq 1$, one can conjugate H into a normal form

$$H \circ \mathcal{T}_r = H_0 + Z + R_r \tag{1.1.2}$$

where

- $\mathcal{T}_r$ is a real analytic symplectic transformation,
- the polynomial Z is of order $r+2$ depending only on the actions $I_j := |u_j|^2$,
- the remainder term R_r has a zero of order $r+3$.

Assuming the initial data is of size $\varepsilon \ll 1$, as a consequence of Birkhoff normal form, the solution remains bounded in the ball of radius 2ε with center at the origin for times of order ε^{-r}. Moreover, the solutions remain close to a torus of maximal dimension at a distance of size ε^{r_1} up to times of order ε^{r_2}, with $r_1 + r_2 = r + 1$. In [Bam03 ; BG06], the authors generalized Birkhoff normal form theory to infinite dimensional Hamiltonian systems. In these papers, the authors applied to nonlinear wave equations and nonlinear Schrödinger equations to obtain long time existence and bounds of solutions. The point here is that instead of considering the whole extension of PDEs, we split the phase variables in two groups : low modes and high modes. Precisely, we fix a positive integer number N then write $u = \overline{u} + \widehat{u}$ with $\overline{u} = \sum_{|j|\leq N} u_j e^{ijx}$ and $\widehat{u} = \sum_{|j|>N} u_j e^{ijx}$. Then one observes that monomials with more than two high mode variables in their expression are not relevant, since their vector field is already small. This was proved in [BG06] using tame inequality, namely

$$\|uv\|_{H^s} \leq C_s(\|u\|_{H^s}\|v\|_{H^1} + \|u\|_{H^1}\|v\|_{H^s}).$$

Note that

$$\|\widehat{u}\|_{H^s} \leq \frac{\|u\|_{H^s}}{N^{s-1}}$$

for $s \geq 1$. This term is negligible when N is sufficiently large. So that if the nonlinearity satisfies a tame modulus condition, it is always possible to put H in the form (1.1.2), where the remainder term R_r is of order $r + 5/2$ and Z is a polynomial of degree $r + 2$ containing only monomials which are "almost resonant". Assuming also the nonresonant condition, then H can be put in integrable Birkhoff normal form, i.e., Z depends only on the actions.

As in the KAM procedure, the proof of Birkhoff normal form theory relates to solve homological equations, likely to solve

$$P + \{H_0, S\} = Z^+ + O(\varepsilon^\alpha)$$

where P is a homogeneous polynomial of order at least three, Z^+ depends only on the actions. Expanding this equation leads us to prove a nonresonant condition for small divisors $\omega \cdot k$ with $0 \neq k \in \mathbb{Z}^\infty$. However, unlike the KAM procedure, the frequency ω does not change during the Birkhoff procedure.

Since we only consider small divisors related to the actions of low modes and maximum two high modes, the nonresonant conditions (at least in one dimension) are usually satisfied. Let's see an example, the nonlinear wave equation (NLW)

$$u_{tt} - \Delta u + V(x)u = g(x, u) \qquad x \in \mathbb{T}$$

where V is a C^∞, 2π periodic potential, having average m, and $g \in C^\infty(\mathbb{T} \times \mathcal{U})$, $\mathcal{U}$ being a neighbourhood of the origin in $\mathbb{R}$. It is proved in [Bam03 ; BG06] that for a large set of m, the frequencies $\omega_j = \sqrt{|j^2| + m}$ of the linear wave operator satisfy a nonresonant condition. Precisely, fix $r \geq 3$, there exist $\gamma > 0$ and $\alpha = \alpha(r)$ such that

$$|\omega_{k_1} + \cdots + \omega_{k_p} - \omega_{\ell_1} - \cdots - \omega_{\ell_q}| \geq \frac{\gamma}{\mu_3(k,\ell)} \qquad (*)$$

where $k = (k_1, \cdots, k_p) \in (\mathbb{Z}^d)^p,\ \ell = (\ell_1, \cdots, \ell_q) \in (\mathbb{Z}^d)^q$ with $p+q \leq r$, and $\{|k_1|, \cdots, |k_p|\} \neq \{|\ell_1|, \cdots, |\ell_q|\}$; $\mu_3(k,\ell)$ denotes the third largest number among the collection $(|k_i|, |\ell_j|)_{i,j}$. The condition (*) appears not only in the case of one dimensional wave equations but also in many different contexts, such as the one dimensional nonlinear Schrödinger equation with external potential [Bam03 ; Bam08 ; BG06] or even in multidimensional space for nonlinear Schrödinger equations [BG06 ; FGL13], and wave equations on Zoll manifolds [BDGS07].

1.1.6 Reducibility

Another topic of discussion among mathematicians is the existence of quasiperiodic solutions for time-forced PDEs. Typically, a time forced nonlinear PDE can be formulated as a fixed point problem, which can be solved via the Newton algorithm. We linearize the equation around an approximate quasi periodic solution u_0, then solve this linear equation to obtain u_1. We continue to linearize the equation around u_1 to get u_2 and then iterate this procedure. Solving the linear equations leads us to considering a time dependent linear operator and its inverse. The idea of reducibility is to conjugate such time dependent linear operator to a time independent diagonal operator.

We first see how this idea works in the context of ordinary differential equations. One considers a linear system of differential equations with periodic coefficients

$$\dot{x} = A(t)x \qquad t \in \mathbb{R};\ x \in \mathbb{R}^n$$

where $A(t)$ is an $n \times n$ periodic with period T, piecewise continuous matrix. Let $\Psi(t)$ be a fundamental matrix solution of this differential equation. Then

$$\Psi(t+T) = \Phi(t)\Psi(0)^{-1}\Psi(T) \qquad \forall t \in \mathbb{R}.$$

Moreover, Gaston Floquet(1833) said that there exists a periodic matrix function $P(t)$ and a constant coefficient matrix B such that

$$\Psi(t) = P(t)e^{tB} \qquad \forall t \in \mathbb{R}.$$

This mapping gives rise to a transformation, which puts the original equation into an autonomous form

$$y = P(t)^{-1}x, \qquad \dot{y} = By.$$

Then, the reducibility problem for forced PDEs, which is more difficult, has been intensively studied. The KAM theory plays a crucial role in proving reducibility results. In one dimensional context, we quote [Kuk93 ; BG01 ; LY10 ; GT11]. In these papers, the authors adapted a KAM procedure to prove the reducibility of Schrödinger equation with time dependent perturbation. All these results are for equations with bounded potential.

To consider Hamiltonian with unbounded perturbation, a breakthrough strategy has been developed in [BBM14 ; BBM16]. The idea is to apply pseudo-differential calculus to reduce the order of the perturbation. Pseudo-differential calculus allows us to reduce the perturbation to an arbitrary smoothing operator. More precisely, let f and g two pseudo-differential operators of order a and b, respectively, then their commutator $[f, g] := fg-gf$ is again a pseudo-differential operator but of order $a + b - 1$, while considering fg and gf separately is of order $a + b$. Apply this to a Hamiltonian with pseudo-differential perturbation $H = h + P$, we regularize it by an analytic, symplectic transformation

$$H \circ \Phi_S = h + P + [h, S] + [P, S] + \frac{1}{2}[[h, S], S] + \frac{1}{2}[[P, S], S] + \cdots$$

Thanks to the properties of pseudo-differential calculus, we can gain one regularity in each step of regularization after we solve the homological equation

$$P + [h, S] + [P, S] = OP^{a+b-1}.$$

The remainder of the new perturbation's terms are even more regular. Iterating this procedure, we can put H in a form with an arbitrary smoothing perturbation. Then a reducibility scheme is obtained by applying KAM theory. The idea of using pseudo-differential calculus has been demonstrated to be extremely useful in one dimensional context (see [BBM16 ; FP14 ; BM20 ; Mon19 ; Bam17 ; FGP18].

In a higher-dimensional context, on the other hand, we know very little. In [BGMR17], the authors proved a reducibility result for the quantum harmonic oscillator with time dependent polynomial perturbation on $\mathbb{R}^n$. In [BLM19], a reducibility result is obtained for transport equation on the d-dimensional torus $\mathbb{T}^d$ with a time quasi periodic unbounded perturbation. (See also [Mon19 ; FGMP19]). In all these results, the integrability of the unperturbed linear system plays a crucial role in controlling the perturbed spectrum.

In high dimensional cases, there is still a serious problem of perturbations. Typically, we associated the perturbation with a matrix operator which leads to, in KAM procedure, the homological equation be solved blockwise. However, the increasing size of the blocks may cause loss of regularity. This obstacle was overcome differently in [EK09], using geometric arguments, and in [GP19 ; FG19], where the authors used another argument.

1.1.7 The Schrödinger equation

The Schrödinger equation, which was named after Erwin Schrödinger in 1925, is a partial differential equation which is essential in quantum mechanics. The Schrödinger equation describes probability waves of a quantum mechanical system. It gives the evolution over time of the wave functions. The wave functions contain physical information of the system, such as position, momentum, energy, velocity or other physical properties. The equation is used extensively in atomic, nuclear, solid-state, and many other physical problems.

The Schrödinger equation in quantum mechanics is a counterpart of Newton's second law in classical mechanics. Considering a particle of mass m_e, its total energy E is sum of the potential energy $V(x)$ at position x and the kinetic energy $\frac{p^2}{2m_e}$

$$\frac{p^2}{2m_e} + V(x) = E.$$

Since energy is conserved, the particle is assumed to be confined to a certain region in space. By replacing p in the above energy equation with a differential operator and using de Broglie relation, Schrödinger showed that the wave function follows a time-independent partial differential equation

$$E\Psi(x) = \left[-\frac{\hbar^2}{2m}\Delta + V(x)\right]\Psi(x)\,.$$

Here Δ is Laplace operator. Eigenvalues associated with eigenfunctions of the linear operator $-\frac{\hbar^2}{2m}\Delta + V(x)$ forms quantum states with discrete state energy. The state energy E is proportional to angular frequency ω. Schrödinger applied his equation to the hydrogen atom. The square of the wave function Ψ^2 gives the probability of finding the electron at position x and time t.

By replacing the energy E in Schrödinger's equation with a time-derivative operator, Schrödinger then generalized his wave equation to describe how a system changes from one state to another

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\Delta + V(x,t)\right]\Psi(x,t)\,.$$

Here, the constant i is the imaginary unit, and $\hbar$ is the reduced Planck constant. The time dependent Schrödinger equation allows us to calculate the probability of a transition between one atomic stationary state and some other state.

Nonlinear Schrödinger equation

The nonlinear Schrödinger (NLS) equation is a nonlinear variation of the linear Schrödinger equation, applicable to both classical mechanics and quantum mechanics. The equation appears in many areas of physics and are analyzed mathematically by mathematicians. It was derived in the studies of the propagation of light in nonlinear optical fibers and planar waveguides, the Langmuir waves in hot plasma, the small-amplitude gravity waves on the surface of deep inviscid water, magnetic spin waves, and many others. Typically, most weakly nonlinear, dispersive, energy-preserving systems appropriately give rise to the NLS equation. Historically, the NLS equations were first derived by Ginzburg and Landau [GL55] in their study of the macroscopic theory of superconductivity and became well-known, especially in connection with the phenomenon of self focusing, after the work about optical beams of Chiao et al [CGT64]. From then until now, the NLS equation has been intensively studied in many areas of physics and mathematics. In the optical context, Hasegawa and Tappert [HT73] first derived the NLS equation taking into account both group velocity dispersion and fiber nonlinearity due to the so-called Kerr effect. Eisenberg et al [ESMBA98] drew a discrete NLS model for a coupled optical waveguides. The NLS equations on a lattice background were first studied by Efremidis et al [EHCFCS03]. The experimental realization of Bose-Einstein condensates generates an NLS equation with external potentials. The NLS equation for small-amplitude water waves was derived by Zakharov [Z68] for the case of infinite depth, then Benney and Roskes [BR69] for the case of finite depth.

In one dimensional context, the cubic NLS equation is integrable, which was solved by Zakharov and Shabat [ZS72] via the Inverse Scattering Transform - which is a nonlinear Fourier Transform. It admits an infinite number of conserved quantities and multisoliton solutions. For the equation with nonlinearity of higher order or in higher dimensional context, it is not integrable, the phenomenon of wave collapse and turbulence can take place.

Different nonlinear terms form different nonlinear Schrödinger equations, among them the cubic NLS and the quintic NLS are most studied. The cubic NLS takes the form

$$i\partial_t\psi = -\Delta\psi + \kappa|\psi|^2\psi$$

here $\partial_t\psi$ is a complex vector field. The equation associates with the Hamiltonian

$$H = \int \mathrm{d}x \left[|\partial_x\psi|^2 + \frac{\kappa}{2}|\psi|^4\right].$$

The equation is called focusing NLS when κ is negative. The focusing NLS has bright soliton solutions (localized in space, and having spatial attenuation towards infinity). This case was solved by use of the inverse scattering transform, see [ZS72]. The equation is called defocusing when κ is positive. The defocusing NLS has dark soliton solutions (having constant amplitude at infinity, and a local spatial dip in amplitude) . See also [GK14 ; GK02 ; KL12 ; Bou99]. In optics, the NLS equation describes the propagation of the wave in fiber optics through a nonlinear medium while for water waves, the solution ψ is related to the amplitude and phase of the water waves. The value of the nonlinearity parameter κ depends on the relative water depth. The NLS equation is focusing on shallow water, with the water depth small compared to the wave length of the water waves, and defocusing on deep water .

Considering purely self-focusing cubic nonlinearity in 2 spatial dimension, it was proved by Vlasov et al [VPT70] , the phenomenon of wave collapse takes place and the light beam blows up in a finite time. This is based on the fact that the Hamiltonian $H = \int \mathrm{d}x \left[|\partial_x\psi|^2 - \frac{1}{2}|\psi|^4\right]$ is negative for suitable initial data, then there exists a finite time T such that the quantity $\int \mathrm{d}x|\partial_x\psi|^2$ blows up.

In d spatial dimensions, one consider the generic power nonlinearity equation

$$i\partial_t\psi = -\Delta\psi - \psi|\psi|^{p-2}, \qquad x \in \mathbb{R}^d.$$

It is well known that there is locally wellposedness for any data in H^s with

$$\begin{cases} s \geq 0 & \text{if } p \leq 2 + \frac{4}{d} \\ s \geq s_*,\ s_* \text{ defined by } p = 2 + \frac{4}{d-2s_*} & \text{if } p > 2 + \frac{4}{d} \end{cases}.$$

Moreover, in H^1, one has the following cases

— Critical ($p = 2 + \frac{4}{d-2}$) : blowup can occur or global solution can exist.

— Subcritical ($p < 2 + \frac{4}{d-2}$) : global solutions exist.

There are many works concerning the theory of existence, uniqueness and long-time dynamical behaviors as well as the regularity problems of the NLS equations. In $\mathbb{R}^d$, we quote

[BGT80 ; GV92 ; G00 ; K87 ; Y87 ; KT98] for the energy subcritical problems, and [Bou99 ; CKSTT08 ; KM06 ; KM10 ; TVZ07 ; TVZ08] for the energy critical cases. For the NLS equations in $\mathbb{R}^d$, the Morawetz's inequalities and Strichartz estimates play a crucial role. The Cauchy problem for the nonlinear Schrödinger equation on torus $\mathbb{T}^d$ was studied by Bourgain [Bou93bb], where he extended the classical Strichartz's inequalities to $\mathbb{T}^d$ in all dimensions. These inequalities are called moment estimate for trigonometric polynomials. In this paper, Bourgain proved that

— ($d = 1$). The NLS equation is locally well-posed for $\psi \in H^s(\mathbb{T})$, provided $p < 2 + \frac{2}{1-2s}$;

— ($d = 2$). The NLS equation is globally well-posed for $p = 4$ with initial data in $H^1(\mathbb{T}^2)$ and sufficiently small L_2-norm. The same result holds for all $\alpha \geq 2$ for sufficiently small H^1-data ;

— ($d = 3$). The NLS equation is globally well-posed for $4 \leq p < 6$ with sufficiently small initial data in $H^1(\mathbb{T}^3)$;

— ($d \geq 4$). The NLS equation is locally well-posed for $4 \leq p < 2 + \frac{4}{d-2s}$ and $s > \frac{3d}{d+4}$.

See also [Bou93aa ; Bou93bb ; Bou13]. In general compact manifolds, the approach to the Strichartz estimates is much different from Bourgain. The Cauchy problem of NLS on general compact manifolds was initiated by Burq et al [BGT02 ; BGT04 ; BGT05 ; BGT09].

As long as the well-posedness (local or global) is proved, one want to study the long time behavior of solutions. The initial datum are assumed to be in a Sobolev space H^s. These initial datum form different invariant tori. Will these tori survive or be destroyed after a long time ? Can we control the exchange of energy between different modes of these tori ? These questions have been studied extensively recently through the use of KAM theory and Birkhoff normal form theory.

1.1.8 The abcd Boussinesq system

In 1757, Euler introduced a set of quasi-linear hyperbolic equations to describe the irrotational waves on the surface of an inviscid fluid under the gravity force. These equations take into account the conservation of mass, momentum and energy while dissipative and surface tension effects are safely ignored. However, in many theoretical, numerical and practical situations, the full Euler equations seem to be more complicated than necessary and further approximated models have been introduced to restricted physical regimes.

In 1872, Joseph Boussinesq, in response to an observation by John Scott Russell of the solitary wave, derived approximations valid for weakly non-linear, small amplitude and fairly long waves in a channel of approximately constant depth h. The approximations are counterparts to the Stokes expansion, which is appropriate for short waves. The Boussinesq systems take into account the vertical structure of horizontal and vertical flow velocity. Denote A the wave amplitude and ℓ the wavelength, the considered situation is that

$$\alpha := \frac{A}{h} \ll 1, \qquad \beta := \frac{h^2}{\ell^2} \ll 1, \qquad S := \frac{\alpha}{\beta} = \frac{A\ell^2}{h^3} \approx 1.$$

Boussinesq derived the one-dimensional wave model

$$w_{tt} = w_{xx} + (w^2)_{xx} + w_{xxxx}$$

or its regularized version

$$w_{tt} = w_{xx} + (w^2)_{xx} + w_{xxtt}$$

and system of two coupled equations

$$\begin{aligned} \eta_t + w_x + (w\eta)_x &= 0 \\ w_t + \eta_x + ww_x + \frac{1}{3}\eta_{xtt} &= 0 \end{aligned}$$

or its regularized version

$$\begin{aligned} \eta_t + w_x + (w\eta)_x &= 0 \\ w_t + \eta_x + ww_x - \frac{1}{3}w_{xxt} &= 0. \end{aligned}$$

These equations were derived directly from the Eulerian equation of the water wave problem. These equations are formally comparable to KDV equations and Kadomtsev–Petviashvili equations. Indeed, there are an overwhelming number of different but formally Boussinesq-type system. These systems may have different mathematical properties.

In [BCS02 ; BCS04], Bona, Chen and Saut derived a family of Boussinesq-type systems, which depend on four parameters a, b, c, d and are called the $abcd-$Boussinesq systems

$$\begin{cases} (1 - b\partial_{xx})\partial_t \eta + \partial_x(a\partial_{xx}u + u + u\eta) = 0 \\ (1 - d\partial_{xx})\partial_t u + \partial_x(c\partial_{xx}\eta + \eta + \frac{1}{2}u^2) = 0 \end{cases}. \tag{1.1.3}$$

Here the independent variable x corresponds to distance along the channel and t is proportional to elapsed time. The quantity $\eta = \eta(x, t)$ corresponds to the depth of the water at the point x and time t. The variable $u(x,t)$ is proportional to the horizontal velocity at the height θh, where θ is a fixed constant in the interval $[0,1]$ and h is the undisturbed water depth. The four parameters a, b, c, d obey the relations

$$\begin{cases} a+b = \frac{1}{2}(\theta^2 - \frac{1}{3}) \\ c+d = \frac{1}{2}(1-\theta^2) \geq 0 \\ a+b+c+d = \frac{1}{3} \end{cases} \tag{1.1.4}$$

When $(a = b = c = 0, d = \frac{1}{3})$, (1.1.3) is the classical Boussinesq system. Higher order nonlinearity system were also derived in [BCS02 ; BCS04]. In fact, different possible values of a, b, c, d form different Boussinesq systems with different properties, such systems are specialized as subclasses : classical Boussinesq system $(a = b = c = 0, d = \frac{1}{3})$, Kaup system $(a = \frac{1}{3}, b = c = d = 0)$, coupled BBM system $(a = c = 0, b = d = \frac{1}{6})$, coupled KdV system $(a = c = \frac{1}{3}, b = d = 0)$, etc. All these models are derived from the full Euler equations for two-dimensional water waves under the force of gravity by truncating a Taylor expansion of the velocity potential. As any PDEs for physical regimes, there arise questions, both theoretical and practical : Well-posedness of initial-value problems, existence of solitary-wave solutions, energy exchange in different Fourier modes, etc.

1.2 Results of the thesis

1.2.1 An unstable three dimensional KAM torus for the quintic NLS on the circle

In chapter 2, we prove a KAM result for the quintic NLS in the circle [N19]. The work presented here is the center of an article published in "Dynamics of Partial Differential Equations (DPDE)". We consider the quintic nonlinear Schrödinger equation on the torus

$$i\partial_t u + \partial_{xx} u = |u^4|u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}. \tag{1.2.1}$$

associated with the Hamiltonian

$$h = \int_{\mathbb{T}} |u_x|^2 + \frac{1}{3}|u|^6 dx$$

and the symplectic form $-\mathrm{i}du\wedge d\overline{u}$. The solution $u(t)$ preserves the mass and the momentum

$$\mathbb{L}=\int_{\mathbb{T}}|u|^2dx,\quad \mathbb{M}=\int_{\mathbb{T}}Im(u\cdot\nabla\overline{u})dx.$$

Let us expand u and $\overline{u}$ in Fourier variables :

$$u(t,x)=\sum_{j\in\mathbb{Z}}a_j(t)e^{ijx},\quad \overline{u}(t,x)=\sum_{j\in\mathbb{Z}}b_j(t)e^{-ijx}.$$

The Hamiltonian h of the system reads :

$$h=\sum_{j\in\mathbb{Z}}j^2a_jb_j+\frac{1}{3}\sum_{j,\ell\in\mathbb{Z}^3;\mathcal{M}(j,l)=0}a_{j_1}a_{j_2}a_{j_3}b_{\ell_1}b_{\ell_2}b_{\ell_3}=N+P,$$

with the symplectic structure $-i\sum_{j\in\mathbb{Z}}da_j\wedge db_j$, here $\mathcal{M}(j,l)=j_1+j_2+j_3-\ell_1-\ell_2-\ell_3$ denotes the momentum of the monomial $a_{j_1}a_{j_2}a_{j_3}b_{\ell_1}b_{\ell_2}b_{\ell_3}$. We are interested in the dynamic behavior near to 0 of solution of (2.1.1) in two specific forms :

$$u(t,x)=a_p(t)e^{ipx}e^{-ip^2t}+a_q(t)e^{iqx}e^{-iq^2t}+\mathcal{O}(\varepsilon),\tag{1.2.2}$$

and

$$u(t,x)=a_p(t)e^{ipx}e^{-ip^2t}+a_q(t)e^{iqx}e^{-iq^2t}+a_m(t)e^{imx}e^{-im^2t}+\mathcal{O}(\varepsilon),\tag{1.2.3}$$

or more precisely, the persistence of two and three dimensional linear invariant tori :

$$\mathbf{T}_c^2(p,q)=\{|a_p|^2=c_1,\ |a_q|^2=c_2\},\tag{1.2.4}$$

$$\mathbf{T}_c^3(p,q,m)=\{|a_p|^2=c_1,\ |a_q|^2=c_2,\ |a_m|^2=c_3\},\tag{1.2.5}$$

with $0<c_1,c_2,c_3\ll 1$.

We begin by demonstrating that all two dimensional tori are linearly stable.

Theorem 1.2.1. *Fix p, $q\in\mathbb{Z}$, and $s>\frac{1}{2}$. There exists $\nu_0>0$, and for $0<\nu<\nu_0$, there exists $\mathcal{D}_\nu\subset[1,2]^2$ asymptotically of full measure (i.e., meas$([1,2]^2\setminus\mathcal{D}_\nu)\to 0$ when $\nu\to 0$) such that for $\rho\in\mathcal{D}_\nu$, equation (1.2.1) admits a solution of the form*

$$u(x)=\sum_{j\in\mathbb{Z}}a_j(t\omega)e^{ijx}$$

where $\{a_j\}_j$ is analytic function from $\mathbb{T}^2$ to ℓ_s^2 satisfying uniformly in $\theta\in\mathbb{T}^2$

$$||a_p|-\sqrt{\nu\rho_1}|^2+||a_q|-\sqrt{\nu\rho_2}|^2+\sum_{j\neq p,q}(1+j^2)^s|a_j|^2=\mathcal{O}(\nu^2).$$

Here ω is a nonresonant vector in $\mathbb{R}^2$ that satisfies

$$\omega = (p^2, q^2) + \mathcal{O}(\nu^2).$$

Furthermore, this solution is linearly stable.

Here we say a solution u is linearly stable if the linearization of the equation at this solution has linear operator whose spectrum contains only pure imaginary eigenvalues.

In the case of three dimensional tori, we will give an example of (p, q, m) and ρ such that for ν small enough the torus $\mathbf{T}^3_{\nu\rho}(p, q, m) = \{|a_p|^2 = \nu\rho_1,\ |a_q|^2 = \nu\rho_2,\ |a_m|^2 = \nu\rho_3\}$ is linearly unstable. Let $\epsilon = 10^{-2}$, denote

$$\mathcal{D} = \mathcal{D}_2 = [2-\epsilon, 2+\epsilon] \times [1-\epsilon, 1+\epsilon] \times [9-\epsilon, 9+\epsilon].$$

Then we have the following theorem

Theorem 1.2.2. *Fix $p = -3$, $q = 10$, $m = -6$, and $s > \frac{1}{2}$. There exists $\nu_0 > 0$, and for $0 < \nu < \nu_0$, there exists $\mathcal{D}_\nu \subset \mathcal{D}$ asymptotically of full measure (i.e., meas$(\mathcal{D} \setminus \mathcal{D}_\nu) \to 0$ when $\nu \to 0$) such that for $\rho \in \mathcal{D}_\nu$, equation (1.2.1) admits a solution of the form*

$$u(x) = \sum_{j\in\mathbb{Z}} a_j(t\omega)e^{ijx} \tag{1.2.6}$$

where $\{a_j\}_j$ is analytic function from $\mathbb{T}^3$ to ℓ^2_s satisfying uniformly in $\theta \in \mathbb{T}^3$

$$||a_p| - \sqrt{\nu\rho_1}|^2 + ||a_q| - \sqrt{\nu\rho_2}|^2 + ||a_m| - \sqrt{\nu\rho_3}|^2 + \sum_{j\neq p,q,m} (1+j^2)^s |a_j|^2 = \mathcal{O}(\nu^2). \tag{1.2.7}$$

Here ω is a non resonant vector in $\mathbb{R}^3$ that satisfies

$$\omega = (3^2, 10^2, 6^2) + \mathcal{O}(\nu^2).$$

Furthermore, this solution is linearly unstable.

Avoiding the case (2.1.7), we can generalize the theorem for all sets of three Fourier modes (p, q, m) which satisfy the system

$$\begin{cases} 2p + q &= m + s + t \\ 2p^2 + q^2 &= m^2 + s^2 + t^2. \end{cases} \tag{1.2.8}$$

for some integer numbers s, t.

Scheme of the proof

Our results base on a Birkhoff normal form procedure and a KAM theorem. In Birkhoff normal form step, one kills the nonresonances of the quintic nonlinearity P. More precise, the original Hamiltonian is transformed into the following form using a canonical transformation

$$\overline{h} = h \circ \tau = N + Z_6 + R_{10},$$

where

— N is the term $N(I) = \sum_{j\in\mathbb{Z}} j^2 |a_j|^2$;
— Z_6 is a homogeneous polynomial of degree 6 containing only the resonant part

$$Z_6 = \sum_{\mathcal{R}} a_{j_1} a_{j_2} a_{j_3} b_{\ell_1} b_{\ell_2} b_{\ell_3}$$

where
$\mathcal{R} = \{(j,\ell) \in \mathbb{Z}^3 \times \mathbb{Z}^3 \; s.t \; j_1 + j_2 + j_3 = \ell_1 + \ell_2 + \ell_3, \quad j_1^2 + j_2^2 + j_3^2 = \ell_1^2 + \ell_2^2 + \ell_3^2\}$;
— R_{10} is the remainder of order 10.

In the KAM procedure, let us write the Hamiltonian as following

$$h = h_0 + f$$
$$h_0 = \Omega(\rho) \cdot r + \sum_{a \in \mathcal{Z}} \Lambda_a(\rho) |\zeta_a|^2.$$

Here

— ρ is a parameter in $\mathcal{D}$, which is compact in the space $\mathbb{R}^n$;
— $r \in \mathbb{R}^n$ and $\theta \in \mathbb{T}^n$ are the action and angle associated with the internal modes $(r,\theta) \in (\mathbb{R}^n \times \mathbb{T}^n, dr \wedge d\theta)$;
— $\zeta = (\zeta_a)_{a\in\mathcal{Z}} \in \mathbb{C}^{\mathcal{Z}}$ are the external modes endowed with the standard complex symplectic structure $-id\zeta \wedge d\eta$, with $\eta = \overline{\zeta}$. Denote also $\omega = (\zeta, \eta)$
— the mappings

$$\Omega : \mathcal{D} \to \mathbb{R}^n, \tag{1.2.9}$$
$$\Lambda_a : \mathcal{D} \to \mathbb{C}, \quad a \in \mathcal{Z}, \tag{1.2.10}$$

are smooth ;
— $f = f(r,\theta,\zeta;\rho)$ is a perturbation, small compare to the integrable part h_0.

For the perturbation f, we define its jet function $f^T(x)$ as the following Taylor polynomial of f at $r=0$ and $\omega=0$

$$f^T(x) = f(0,\theta,0) + d_r f(0,\theta,0)\cdot r + d_\omega f(0,\theta,0)[\omega] + 1/2 d_\omega^2 f(0,\theta,0)[\omega,\omega].$$

By applying iteratively a KAM scheme, we put the Hamiltonian into a normal form

$$(h_0 + f)\circ\Phi = \tilde{h} + g$$

with $\tilde{h} = \Omega(\rho)\cdot r + \langle \zeta_{\mathcal{L}}, Q(\rho)\eta_{\mathcal{L}}\rangle + 1/2\langle \omega_{\mathcal{F}}, K(\rho)\omega_{\mathcal{F}}\rangle$ on normal form, and the jet part of g is vanishing, i.e., $g^T \equiv 0$. The set $\mathcal{L}$ corresponds to elliptic directions, while the set $\mathcal{F}$ corresponds to hyperbolic directions. The considered torus is linearly stable if and only if $\mathcal{F} = \varnothing$. The KAM procedure requires hypotheses on small divisors, the conservation of the mass and the momentum give us a good tool to estimate them. This is verified precisely in the appendix.

1.2.2 Reducibility of Schrödinger equation on a Zoll manifold with unbounded potential

In chapter 3, we are interested in the reducibility of Schrödinger equation on a Zoll manifold [FGN20]. Precisely, we prove a reducibility result for the linear Schrödinger equation on a Zoll manifold with quasi-periodic in time pseudo-differential perturbation of order less or equal than $\frac{1}{2}$. The work presented here has been published in the "Journal of Mathematical Physics".

In fact, we are considering the following linear Schrödinger equation

$$i\partial_t u = \Delta_g u + \varepsilon W(\omega t)u, \qquad (t,x)\in\mathbb{R}\times M^n \tag{1.2.11}$$

where $\varepsilon > 0$ is a small parameter, $\omega \in \mathbb{R}^d, d \geq 1$, is a frequency vector, M^n is a Zoll manifold and Δ_g is the positive Laplace-Beltrami operator defined on M^n. We recall that a Zoll manifold of dimension $n \in \mathbb{N}$ is a compact Riemannian manifold such that all the geodesic curves have the same period T, assuming $T = 2\pi$. A typical example of Zoll manifold is the sphere $\mathbb{S}^n$. The linear operator W is a pseudo-differential operator of order $\delta \leq \frac{1}{2}$. We denote $\mathcal{A}_m$ the class of pseudo-differential operators of order $m\in\mathbb{R}$, then $W \in C^\infty(\mathbb{T}^d, \mathcal{A}_\delta)$. We consider the solutions in the Sobolev space defined as $H^s(M^n) := dom(\sqrt{1+\Delta_g})^s$.

The purpose of this chapter is to find a transformation that puts the non-autonomous equation (1.2.11) into an autonomous form .

Our main result is the following.

Theorem 1.2.3. *Let $0 < \alpha < 1$ and $\delta \in \mathbb{R}$, $\delta \le 1/2$. Assume that the map $\varphi \mapsto W(\varphi, \cdot) \in \mathcal{A}_\delta$ is C^∞ in $\varphi \in \mathbb{T}^d$. Then for any $s \in \mathbb{R}$, $s > n/2$ there exists $\varepsilon_0 > 0$ and $C > 0$ such that, for any $0 < \varepsilon \le \varepsilon_0$ there is a set $\mathcal{O}_\varepsilon \subset [1/2, 3/2]^d \subset \mathbb{R}^d$ with*

$$\operatorname{meas}([1/2,3/2]^d \setminus \mathcal{O}_\varepsilon) \le C\varepsilon^\alpha \tag{1.2.12}$$

such that the following holds. For any $\omega \in \mathcal{O}_\varepsilon$ there exists a family of linear isomorphism $\Psi(\varphi) \in \mathcal{L}(H^s(\mathtt{M}^n))$ and a Hermitian operator $Z \in \mathcal{A}_\delta$ commuting with the Laplacian[1] *and satisfying*

$$\|Z\|_{\mathcal{L}(H^s(\mathtt{M}^n), H^{s-\delta}(\mathtt{M}^n))} \le C\varepsilon\,. \tag{1.2.13}$$

Furthermore
- *$\Psi(\varphi)$ is unitary on $L^2(\mathtt{M}^n)$;*
- *for any $\frac{n}{2} < s' \le s$ and any $\omega \in \mathcal{O}_\varepsilon$*

$$\begin{aligned} \|\Psi(\varphi) - \mathrm{Id}\|_{\mathcal{L}(H^{s'}(\mathtt{M}^n), H^{s'-\delta}(\mathtt{M}^n))} \qquad\qquad & \\ + \|\Psi(\varphi)^{-1} - \mathrm{Id}\|_{\mathcal{L}(H^{s'}(\mathtt{M}^n), H^{s'-\delta}(\mathtt{M}^n))} &\le C\varepsilon^{1-\alpha}\,, \\ \|\Psi(\varphi)\|_{\mathcal{L}(H^{s'}(\mathtt{M}^n))} + \|\Psi(\varphi)^{-1}\|_{\mathcal{L}(H^{s'}(\mathtt{M}^n))} &\le 1 + C\varepsilon^{1-\alpha}\,, \end{aligned} \tag{1.2.14}$$

- *for any $\frac{n}{2} < s' \le s$ and any $\omega \in \mathcal{O}_\varepsilon$ the map $t \mapsto u(t, \cdot) \in H^{s'}(\mathtt{M}^n)$ solves* (3.1.1) *if and only if the map $t \mapsto v(t, \cdot) := \Psi(\omega t)u(t, \cdot)$ solves the autonomous equation*

$$\mathrm{i}\partial_t v = \Delta_g v + \varepsilon Z(v)\,. \tag{1.2.15}$$

As a consequence of reducibility, one proves the existence of almost-periodic solution. Precisely, one has the following corollary.

Corollary 1.2.4. *Let $W \in C^\infty(\mathbb{T}^d; \mathcal{A}_\delta)$ with $\delta \le 1/2$. Then, for any $s \in \mathbb{R}$, $s > n/2$ there exists $\varepsilon_0 > 0$ and $C > 0$ such that, for any $0 < \varepsilon \le \varepsilon_0$ there is a set $\mathcal{O}_\varepsilon \subset [1/2, 3/2]^d \subset \mathbb{R}^d$ satisfying* (1.2.12) *such that for any $\omega \in \mathcal{O}_\varepsilon$ the flow generated by the* (1.2.11) *equation is bounded in $H^s(\mathtt{M}^n)$.*
More precisely, if $u_0 \in H^s(\mathtt{M}^n)$ then there exists a unique solution $u \in C^1\Big(\mathbb{R}\,;\, H^s(\mathtt{M}^n)\Big)$ of (1.2.11) *such that $u(0) = u_0$. Moreover, u is almost-periodic in time and satisfies*

$$(1 - \varepsilon C)\|u_0\|_{H^s} \le \|u(t)\|_{H^s} \le (1 + \varepsilon C)\|u_0\|_{H^s}, \quad \forall\, t \in \mathbb{R}\,, \tag{1.2.16}$$

1. actually $[\Delta_g, Z] = 0$ on sphere while on Zoll manifold we have Z and Δ_g can be diagonalized in the same basis of $L^2(\mathtt{M}^n)$.

for some $C = C(s) > 0$.

We also mention the reducibility result on the torus $\mathbb{T}^n$ in [BLM19], where the authors study the transport equations on the torus, which is an integrable system, and the recent reducibility result for the Schrödinger equation on the Sphere [FG19], where the authors consider quasi-periodic in time odd perturbations of order $< 1/2$ and, in particular, do not require pseudo-differential calculus.

Our result gives an idea to approach the nonlinear Schrödinger equation. Consider the non linear Schrödinger equation

$$\mathrm{i}\partial_t u = \Delta_g u + mu + \varepsilon |u|^2 u\,, \qquad u = u(t,x)\,, \quad t \in \mathbb{R}\,, \quad x \in \mathtt{M}^n\,,$$

we would like to solve this equation using Newton's method. Starting with an approximate solution u_0, we linearize the NLS equation around u_0 and solve the linear equation to obtain u_1, do this again to obtain u_2 and iterate this procedure to obtain a convergent sequence of solutions. However, this approach has to face some obstacles, which we mention in chapter 3. First, linearizing the cubic nonlinearity at the point u in the direction h, one obtains $2|u|^2 h + u^2 \overline{h}$. As first step of the regularization procedure, one need to eliminate $\overline{h}$. The major problems regard the minimal regularity of the potential $W(\omega t)$, which is now $|u_0(t)|^2$, and the study of the small divisors relating in KAM procedure. In the linear Schrödinger equation, we need some requirements for the regularity of the potential and small divisors, which do not persist in Newton scheme.

Scheme of the proof

The result is proven in two steps : regularization step and KAM step. In regularization step, the pseudo-differential calculus is used to transform time-dependent, unbounded potential system in a system with a time-dependent, smoothing perturbation. Then we use a KAM procedure on infinite dimensional matrices to put the equation into an autonomous form.

In the regularization step, we prove that we can transform (by using a symplectic map : $u = \Phi(v)$) the original Schrödinger equation into a new one

$$\mathrm{i}\partial_t v = \Delta_g v + \varepsilon(Z + R(\omega t))v\,, \tag{1.2.17}$$

where Z is a pseudo-differential operator of order δ independent on time and commuting with Δ_g and R is a ρ-regularizing operator in $\mathcal{L}(H^s(\mathtt{M}^n), H^{s+\rho}(\mathtt{M}^n))$ with ρ arbitrary large. In fact, the regularization step consists of two parts : averaging the pseudo-differential operators, the averaged operators correspond to diagonalized block matrices, and eliminating the time in averaged operators.

The idea of averaging pseudo-differential operators is based on the fact that we can write $\sqrt{\Delta_g} = K_0 - Q$ where Q is a pseudo-differential operator of order -1 chosen (following [**Colin**]) in such a way that the spectrum of K_0 is included in $\mathbb{N} + \lambda$ for some constant $\lambda \in \mathbb{R}^+ (\lambda > 0)$.

This property makes the K_0 flow periodic and leads us to the fact that if A is a pseudo-differential operator, then its average with respect to the flow of K_0 is given by $\langle A \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}\tau K_0} A e^{\mathrm{i}\tau K_0} d\tau$. This idea was already used in a pioneering work of Weinstein [Wein77]. Let us see how this works for the original Schrödinger equation. Let us write $H = H_0 + V(t)$ where $H_0 = \Delta_g$, and $V(t) = \varepsilon W(\omega t)$ is a pseudo-differential operator of order δ. Denote $Y = \frac{1}{2\pi}\int_0^{2\pi} \tau(V - \langle V \rangle)(\tau) d\tau \in \mathcal{A}_\delta$, and $S = \frac{1}{4}(YK_0^{-1} + K_0^{-1}Y)$ a pseudo-differential operator of order $\delta - 1$, then S solves the following homological equation

$$V + \mathrm{i}[S, K_0^2] = \langle V \rangle + \text{order} \quad \delta - 2.$$

Then the flow generated by S $\Psi_S = e^{\mathrm{i}S(t)}$ conjugates the original Hamiltonian H to a new one $H^+(t)$ with

$$H^+ = H^0 + \langle V(t) \rangle + \text{order } \delta - \nu$$

where $\nu = min(1, 2-\delta)$. Thus if $\delta < 2$, we have a better equation. In the time eliminating step, we find a Lie transformation $\Psi_T = e^{\mathrm{i}T}$ that kills the time in $Z = \langle V \rangle$. This time eliminating step requires a non resonance hypothesis on the frequency vector ω of form

$$\mid \omega \cdot k \mid \geq \frac{\gamma}{|k|^\alpha} \qquad k \in \mathbb{Z}^n.$$

We then alternate the averaging procedure with the time elimination procedure.

After the regularizing procedure, we do a KAM procedure to kill the remainder term R in (1.2.17) which still depends on time but is now a arbitrary smoothing operator. We coincide the operator Z, and the remainder term R with their matrix representation. The matrix Z is block-diagonal, and thus, the spectrum of $\Delta_g + Z$ preserves the cluster structure inherited from Δ_g on the Zoll manifold. We also have a link between $\rho-$smoothing operators, and $\beta-$regularizing matrices. The KAM procedure consists in solving homological equations, which are solved blockwise. The increasing size of the blocks may generate loss of regularity, but this loss is acceptable since R is a regularizing operator. We also notice that the new remainder term R_+ after a KAM step is estimated by a tame inequality with two different norms, a low $s-$decay norm and a high $s+b-$decay norm. This tame estimation allows to obtain a convergent scheme for the sequence of remainders R_k.

1.2.3 Birkhoff normal form for abcd Boussinesq system on the circle

In chapter 4, we investigate the long time behavior of *abcd* Boussinesq system on the circle [N21]. Precisely, we consider the system

$$\begin{cases} (1-b\partial_{xx})\partial_t\eta + \partial_x(a\partial_{xx}u + u + u\eta) = 0 \\ (1-d\partial_{xx})\partial_t u + \partial_x(c\partial_{xx}\eta + \eta + \frac{1}{2}u^2) = 0 \end{cases} \qquad (t,x) \in \mathbb{R}\times\mathbb{T}. \tag{1.2.18}$$

Where η, u are real functions with zero average

$$\int_{\mathbb{T}} \eta(t,x)\ \mathrm{d}x = \int_{\mathbb{T}} u(t,x)\ \mathrm{d}x = 0.$$

The system was derived by Bona, Chen and Saut [BCS02 ; BCS04], in the vein of the original Boussinesq system, to describe the two dimensional, incompressible and irrotational water wave in the shallow water regime. The two functions $\eta(x,t)$ and $u(x,t)$ describe the behavior of water in the vertical direction and horizontal direction at the position x and at time t. Four parameters a,b,c,d satisfy the consistency conditions (1.1.4). In fact, the system (1.2.18) has different properties when the four parameters a,b,c,d vary. In this chapter, we study the system in the "generic Hamiltonian" case, namely the case $b=d>0, a,c<0$.

Expand the solution in Fourier variables, one has

$$u(x) = \sum_{k\in\mathbb{Z}^*} u_k e^{2i\pi kx}, \qquad \eta(x) = \sum_{k\in\mathbb{Z}^*} \eta_k e^{2i\pi kx} \tag{1.2.19}$$

note that $\overline{u}_k = u_{-k}$, $\overline{\eta}_k = \eta_{-k}$ since u and η are real, then (1.2.18) reads [2]

$$\begin{cases} \partial_t\eta_k = -\frac{i2\pi k}{1+4\pi^2bk^2}((1-4\pi^2ak^2)u_k + \sum_{j+l=k} u_j\eta_l) \\ \partial_t u_k = -\frac{i2\pi k}{1+4\pi^2bk^2}((1-4\pi^2ck^2)\eta_k + \frac{1}{2}\sum_{j+l=k} u_j u_l) \end{cases} \qquad k \in \mathbb{Z}^*. \tag{1.2.20}$$

The couple solutions (η,u) can be identified with their Fourier expansions $(\eta,u) = (\eta_k,u_k)_{k\in\mathbb{Z}*}$. We study these solutions on the Sobolev space $(s \geq 0)$

2. Here $\mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$

$$H^s := \{z = (u,v) = (u_k, v_k)_{k\in\mathbb{Z}^*} \in \ell^2(\mathbb{T}) \times \ell^2(\mathbb{T}) \mid \|z\|_s^2 = \sum_{k\in\mathbb{Z}^*} |k|^{2s}(|u_k|^2 + |v_k|^2) < \infty\}. \tag{1.2.21}$$

One prove that for a long time, the energy exchanges mainly in same Fourier modes η_k of horizontal velocity and Fourier modes u_k of the depth. Precisely, one has the following informal theorem

Theorem 1.2.5. *Let fix $r \geq 1$, fix $b = d > 0$, for any sufficiently large s and for almost all values of a, c, assume that the initial data $\|(u,v)(x,0)\|_s = \mu$ is small, i.e., $\mu \ll 1$, then there exists a constant $C = C(r,s,b)$ such that*

$$\partial_t(\alpha_k^2|u_k|^2 + \alpha_k^{-2}|\eta_k|^2) \leq \mu^{r+\frac{5}{2}} \qquad \textit{for}\, t \leq C\mu^{-r+\frac{3}{2}}$$

Here $\alpha_k = (\frac{1-4\pi^2 ak^2}{1-4\pi^2 ck^2})^{\frac{1}{4}}$. And thus,

$$\sum_{k\in\mathbb{Z}^*}(\alpha_k^2|u_k|^2 + \alpha_k^{-2}|\eta_k|^2) \leq 2\mu \qquad \textit{for}\, t \leq C\mu^{-r+\frac{3}{2}}.$$

This result is in fact a corollary of a Birkhoff normal form result stated in chapter 4. The appearance of the scalar α_k is unusual. In fact, before stating the Birkhoff normal form theorem, we need to conjugate the original solutions to a new form, which follows a more convenient Hamiltonian equation. One has

$$\psi_k = \frac{1}{\sqrt{2}}(\alpha_k u_k + \alpha_k^{-1}\eta_k), \quad \phi_k = \frac{1}{\sqrt{2}}(\alpha_k u_k - \alpha_k^{-1}\eta_k)$$

Then the system can be written as

$$\begin{cases} \partial_t \psi_k = -iD_k \nabla_{\psi_{-k}} H \\ \partial_t \phi_k = iD_k \nabla_{\phi_{-k}} H \end{cases} \qquad k \in \mathbb{Z}^*, \tag{1.2.22}$$

where H is the Hamiltonian

$$H = H_0 + P$$

with [3]

$$H_0 = \sum_{k\in\mathbb{N}^*} \omega_k(|\psi_k|^2 + |\phi_k|^2)$$
$$P = \frac{1}{4\sqrt{2}} \sum_{j+l+h=0} \alpha_h \alpha_j^{-1} \alpha_l^{-1} (\psi_j + \phi_j)(\psi_l + \phi_l)(\phi_h - \psi_h)$$

where

$$D_k = \frac{2\pi k}{1 + 4b\pi^2 k^2}, \quad \omega_k = \sqrt{(1 - 4a\pi^2 k^2)(1 - 4c\pi^2 k^2)}.$$

Each couple Fourier modes (ψ_k, ϕ_k) is associated with a frequency

$$\Omega_k = D_k \omega_k = \frac{2\pi k}{1 + 4b\pi^2 k^2} \sqrt{pk^4 + ek^2 + 1}$$

where $p = 16\pi^4 ac, e = -4\pi^2(a+c)$. Since $b = d$ is fixed and $a+b+c+d = \frac{1}{3}$, one has that $e = 4\pi^2(2b - \frac{1}{3})$ is fixed and p is bounded in a segment $\mathcal{I}_b := (0, 16\pi^4(b - \frac{1}{6})^2)$. Denote $B_s(\mu)$ a ball of radius μ in H^s norm, center at origin and $N_k := |\psi_k|^2 + |\phi_k|^2$ the action at mode k. One has the following Birkhoff normal form result

Theorem 1.2.6. *Let $r \geq 1$, $s \in \mathbb{R}$ sufficiently large and $0 < \mu \ll 1$, then there exists a subset $\mathcal{I}^\mu \subset \mathcal{I}_b$ asymptotically of full measure, and a constant $C = C(r, s, b)$ such that for any $p \in \mathcal{I}^\mu$, for $|t| \leq \mu^{-r+3/2}$, there exists a transformation $\mathcal{T} : B_s(\mu/3) \to B_s(\mu)$ satisfying*

$$H \circ \mathcal{T} = H_0 + Z + \mathcal{R}. \tag{1.2.23}$$

here Z is a polynomial of degree at most $r + 2$ that commutes with the actions N_k, i.e.,

$$\{Z, N_k\} = 0, \qquad \forall k \in \mathbb{Z}^* \tag{1.2.24}$$

and $\mathcal{R} \in C^\infty(B_s(\mu))$ fulfills the estimate

$$\sup_{\|(\psi,\phi)\|_s \leq \mu/3} \|X_{\mathcal{R}}\|_s \leq C\mu^{r+\frac{3}{2}}. \tag{1.2.25}$$

3. $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$

The canonical transformation and its inverse are close to identity

$$\sup_{\|(\phi,\psi)\|_s\le\mu/3} \|(\phi,\psi)-\mathcal{T}(\phi,\psi)\|_s \le C_s\mu^2 \tag{1.2.26}$$

$$\sup_{\|(\phi,\psi)\|_s\le\mu/3} \|(\phi,\psi)-\mathcal{T}^{-1}(\phi,\psi)\|_s \le C_s\mu^2. \tag{1.2.27}$$

Scheme of the proof

The idea to prove Birkhoff normal form result is to search for iterative changes of variables $\mathcal{T}_n = \Phi^t_{\chi_n}\,|_{t=1}$, $1\le n\le r$ that put the original Hamiltonian into better and better forms. That is

$$H\circ\mathcal{T}_n = H_0 + Z_n + R_{n+1}. \tag{1.2.28}$$

Where Z_n is a normal form, which does not affect to the behavior of the solutions, and R_{n+1} is a perturbation term which is of the size $\mu^{n+\frac{3}{2}}$. At each step of Birkhoff normal form procedure, one need to solve a homological equation

$$\{H_0,\chi\}+Z=f \tag{1.2.29}$$

with Z is in a normal form and f is a polynomial remainder term. We expand f in Taylor series

$$f(\phi,\psi)=\sum_{j,l} f_{j,l}\Pi_{k,h}\psi_k^{j_k}\phi_h^{l_h}$$

and similarly for χ, Z. The homological equation becomes

$$(\Omega(j-l))\chi_{j,l}+Z_{j,l}=\sum_{k\in\mathbb{N}^*}(\Omega_k(j_k-j_{-k}-l_k+l_{-k}))\chi_{j,l}+Z_{j,l}=f_{j,l}. \tag{1.2.30}$$

Here we use

$$\{H_0,\psi_k^{j_k}\phi_h^{l_h}\}=(\Omega_k j_k-\Omega_h l_h)\psi_k^{j_k}\phi_h^{l_h}, \qquad \Omega_k=-\Omega_{-k}.$$

The result is based on a so-called tame inequality

$$\|uv\|_s\le C(\|u\|_s\|v\|_1+\|u\|_1\|v\|_s)$$

for some constant $C>0$ and s. This inequality was introduced in [BG06],see also [GA91 ; H76], where the authors proved Birkhoff normal form for many partial differential equations. This inequality allows us to ignore all monomials with more than two high modes. Indeed, consider a function u depending on only high modes, i.e., $u=\sum_{k\ge N}u_k e^{ikx}$ with a large number N, one has

$$\|u\|_1 \leq \frac{\|u\|_s}{N^{s-1}}.$$

In our procedure, the polynomial f satisfies the tame inequality. Indeed, according to [BG06], it is enough to prove the tame property of the original nonlinearity $f = P$.

For monomials containing at most two high modes, one proves a nonresonant condition for frequencies, that is

$$|\Omega(j-l)| \geq \frac{\kappa}{N^\alpha}$$

for some constants κ, α and N. Consider r frequencies $\Omega_{j_1}, \ldots, \Omega_{j_r}$ with $j_1 < j_2 < \ldots < j_r \leq N$, as functions of p, then the corresponding determinant is bounded from below by $\frac{1}{N^\alpha}$ with a constant $\alpha = \alpha(r)$. Combine this with a theorem introduced in [XYQ97], which says that if $|g^{(r)}(p)| \geq d$ then $|g(p)| \geq h$ except for a small set of p, one has an estimate for small divisor $\Omega(j-l)$ for most value of p.

This nonresonant condition allows us to estimate the solution χ and Z of the homological equation and continue the Birkhoff procedure. Precisely, one has

$$\langle |X_\chi| \rangle_{s,R} \leq C\frac{N^\alpha}{\kappa}\langle |X_f| \rangle_{s,R}, \qquad \langle |X_Z| \rangle_{s,R} \leq C\langle |X_f| \rangle_{s,R}. \tag{1.2.31}$$

for some constant C. Here X_χ denotes the vector field generated by χ and $\langle |\cdot| \rangle_{s,R}$ denotes a norm in Sobolev space of the vector field with variables bounded in the ball with center in the origin and radius R.

2

An unstable three dimensional KAM torus for the quintic NLS

2.1 Introduction

We consider the nonlinear Schrödinger equation on the torus

$$i\partial_t u + \partial_{xx} u = |u|^4 u, \quad (t,x) \in \mathbb{R} \times \mathbb{T} \tag{2.1.1}$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. This is an infinite dimensional dynamic system on the phase space $(u, \overline{u}) \in L^2(\mathbb{T})$ endowed with the symplectic form $-idu \wedge d\overline{u}$. The flow $u(t)$ preserves the Hamiltonian

$$h = \int_{\mathbb{T}} |u_x|^2 + \frac{1}{3}|u|^6 dx,$$

and also, the mass and the momentum

$$\mathbb{L} = \int_{\mathbb{T}} |u|^2 dx, \quad \mathbb{M} = \int_{\mathbb{T}} Im(u \cdot \nabla \overline{u}) dx.$$

Let us expand u and $\overline{u}$ in Fourier basis :

$$u(t,x) = \sum_{j\in\mathbb{Z}} a_j(t) e^{ijx}, \quad \overline{u}(t,x) = \sum_{j\in\mathbb{Z}} b_j(t) e^{-ijx}.$$

In this variables, the symplectic structure becomes $-i\sum_{j\in\mathbb{Z}} da_j \wedge db_j$. The Hamiltonian h of the system reads

$$h = \sum_{j\in\mathbb{Z}} j^2 a_j b_j + \frac{1}{3} \sum_{j,\ell\in\mathbb{Z}^3; \mathcal{M}(j,l)=0} a_{j_1} a_{j_2} a_{j_3} b_{\ell_1} b_{\ell_2} b_{\ell_3} = N + P,$$

and the mass and the momentum

$$\mathbb{L} = \sum_{j\in\mathbb{Z}} a_j b_j, \quad \mathbb{M} = \sum_{j\in\mathbb{Z}} j a_j b_j,$$

here $\mathcal{M}(j,l) = j_1 + j_2 + j_3 - \ell_1 - \ell_2 - \ell_3$ denotes the momentum of the monomial $a_{j_1}a_{j_2}a_{j_3}b_{\ell_1}b_{\ell_2}b_{\ell_3}$. We can rewrite equation (2.1.1) into a system of infinite number of equations

$$\begin{cases} i\dot{a_j} & = j^2 a_j + \frac{\partial P}{\partial b_j} \quad j \in \mathbb{Z}, \\ -i\dot{b_j} & = j^2 b_j + \frac{\partial P}{\partial a_j} \quad j \in \mathbb{Z}. \end{cases} \tag{2.1.2}$$

In this article, we are interested in the dynamic behavior near to 0 of solution of (2.1.1) in two specific forms :

$$u(t,x) = a_p(t)e^{ipx}e^{-ip^2t} + a_q(t)e^{iqx}e^{-iq^2t} + \mathcal{O}(\varepsilon), \tag{2.1.3}$$

and

$$u(t,x) = a_p(t)e^{ipx}e^{-ip^2t} + a_q(t)e^{iqx}e^{-iq^2t} + a_m(t)e^{imx}e^{-im^2t} + \mathcal{O}(\varepsilon), \tag{2.1.4}$$

or more precisely the persistence of two and three dimensional linear invariant tori :

$$\mathbf{T}^2_c(p,q) = \{|a_p|^2 = c_1,\ |a_q|^2 = c_2\}, \tag{2.1.5}$$

$$\mathbf{T}^3_c(p,q,m) = \{|a_p|^2 = c_1,\ |a_q|^2 = c_2,\ |a_m|^2 = c_3\}, \tag{2.1.6}$$

with $0 < c_1, c_2, c_3 \ll 1$.
The first result of this paper is stated for two dimensional tori.

Theorem 2.1.1. Fix p, $q \in \mathbb{Z}$, and $s > \frac{1}{2}$. There exists $\nu_0 > 0$, and for $0 < \nu < \nu_0$, there exists $\mathcal{D}_\nu \subset [1,2]^2$ asymptotically of full measure (i.e. $meas([1,2]^2 \setminus \mathcal{D}_\nu) \to 0$ when $\nu \to 0$) such that for $\rho \in \mathcal{D}_\nu$, equation (2.1.1) admits a solution of the form

$$u(x) = \sum_{j\in\mathbb{Z}} a_j(t\omega)e^{ijx}$$

where $\{a_j\}_j$ are analytic functions from $\mathbb{T}^2$ to ℓ^2_s satisfying uniformly in $\theta \in \mathbb{T}^2$

$$||a_p| - \sqrt{\nu\rho_1}|^2 + ||a_q| - \sqrt{\nu\rho_2}|^2 + \sum_{j\neq p,q}(1+j^2)^s|a_j|^2 = \mathcal{O}(\nu^2).$$

Here ω is a nonresonant vector in $\mathbb{R}^2$ that satisfies

$$\omega = (p^2,q^2) + \mathcal{O}(\nu^2).$$

Furthermore, this solution is linearly stable.

Remark 2.1.2. — Here, the notation nonresonant means that there is no $\{0,0\} \neq \ell \in$

$\mathbb{Z}$š such that $\omega \cdot \ell = 0$.

— u is linearly stable if the linear equation system obtained by linearizing the system (2.1.2) on this solution has the form $\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = A \begin{pmatrix} a \\ b \end{pmatrix}$, where A is a linear operator whose all the eigenvalues have negative real part. By contrast, it is linearly unstable if the spectrum of A contains eigenvalues with positive real part.

For three dimensional tori, it is too complicated [1] to consider the general case. In order to apply KAM theorem 2.2.2, we avoid the case where there is $\ell \in \mathbb{Z}$ solving equation [2]

$$\begin{cases} 2j_1 + j_2 &= 2j_3 + \ell \\ 2j_1^2 + j_2^2 &= 2j_3^2 + \ell^2. \end{cases} \tag{2.1.7}$$

In this paper, we will give here an example of (p, q, m) and ρ such that for ν small enough the torus $\mathbf{T}^3_{\nu\rho}(p, q, m) = \{|a_p|^2 = \nu\rho_1,\ |a_q|^2 = \nu\rho_2,\ |a_m|^2 = \nu\rho_3\}$ is linearly unstable. For $\epsilon = 10^{-2}$, denote

$$\mathcal{D} = \mathcal{D}_2 = [2 - \epsilon, 2 + \epsilon] \times [1 - \epsilon, 1 + \epsilon] \times [9 - \epsilon, 9 + \epsilon].$$

Theorem 2.1.3. Fix $p = -3$, $q = 10$, $m = -6$, and $s > \frac{1}{2}$. There exists $\nu_0 > 0$, and for $0 < \nu < \nu_0$, there exists $\mathcal{D}_\nu \subset \mathcal{D}$ asymptotically of full measure (i.e. $meas(\mathcal{D} \setminus \mathcal{D}_\nu) \to 0$ when $\nu \to 0$) such that for $\rho \in \mathcal{D}_\nu$, equation (2.1.1) admits a solution of the form

$$u(x) = \sum_{j \in \mathbb{Z}} a_j(t\omega) e^{ijx} \tag{2.1.8}$$

where $\{a_j\}_j$ are analytic functions from $\mathbb{T}^3$ to ℓ^2_s satisfying uniformly in $\theta \in \mathbb{T}^3$

$$||a_p| - \sqrt{\nu\rho_1}|^2 + ||a_q| - \sqrt{\nu\rho_2}|^2 + ||a_m| - \sqrt{\nu\rho_3}|^2 + \sum_{j \neq p,q,m} (1 + j^2)^s |a_j|^2 = \mathcal{O}(\nu^2). \tag{2.1.9}$$

Here ω is a non resonant vector in $\mathbb{R}^3$ that satisfies

$$\omega = (3^2, 10^2, 6^2) + \mathcal{O}(\nu^2).$$

Furthermore, this solution is linearly unstable.

In order to prove Theorems 2.1.1, 2.1.3, we follow a general strategy developed in [GD18] for a system of coupled nonlinear Schrödinger equations on the torus. Firstly,

1. the difficulty is to verify KAM hypotheses

2. in this case, the linear part $a^2_{j_1} a_{j_2} b^2_{j_3} b_\ell + b^2_{j_1} b_{j_2} a^2_{j_3} a_\ell$ of the mode ℓ would create the instability, and the energy would soon transfer mainly between four modes p, q, m, ℓ, which was studied carefully in [GT12].

we apply a Birkhoff normal form procedure (Proposition 2.3.1) to kill the nonresonance of P. Then we use sympletic changes of variables to diagonalize the effective part into the form of h_0. The main difference between Theorem 2.1.1 and Theorem 2.1.3 is for the linear stability of the solution, which is explained by the presence of hyperbolic directions on the torus $\mathbf{T}^3_{\nu\rho}(-3, 10, -6)$. In section 3, we will see that in this case the energy will drain out of these three modes into two exterior modes $\{9, 1\}$. Since the proof bases on KAM theorem in [GD18], readers are suggested to take a look at the original statement for further understanding.

The study of finite dimensional tori in an infinite dimensional phase space was pioneered by J. Bourgain [Bou98] in 1988. However, the existence of unstable KAM tori in one dimensional context was first proved by B. Grébert and V. Vilaça da Rocha [GD18] in 2017, where they studied the system of coupled nonlinear Schrödinger equations on the torus. For the equation (2.1.1), in case of $u(0, x)$ supported mainly in four modes (p, q, m, s), which satisfy such a relation in (2.1.7), the study of solution was studied carefully in [GT12] and [HP17]. In particular, in [HP17] they proved the recurrent exchange of energy between those modes.

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2.2 KAM theorem

In order to proof Theorems 2.1.1 and 2.1.3, we recall a KAM theorem stated in [GD18]. We consider a Hamiltonian $h = h_0 + f$, where h_0 is a quadratic Hamiltonian in normal form

$$h_0 = \Omega(\rho) \cdot r + \sum_{a \in \mathcal{Z}} \Lambda_a(\rho) |\zeta_a|^2. \tag{2.2.1}$$

Here

- ρ is a parameter in $\mathcal{D}$, which is a compact in the space $\mathbb{R}^n$;
- $r \in \mathbb{R}^n$ are the actions corresponding to the internal modes $(r, \theta) \in (\mathbb{R}^n \times \mathbb{T}^n, dr \wedge d\theta)$;
- $\mathcal{L}$ and $\mathcal{F}$ are respectively infinite and finite sets, $\mathcal{Z}$ is the disjoint union $\mathcal{L} \cup \mathcal{F}$;
- $\zeta = (\zeta_a)_{a \in \mathcal{Z}} \in \mathbb{C}^{\mathcal{Z}}$ are the external modes endowed with the standard complex symplectic structure $-id\zeta \wedge d\eta$. The external modes decomposes in a infinite part $\zeta_{\mathcal{L}} = (\zeta_a)_{a \in \mathcal{L}}$, corresponding to elliptic directions, which means $\Lambda_a \in \mathbb{R}$ for $a \in \mathcal{L}$,

and a finite part $\zeta_{\mathcal{F}} = (\zeta_a)_{a\in\mathcal{F}}$, corresponding to hyperbolic directions, which means $\operatorname{Im}\Lambda_a \neq 0$ for $a \in \mathcal{F}$;

— $\mathcal{L}$ has a clustering structure $\mathcal{L} = \cup_{j\in\mathbb{N}}\mathcal{L}_j$, where $\mathcal{L}_j$ are finite sets of cardinality $d_j \leq d < \infty$. If $a \in \mathcal{L}_j$, we denote $[a] = \mathcal{L}_j$ and $w_a = j$, for $a \in \mathcal{F}$ we set $w_a = 1$;

— the mappings

$$\Omega : \mathcal{D} \to \mathbb{R}^n, \tag{2.2.2}$$

$$\Lambda_a : \mathcal{D} \to \mathbb{C}, \quad a \in \mathcal{Z}, \tag{2.2.3}$$

are smooth ;

— $f = f(r, \theta, \zeta; \rho)$ is a perturbation, small compared to the integrable part h_0.

Linear space Let $s \geq 0$, we consider the complex weighted ℓ_2- space

$$Z_s = \{\zeta = (\zeta_a \in \mathbb{C},\, a \in \mathcal{Z}) \mid \|\zeta\|_s < \infty\},$$

where

$$\|\zeta\|_s = \sum_{a\in\mathcal{Z}} |\zeta_a|^2 w_a^{2s}.$$

Similarly we define

$$Y_s = \{\zeta_{\mathcal{L}} = (\zeta_a \in \mathbb{C},\, a \in \mathcal{L}) \mid \|\zeta_{\mathcal{L}}\|_s < \infty\},$$

with the same norm. We endow $Z_s \times Z_s$ and $Y_s \times Y_s$ with the symplectic structure $-id\zeta\wedge d\eta$, with $\eta = \overline{\zeta}$.

A class of Hamiltonian functions. Denote $\omega = (\zeta, \eta)$. On the space

$$\mathbb{C}^n \times \mathbb{C}^n \times (Z_s \times Z_s)$$

we define the norm

$$\|(r, \theta, \omega)\|_s = \max\left(|r|, |\theta|, \|\zeta\|_s\right).$$

For $\sigma > 0$ we denote

$$\mathbb{T}^n_\sigma = \{\theta \in \mathbb{C}^n : |\operatorname{Im}\theta| < \sigma\}/2\pi\mathbb{Z}^n.$$

For $\sigma, \mu \in (0, 1]$ and $s \geq 0$ we set

$$\mathcal{O}^s(\sigma, \mu) = \{r \in \mathbb{C}^n : |r| < \mu^2\} \times \mathbb{T}^n_s \times \{\omega \in Z_s \times Z_s : \|\zeta\|_s < \mu\}.$$

We will denote points in $\mathcal{O}^s(\sigma,\mu)$ as $x = (r,\theta,\omega)$. Let $f : \mathcal{O}^0(\sigma,\mu) \times \mathcal{D} \to \mathbb{C}$ be a C^1 function [3], real holomorphic in the first variable x, such that for all $\rho \in \mathcal{D}$, $x \in \mathcal{O}^s(\sigma,\mu)$:

$$\nabla_\omega f(x,\rho) \in Z_s \times Z_s$$

and

$$\nabla^2_{\omega_{\mathcal{L}}\omega_{\mathcal{L}}} f(x,\rho) \in \mathcal{L}(Y_s, Y_s)$$

are real holomorphic functions. We denote by $\mathcal{T}^s(\sigma,\mu,\mathcal{D})$ this set of functions. For $f \in \mathcal{T}^s(\sigma,\mu,\mathcal{D})$, we define

$$|\partial^j_\rho f|_{\sigma,\mu,\mathcal{D}} = \sup_{x\in\mathcal{O}^s(\sigma,\mu);\,\rho\in\mathcal{D}} \max(|\partial^j_\rho f|, \mu \left\|\partial^j_\rho \nabla_\omega f(x,\rho)\right\|_s, \mu^2 \left\|\nabla^2_{\omega_{\mathcal{L}}\omega_{\mathcal{L}}} \partial^j_\rho f(x,\rho)\right\|),$$

and

$$[f]^s_{\sigma,\mu,\mathcal{D}} = \max_{j=0,1}(|\partial^j_\rho f|_{\sigma,\mu,\mathcal{D}}).$$

Jet functions For any $f \in \mathcal{T}^s(\sigma,\mu,\mathcal{D})$, we define its jet $f^T(x)$ as the following Taylor polynomial of f at $r = 0$ and $\omega = 0$

$$f^T(x) = f(0,\theta,0) + d_r f(0,\theta,0)\cdot r + d_\omega f(0,\theta,0)[\omega] + 1/2 d^2_\omega f(0,\theta,0)[\omega,\omega].$$

Infinite matrices For the elliptic variables, we denote by $\mathcal{M}_s$ the set of infinite matrices $A : \mathcal{L}\times\mathcal{L} \to \mathbb{C}$ such that A maps linearly Y_s into Y_s. We provide $\mathcal{M}_s$ with the operator norm

$$|A|_s = \|A\|_{\mathcal{L}(Y_s,Y_s)}.$$

We say that a matrix $A \in \mathcal{M}_s$ is on normal form if it is block diagonal and Hermitian, i.e.

$$A^\beta_\alpha = 0 \quad \text{for } [\alpha] \neq [\beta] \quad \text{and } A^\beta_\alpha = \overline{A^\alpha_\beta} \quad \text{for } \alpha,\beta \in \mathcal{L}.$$

In particular, if $A \in \mathcal{M}_s$ is on normal form, its eigenvalues are real.

Normal form A quadratic Hamiltonian function is on normal form if it reads

$$h = \Omega(\rho)\cdot r + \langle \zeta_{\mathcal{L}}, Q(\rho)\eta_{\mathcal{L}}\rangle + 1/2\langle \omega_{\mathcal{F}}, K(\rho)\omega_{\mathcal{F}}\rangle$$

3. C^1 regularity with respect to ρ in the Whitney sense

for some vector function $\Omega(\rho) \in \mathbb{R}^n$, some matrix functions $Q(\rho) \in \mathcal{M}_s$ on normal form and $K(\rho)$ is a matrix $\mathcal{F} \times \mathcal{F} \to \mathbb{C}$ symmetric in the following sense :$K_\alpha^\beta = {}^t K_\beta^\alpha$.

Poisson brackets The Poisson brackets of two Hamiltonian functions is defined by

$$\{f, g\} = \nabla_\theta f \cdot \nabla_r g - \nabla_r f \cdot \nabla_\theta g - i\langle \nabla_\omega f, J \nabla_\omega g\rangle.$$

Remark 2.2.1. A function f is preserved under the flow $u(t)$ if and only if it commutes with h i.e. $\{f, h\} = 0$. By this, we have

$$\{\mathbb{L}, h\} = \{\mathbb{M}, h\} = 0.$$

Hypothesis A0 There exists a constant $C > 0$ such that

$$|\Lambda_a - |w_a|^2| \leq C, \, \forall a \in \mathcal{L}.$$

Hypothesis A1 There exists $\delta > 0$ such that

$$\begin{aligned} |\Lambda_a| &\geq \delta, \quad \forall a \in \mathcal{L}; \\ |\operatorname{Im} \Lambda_a| &\geq \delta, \quad \forall a \in \mathcal{F}; \\ |\Lambda_a - \Lambda_b| &\geq \delta, \quad \forall a, b \in \mathcal{Z}, \; [a] \neq [b]; \\ |\Lambda_a + \Lambda_b| &\geq \delta, \quad \forall a, b \in \mathcal{L}. \end{aligned}$$

Hypothesis A2 There exists $\delta > 0$ such that for all Ω $\delta\breve{}$close to Ω_0 in C^1 norm and for all $k \in \mathbb{Z}^n \backslash \{0\}$:

1. either

$$|\Omega(\rho) \cdot k| \geq \delta \quad \forall \rho \in \mathcal{D},$$

   or there exists a unit vector $z = z(k) \in \mathbb{R}^n$ such that

$$(\nabla_\rho \cdot z)\,(\Omega(\rho) \cdot k) \geq \delta \quad \forall \rho \in \mathcal{D};$$

2. for all $a \in \mathcal{L}$ either

$$|\Omega(\rho) \cdot k + \Lambda_a| \geq \delta \quad \forall \rho \in \mathcal{D},$$

   or there exists a unit vector $z = z(k) \in \mathbb{R}^n$ such that

$$(\nabla_\rho \cdot z)\,(\Omega(\rho) \cdot k + \Lambda_a) \geq \delta \quad \forall \rho \in \mathcal{D};$$

3. for all α, $\beta \in \mathcal{L}$ and $a \in [\alpha]$, $b \in [\beta]$ either

$$|\Omega(\rho) \cdot k + \Lambda_a \pm \Lambda_b| \geq \delta \quad \forall \rho \in \mathcal{D},$$

or there exists a unit vector $z = z(k) \in \mathbb{R}^n$ such that

$$(\nabla_\rho \cdot z)\,(\Omega(\rho) \cdot k + \Lambda_a \pm \Lambda_b) \geq \delta \quad \forall \rho \in \mathcal{D};$$

4. for all $a, b \in \mathcal{F}$

$$|\Omega(\rho) \cdot k + \Lambda_a \pm \Lambda_b| \geq \delta.$$

Theorem 2.2.2 (KAM theorem)**.** Assume that hypothesis A0, A1, A2 are satisfied, $f \in \mathcal{T}^s(\sigma, \mu, \mathcal{D})$, f commutes with $\mathcal{L}$, $\mathcal{M}$ and $s > 1/2$. Let $\gamma > 0$, there exists a constant C_0 such that if

$$[f]^s_{\sigma,\mu,\mathcal{D}} \leq C_0\delta, \quad \varepsilon := [f^T]^s_{\sigma,\mu,\mathcal{D}} \leq C_0\delta^{1+\gamma}, \tag{2.2.4}$$

then there exists a Cantor set $\mathcal{D}' \subset \mathcal{D}$ asymptotically of full measure (i.e. $meas(\mathcal{D}\backslash\mathcal{D}') \to 0$ when $\varepsilon \to 0$) and there exists a symplectic change of variables $\Phi : \mathcal{O}^s(\sigma/2, \mu/2) \to \mathcal{O}^s(\sigma, \mu)$ such that for all $\rho \in \mathcal{D}'$

$$(h_0 + f) \circ \Phi = \tilde{h} + g$$

with $\tilde{h} = \Omega(\rho){\cdot}r + \langle\zeta_{\mathcal{L}}, Q(\rho)\eta_{\mathcal{L}}\rangle + 1/2\langle\omega_{\mathcal{F}}, K(\rho)\omega_{\mathcal{F}}\rangle$ on normal form, and $g \in \mathcal{T}^s(\sigma/2, \mu/2, \mathcal{D}')$ with $g^T \equiv 0$. Furthermore there exists $C > 0$ such that for all $\rho \in \mathcal{D}'$

$$|\Omega - \Omega_0| \leq C\varepsilon, \quad |Q - diag\,(\Lambda_a,\, a \in \mathcal{L})\,| \leq C\varepsilon, \quad |JK - diag\,(\Lambda_a, a \in \mathcal{F})\,| \leq C\varepsilon.$$

As a dynamic consequence $\Phi\left(\{0\} \times \mathbb{T}^n \times \{0\}\right)$ is an invariant torus for $h_0 + f$and this torus is linearly stable if and only if $\mathcal{F} = \varnothing$ (see [GD18]).

Here, the matrix J is of the form,

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

where I is identity matrix of size $\#F$.

Remark 2.2.3. In [GD18], they constrained f in a restricted class instead of using commutation of f with $\mathcal{L}$, $\mathcal{M}$ since they considered a system of coupled NLS equation with more complicated nonlinearities.

2.3 Applications

The Birkhoff normal form procedure. We recall a result proved in [GT12].

Proposition 2.3.1. There exist a canonical change of variable τ from $\mathcal{O}^s(\sigma,\mu)$ into $\mathcal{O}^s(2\sigma,2\mu)$ such that

$$\overline{h} = h\circ\tau = N + Z_6 + R_{10},$$

where

— N is the term $N(I)=\sum_{j\in\mathbb{Z}} j^2|a_j|^2$;

— Z_6 is the homogeneous polynomial of degree 6

$$Z_6 = \sum_{\mathcal{R}} a_{j_1}a_{j_2}a_{j_3}b_{\ell_1}b_{\ell_2}b_{\ell_3}$$

where
$\mathcal{R}=\{(j,\ell)\in\mathbb{Z}^3\times\mathbb{Z}^3\ s.t\ j_1+j_2+j_3=\ell_1+\ell_2+\ell_3,\quad j_1^2+j_2^2+j_3^2=\ell_1^2+\ell_2^2+\ell_3^2\}$;

— R_{10} is the remainder of order 10, i.e a Hamiltonian satisfying

$$\|X_{R_{10}}(x)\|_s \le C\,\|x\|_s^9$$

for all $x\in\mathcal{O}^s(\sigma,\mu)$;

— τ is close to the identity : there exists a constant C such that

$$\|\tau(x)-x\|\le C\,\|x\|^2$$

for all $x\in\mathcal{O}^s(\sigma,\mu)$.

Henceforth, since we do not care about constant, we shall write $a\lesssim b$ in order to say $a\leqslant Cb$.

Persistence of 2 dimensional tori.

Firstly, we want to study the persistence of the two dimensional invariant torus $\mathbf{T}^2_{\nu\rho}(p,q)$ for equation (2.1.1) for ν small. Choose

$$\begin{cases} a_p &= (\nu\rho_1 + r_1(t))^{\frac12}\, e^{i\theta_1(t)} =: \sqrt{I_p}e^{i\theta_1(t)}\\ a_q &= (\nu\rho_2 + r_2(t))^{\frac12}\, e^{i\theta_2(t)} =: \sqrt{I_q}e^{i\theta_2(t)}\\ a_j &= \zeta_j \qquad j\neq p,q, \end{cases}$$

where $\{\rho_1,\rho_2\}\in[1,2]^2=\mathcal{D}$ and ν is a small parameter.The canonical symplectic structure now becomes

$$-id\zeta \wedge d\eta - dI \wedge d\theta$$

with $I = (I_1, I_2)$, $\theta = (\theta_1, \theta_2)$, $\zeta = (\zeta_j)_j$ and $\eta = (\eta_j)_j = (\bar{\zeta}_j)_j$.
Let

$$\mathbf{T}_\rho^{lin} := \{(I, \theta, \zeta) || I - \nu\rho| = 0,\ |\mathrm{Im}\,\theta| < \sigma,\ \|\zeta\|_s = 0\}$$

and its neighborhood

$$\mathbf{T}_\rho(\nu, \sigma, \mu, s) := \{(I, \theta, \zeta) || I - \nu\rho| < \nu\mu^2,\ |\mathrm{Im}\,\theta| < \sigma,\ \|\zeta\|_s < \nu^{1/2}\mu\}.$$

We want to study the persistence of torus $\mathbf{T}_\rho(\nu, \sigma, \mu, s)$. Indeed we have

$$\mathbf{T}_\rho(\nu, \sigma, \mu, s) \approx \mathcal{O}^s(\sigma, \nu^{1/2}\mu) = \{(r, \theta, \zeta) || r| < \nu\mu^2,\ |\mathrm{Im}\,\theta| < \sigma,\ \|\zeta\|_s < \nu^{1/2}\mu\}.$$

By Theorem 2.3.1 we have

$$h \circ \tau = N + Z_6 + R_{10}.$$

We see that the term N contributes to the effective part and the term R_{10} contributes to the remainder term f. So we just need to focus on the term Z_6. Let us split it :

$$Z_6 = Z_{0,6} + Z_{1,6} + Z_{2,6} + Z_{3,6}.$$

Here $Z_{0,6}$, $Z_{1,6}$, $Z_{2,6}$ are homogeneous polynomial of degree 6 which contain respectively external modes of order $0, 1, 2$. $Z_{3,6}$ is a homogeneous polynomial of degree 6 contains external modes of at least order 3, this term contributes the remainder term.
Thank to Lemma 2.2 in [GT12], the term $Z_{1,6} = 0$. We have

$$\begin{aligned}
Z_{0,6} &= |a_p|^6 + |a_q|^6 + 9\left(|a_p|^4|a_q|^2 + |a_p|^2|a_q|^4\right) \\
&= (\nu\rho_1 + r_1)^3 + (\nu\rho_2 + r_2)^3 + 9\,(\nu\rho_1 + r_1)\,(\nu\rho_2 + r_2)\,(\nu\rho_1 + r_1 + \nu\rho_2 + r_2) \\
&= \nu^3(\rho_1^3 + \rho_2^3 + 9\rho_1^2\rho_2 + 9\rho_2^2\rho_1) + 3\nu^2\left(r_1(\rho_1^2 + 6\rho_1\rho_2 + 3\rho_2^2) + r_2(\rho_2^2 + 6\rho_1\rho_2 + 3\rho_1^2)\right) \\
&\quad + \text{jet free}
\end{aligned}$$

where the notation “jet free” means that the remaining Hamiltonian has a vanishing jet.
For the term $Z_{2,6}$, there are two cases that can happen.
First case

We assume that there is no solution[4] $\{s,t\} \neq \{p,q\}$ for

$$\begin{cases} 2p+s & = 2q+t \\ 2p^2+s^2 & = 2q^2+t^2. \end{cases} \tag{2.3.1}$$

Hence

$$Z_{2,6} = Z^1_{2,6} = 9\left(|a_p|^4+|a_q|^4+4|a_p|^2|a_q|^2\right)\sum_{j\neq p,q}|a_j|^2 = 9\nu^2\left(\rho_1^2+\rho_2^2+4\rho_1\rho_2\right)\sum_{j\neq p,q}|\zeta_j|^2 + \text{jet free}.$$

Hence

$$h\circ\tau = h^e + R$$

where the effective Hamiltonian h^e reads

$$\begin{aligned} h^e &= \left(p^2+3\nu^2\left(\rho_1^2+3\rho_2^2+6\rho_1\rho_2\right)\right)r_1 + \left(q^2+3\nu^2\left(\rho_2^2+3\rho_1^2+6\rho_1\rho_2\right)\right)r_2 \\ &\quad + \sum_j \left(j^2+9\nu^2\left(\rho_1^2+\rho_2^2+4\rho_1\rho_2\right)\right)|\zeta_j|^2 \\ &= \Omega(\rho)\cdot r + \sum_{j\neq p,q}\Lambda_j|\zeta_j|^2 \end{aligned}$$

where

$$\Omega(\rho) = \begin{pmatrix} p^2+3\nu^2\left(\rho_1^2+3\rho_2^2+6\rho_1\rho_2\right) \\ q^2+3\nu^2\left(\rho_2^2+3\rho_1^2+6\rho_1\rho_2\right) \end{pmatrix}$$

and

$$\Lambda_j = j^2+9\nu^2\left(\rho_1^2+\rho_2^2+4\rho_1\rho_2\right).$$

The remainder term R reads

$$\begin{aligned} R &= R_{10}+Z_{3,6}+3\nu\rho_1 r_1^2+r_1^3+3\nu\rho_2 r_2^2+r_2^3+9r_1r_2(r_1+r_2) \\ &\quad + \left(r_1^2+r_2^2+2\nu(\rho_1+2\rho_2)r_1+2\nu(\rho_2+2\rho_1)r_2\right)\sum_{j\neq p,q}|\zeta_j|^2. \end{aligned}$$

In order to work on $\mathcal{O}^s(\sigma,\mu)$ we use the rescaling

$$\Psi : r\mapsto \nu r,\ \zeta\mapsto \nu^{1/2}\zeta. \tag{2.3.2}$$

4. it happens when $q-p$ is odd

The symplectic structure now becomes

$$-\nu dr \wedge d\theta - i\nu d\zeta \wedge d\eta.$$

By definition, this change of variables maps $\mathcal{O}^s(\sigma,\mu)$ to a neighborhood of $\mathbf{T}_\rho(\nu,\sigma,\mu,s)$. Since τ is close to identity, the change of variables $\Phi_\rho = \tau\circ\Psi$ maps $\mathcal{O}^s(\sigma,\mu)$ to $\mathbf{T}_\rho(\nu,2\sigma,2\mu,s)$. By this change of variables, we have

$$h\circ\Phi_\rho - C = (h^e + R)\circ\Psi = \nu h_0 + \nu f$$

where C is a constant, h_0 and f are defined by

$$h_0 = \frac{1}{\nu}h^e\circ\Psi \qquad f = \frac{1}{\nu}R\circ\Psi.$$

By Theorem 2.3.1, $R_{10}\in\mathcal{T}^s(\sigma,\nu^{1/2}\mu,\mathcal{D})$. We check that the remaining part of f is in $\mathcal{T}^s(\sigma,\mu,\mathcal{D})$. By construction, f commutes[5] with $\mathbb{L}$ and $\mathbb{M}$. To estimate the norm of f, notice that R contains only term of order at least 3 in ν and $R^T = R^T_{10}$ is of order $9/2$ in ν, so that

$$[f]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^2$$

and

$$[f^T]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^{7/2}.$$

So we have proved :

Theorem 2.3.2. Assume that for $p,q\in\mathbb{Z}$ there do not exist s,t solving the equation (2.3.1). Then, the change of variables $\Phi_\rho = \tau\circ\Psi$ is real holomorphic, symplectic and analytically depending on ρ satisfying

— $\Phi_\rho : \mathcal{O}^s(\sigma,\mu)\to\mathbf{T}_\rho(\nu,2\sigma,2\nu,s)$;

— Φ_ρ puts the Hamiltonian h in normal form in the following sense :

$$\frac{1}{\nu}(h\circ\Phi_\rho - C) = h_0 + f$$

where C is a constant and the effective part h_0 of the Hamiltonian reads

$$h_0 = \Omega(\rho)\cdot r + \sum_{j\neq p,q}\Lambda_j|\zeta_j|^2$$

5. since h commutes with $\mathbb{L}$, $\mathbb{M}$ and all the changes of variables are symplectic

with

$$\Omega(\rho) = \begin{pmatrix} p^2 + 3\nu^2 \left(\rho_1^2 + 3\rho_2^2 + 6\rho_1\rho_2\right) \\ q^2 + 3\nu^2 \left(\rho_2^2 + 3\rho_1^2 + 6\rho_1\rho_2\right) \end{pmatrix}$$

and

$$\Lambda_j = j^2 + 9\nu^2 \left(\rho_1^2 + \rho_2^2 + 4\rho_1\rho_2\right);$$

— The remainder term f belongs to $\mathcal{T}^s(\sigma, \mu, \mathcal{D})$ and satisfies

$$[f]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^2$$

and

$$[f^T]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^{7/2}.$$

Second case

Assume that there are[6] $s, t \neq p, q$ solving (2.3.1), hence

$$Z_{2,6} = Z^1_{2,6} + 9(a_p^2 a_s b_q^2 b_t + b_p^2 b_s a_q^2 a_t) = Z^1_{2,6} + Z_{s,t}$$

For the second term, let us rewrite it

$$9(\nu\rho_1 + r_1)(\nu\rho_2 + r_2)\left(e^{2i(\theta_1-\theta_2)}\zeta_s\eta_t + e^{-2i(\theta_1-\theta_2)}\eta_s\zeta_t\right)$$

The effective part of this term is just given by

$$9\nu^2\rho_1\rho_2\left(e^{2i(\theta_1-\theta_2)}\zeta_s\eta_t + e^{-2i(\theta_1-\theta_2)}\eta_s\zeta_t\right).$$

Notice that

$$\{I_s, \zeta_s\eta_t + \eta_s\zeta_t\} = \{I_t, \zeta_s\eta_t + \eta_s\zeta_t\} = 0.$$

This gives us a clue that the above term does not effect to the stability of the solution. In order to kill the angles, we introduce the symplectic change of variables Ψ_{angles} :

6. in this case, $\{p, q, s, t\}$ is of the form $\{p, p+2n, p+3n, p-n\}$

$\mathcal{O}^s(\sigma,\mu) \to \mathcal{O}^s(\sigma,\mu)$, $(r_1, r_2, \theta, \zeta) \mapsto (r_1', r_2', \theta, \zeta')$ defined by

$$\begin{cases} \zeta_s' &= e^{2i(\theta_1-\theta_2)}\zeta_s \\ \zeta_t' &= \zeta_t \\ \zeta_j' &= \zeta_j, \quad j \neq s,t,p,q \\ r_1' &= r_1 - 2|\zeta_s|^2 \\ r_2' &= r_2 + 2|\zeta_s|^2. \end{cases}$$

By this change of variables

$$\tilde{h} = \overline{h} \circ \Psi_{angles} = C + h^e + R.$$

Here C is a constant given by

$$C = \nu^3(\rho_1^3 + \rho_2^3 + 9\rho_1^2\rho_2 + 9\rho_2^2\rho_1) + 9(\nu p^2 \rho_1 + \nu q^2 \rho_2).$$

The effective Hamiltonian h^e reads

$$\begin{aligned} h^e = &\left(p^2 + 3\nu^2\left(\rho_1^2 + 3\rho_2^2 + 6\rho_1\rho_2\right)\right) r_1' + \left(q^2 + 3\nu^2\left(\rho_2^2 + 3\rho_1^2 + 6\rho_1\rho_2\right)\right) r_2' \\ &+ \sum_{j\neq p,q,s,t} \left(j^2 + 9\nu^2\left(\rho_1^2 + \rho_2^2 + 4\rho_1\rho_2\right)\right)|\zeta_j'|^2 + \left(t^2 + 9\nu^2\left(\rho_1^2 + \rho_2^2 + 4\rho_1\rho_2\right)\right)|\zeta_t'|^2 \\ &+ \left(s^2 + 2p^2 - 2q^2 + \nu^2\left(21\rho_2^2 - 3\rho_1^2 + 36\rho_1\rho_2\right)\right)|\zeta_s'|^2 + 9\nu^2\rho_1\rho_2(\zeta_s'\eta_t' + \eta_s'\zeta_t'). \end{aligned}$$

It is on normal form

$$\Omega(\rho)\cdot r + \sum_{j\neq p,q,s,t} \Lambda_j|\zeta_j'|^2 + \Lambda_s|\zeta_s'|^2 + \Lambda_t|\zeta_t'|^2 + 9\nu^2\rho_1\rho_2(\zeta_s'\eta_t' + \eta_s'\zeta_t')$$

where $\Omega(\rho)$ and Λ_j are defined as in the first case except

$$\Lambda_s = t^2 + \nu^2\left(21\rho_2^2 - 3\rho_1^2 + 36\rho_1\rho_2\right).$$

In order to diagonalize h^e, we use a symplectic change of variables of the form

$$\begin{cases} \zeta_{t^+} &= \frac{1}{\sqrt{1+\alpha^2}}(\zeta_t' + \alpha\zeta_s') \\ \zeta_{t^-} &= \frac{1}{\sqrt{1+\alpha^2}}(\zeta_s' - \alpha\zeta_t') \end{cases}$$

with $\alpha = \frac{-2\rho_1^2+2\rho_2^2+\sqrt{4\rho_1^4+2\rho_1^2\rho_2^2+4\rho_2^4}}{3\rho_1\rho_2}$. Then h^e can be diagonalized as

$$\Omega(\rho)\cdot r + \sum_{j\neq p,q,s,t} \Lambda_j|\zeta_j|^2 + \Lambda_{t^+}|\zeta_{t^+}|^2 + \Lambda_{t^-}|\zeta_{t^-}|^2$$

where

$$\begin{cases} \Lambda_{t^+} &= \Lambda_t - 9\nu^2\rho_1\rho_2\alpha \\ \Lambda_{t^-} &= \Lambda_s + 9\nu^2\rho_1\rho_2\alpha. \end{cases}$$

The remainder term R reads

$$\begin{aligned} R = R_{10} \circ \Psi_{angles} + Z_{3,6} \circ \Psi_{angles} + 3\nu\rho_1 r_1^2 + r_1^3 + 3\nu\rho_2 r_2^2 + r_2^3 \\ + 9r_1r_2(r_1+r_2) + \Big(r_1^2 + r_2^2 + 2\nu(\rho_1+2\rho_2)r_1 + 2\nu(\rho_2+2\rho_1)r_2\Big) \sum_{j\neq p,q} |\zeta_j|^2 \end{aligned}$$

with $r_1 = r_1' + 2|\zeta_s|^2,\ r_2 = r_2' - 2|\zeta_s|^2$.
Using the rescaling Ψ introduced in (2.3.2), we get

$$(h^e + R) \circ \Psi = \nu h_0 + \nu f.$$

Since $\Psi_{angles} : \mathcal{O}^s(\sigma,\mu) \to \mathcal{O}^s(\sigma, 3\mu)$ and τ is closed to identity, we have $\tau \circ \Psi_{angles} \circ \Psi : \mathcal{O}^s(\sigma,\mu) \to \mathbf{T}_\rho(\nu, 2\sigma, 4\mu, s)$. The study of f is the same as in the previous case. Then we get :

Theorem 2.3.3. Assume that $p,\ q,\ s,\ t$ satisfy the equation 2.3.1. The change of variables $\Phi_\rho = \tau \circ \Psi_{angles} \circ \Psi$ is a real holomorphic transformations, analytically depending on ρ satisfying

— $\Phi_\rho : \mathcal{O}^s(\sigma,\mu) \to \mathbf{T}_\rho(\nu, 2\sigma, 4\mu, s)$;
— Φ_ρ puts the Hamiltonian h in normal form in the following sense :

$$\frac{1}{\nu}(h \circ \Phi_\rho - C) = h_0 + f$$

where C is a constant and the effective part h_0 of the Hamiltonian reads

$$h_0 = \Omega(\rho) \cdot r + \sum_{j \neq p,q,s,t} \Lambda_j |\zeta_j|^2 + \Lambda_{t^+}|\zeta_{t^+}|^2 + \Lambda_{t^-}|\zeta_{t^-}|^2$$

with

$$\Omega(\rho) = \begin{pmatrix} p^2 + 3\nu^2\left(\rho_1^2 + 3\rho_2^2 + 6\rho_1\rho_2\right) \\ q^2 + 3\nu^2\left(\rho_2^2 + 3\rho_1^2 + 6\rho_1\rho_2\right) \end{pmatrix}$$

and

$$\Lambda_j = j^2 + 9\nu^2\left(\rho_1^2 + \rho_2^2 + 4\rho_1\rho_2\right),$$

— The remainder term f belongs to $\mathcal{T}^s(1,1,\mathcal{D})$ and satisfies

$$[f]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^2$$

and

$$[f^T]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^{7/2}.$$

Now we can finish the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1. By Theorem 2.3.2 and 2.3.3, there exists a symplectic change of variables Φ_ρ, on a asymtotical set $\mathcal{D}_\nu \subset \mathcal{D} = [1,2]^2$ that puts the Hamiltonian $h = N+P$ in normal form $h_0 + f$. In appendix A we verify that $h_0 + f$ satisfies the hypotheses of KAM theorem 2.2.2 for $\delta = \nu^2$, $\varepsilon = \nu^{7/2} = \delta^{7/4}$ and $\Omega_0 = \omega = (p^2, q^2) + O(\nu^2)$. Since the hyperbolic set $\mathcal{F}$ is empty, $\Phi_\rho^{-1} \circ \mathbf{T}^{lin}_\rho$ is an invariant KAM torus that is linearly stable.

□

Persistence of 3 dimensional tori. Assume that

$$\begin{cases} a_p &= (\nu\rho_1 + r_1(t))^{\frac{1}{2}}\, e^{i\theta_1(t)} =: \sqrt{I_p}e^{i\theta_1(t)} \\ a_q &= (\nu\rho_2 + r_2(t))^{\frac{1}{2}}\, e^{i\theta_2(t)} =: \sqrt{I_q}e^{i\theta_2(t)} \\ a_m &= (\nu\rho_3 + r_3(t))^{\frac{1}{2}}\, e^{i\theta_3(t)} =: \sqrt{I_m}e^{i\theta_3(t)} \\ a_j &= \zeta_j \qquad j \in \mathbb{Z}\setminus\{p,q,m\} \end{cases}$$

where $\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{D} \subset \mathbb{R}^3$ and ν is a small parameter. The canonical symplectic structure now becomes

$$-id\zeta \wedge d\eta - dI \wedge d\theta$$

with $I = (I_p, I_q, I_m)$, $\theta = (\theta_1, \theta_2, \theta_3)$, $\zeta = (\zeta_j)_{j\in\mathbb{Z}\setminus\{p,q,m\}}$ and $\eta = (\eta_j)_{j\in\mathbb{Z}\setminus\{p,q,m\}} = (\overline{\zeta}_j)_{j\in\mathbb{Z}\setminus\{p,q,m\}}$. The same as in two-modes case, we have

$$\overline{h} := h \circ \tau = N + Z_6 + R_{10}.$$

We see that as in the previous case, the term N contributes to the effective Hamiltonian h_0 and the term R_{10} contributes to the remainder term f. So we just need to focus on the term Z_6. Let us split it :

$$Z_6 = Z_{0,6} + Z_{1,6} + Z_{2,6} + Z_{3,6}.$$

Here, $Z_{0,6}$ is a homogeneous polynomial of degree 6 which just contains inner modes (p,q,m) ; $Z_{1,6}$, $Z_{2,6}$ are homogeneous polynomials of degree 6 which contain outer modes of order 1 and 2. $Z_{3,6}$ is a homogeneous polynomial of degree 6 contains outer modes of at least order 3, this term contributes the remainder term. We have :

$$Z_{0,6} = |a_p|^6 + |a_q|^6 + |a_m|^6 + 9 \sum_{j,\ell \in \{p,q,m\}} |a_j|^4 |a_\ell|^2 + 36 |a_p|^2 |a_q|^2 |a_m|^2$$

Even if it looks a bit more complicated, we deal with $Z_{0,6}$ as in the previous case. We assume that there is no solution to (2.1.7), so that $Z_{1,6} = 0$. For $Z_{2,6}$, we have

$$\begin{aligned} Z_{2,6} = \sum_{j_1,j_2,\ell} |a_{j_1}|^2 |a_{j_2}|^2 |a_\ell|^2 &+ \sum_{s_1,t_1 \in \mathcal{A}} \left(a_{j_3}^2 a_{s_1} b_{j_4}^2 b_{t_1} + b_{j_3}^2 b_{s_1} a_{j_4}^2 a_{t_1} \right) \\ &+ \sum_{s_2,t_2 \in \mathcal{B}} \left(a_{j_5}^2 a_{j_6} b_{j_7} b_{s_2} b_{t_2} + b_{j_5}^2 b_{j_6} a_{j_7} a_{s_2} a_{t_2} \right) \\ &+ \sum_{s_3,t_3 \in \mathcal{C}} \left(a_{j_9}^2 a_{s_3} b_{j_8} b_{j_{10}} b_{t_3} + b_{j_9}^2 b_{s_3} a_{j_8} a_{j_{10}} a_{t_3} \right) \\ &+ \sum_{s_4 \in \mathcal{E}} \left(a_{j_{11}}^2 a_{j_{12}} b_{j_{13}} b_{s_4}^2 + b_{j_{11}}^2 b_{j_{12}} a_{j_{13}} a_{s_4}^2 \right) \end{aligned}$$

with $j_i \in \{p,q,m\}$, $s_i, t_i \notin \{p,q,m\}$ and $s_i \neq t_i$. The sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{E}$ are given by

$$\mathcal{A} \leftrightarrow \begin{cases} 2j_3 + s_1 &= 2j_4 + t_1 \\ 2j_3^2 + s_1^2 &= 2j_4^2 + t_1^2 \end{cases} \qquad \mathcal{B} \leftrightarrow \begin{cases} 2j_5 + j_6 &= j_7 + s_2 + t_2 \\ 2j_5^2 + j_6^2 &= j_7^2 + s_2^2 + t_2^2 \end{cases}$$

$$\mathcal{C} \leftrightarrow \begin{cases} 2j_9 + s_3 &= j_8 + j_{10} + t_3 \\ 2j_9^2 + s_3^2 &= j_8^2 + j_{10}^2 + t_3^2 \end{cases} \qquad \mathcal{E} \leftrightarrow \begin{cases} 2j_{11} + j_{12} &= j_{13} + 2s_4 \\ 2j_{11}^2 + j_{12}^2 &= j_{13}^2 + 2s_4^2. \end{cases}$$

Assume that $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{E}$ are disjoint [7] i.e. there is no s or t appearing in two of these sets. We shall deal with each term one by one (in case it's not empty).

The first term just depends on the actions, and we have

$$|a_{j_1}|^2 |a_{j_2}|^2 |a_\ell|^2 = \nu^2 \rho_{j_1} \rho_{j_2} |\zeta_\ell|^2 + \text{jet free}.$$

The second and the fourth term are similar, since their effective parts are all of the form

$$9 e^{i\alpha} \zeta_s \eta_t + 9 e^{-i\alpha} \eta_s \zeta_t.$$

7. this is the case for the example considered in Theorem 2.1.3

The idea to deal with these two terms is the same as that in the two-modes case. Since

$$\{I_s + I_t, \zeta_s\eta_t\} = \{I_s + I_t, \zeta_t\eta_s\} = 0,$$

these terms do not affect the stability of the flow. Since $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{E}$ are disjoint, and as in the two-modes case, a change of variables that used to deal with a pair s, t only affects that modes, i.e the changes of variables commute. We denote Φ_1 as the composition of all changes of variables used to deal with the sets $\mathcal{A}$ and $\mathcal{C}$.

For the third term, its effective parts are of the form

$$18\nu^2\rho_{j_5}\sqrt{\rho_{j_6}\rho_{j_7}}(e^{i\alpha}\zeta_s\zeta_t + e^{-i\alpha}\eta_s\eta_t)$$

where $\alpha = \theta_{j_7} - \theta_{j_6} - 2\theta_{j_5}$. For explicitness, we will consider the case $j_5 = p,\ j_6 = q,\ j_7 = m$, and $s,\ t$ solve the following equation

$$\begin{cases} 2p + q & = m + s + t \\ 2p^2 + q^2 & = m^2 + s^2 + t^2, \end{cases} \tag{2.3.3}$$

then $\alpha = \theta_3 - \theta_2 - 2\theta_1$. An example for this could be $(p, q, m, s, t) = (-3, 10, -6, 1, 9)$. In order to kill the angles, we introduce the symplectic change of variables $\Psi_{ang,1} : \mathcal{O}^s(\sigma, \mu) \to \mathcal{O}^s(\sigma, 3\mu)$; $(r, \theta, \zeta) \mapsto (r', \theta, \zeta')$ defined by

$$\begin{cases} \zeta'_s & = ie^{-i\alpha}\eta_s \qquad \eta'_s = ie^{i\alpha}\zeta_s \\ \zeta'_t & = \zeta_t \qquad\qquad \eta'_t = \eta_t \\ \zeta'_j & = \zeta_j, \qquad\qquad \eta'_j = \eta_j \quad j \neq s, t, p, q \\ r'_1 & = r_1 + 2|\zeta_s|^2 \\ r'_2 & = r_2 + |\zeta_s|^2, \\ r'_3 & = r_3 - |\zeta_s|^2. \end{cases}$$

The effective part related to s, t is of the form

$$\Lambda_s|\zeta'_s|^2 + \Lambda_t|\zeta'_t|^2 - 18i\nu^2\rho_1\sqrt{\rho_2\rho_3}(\zeta'_s\eta'_t + \eta'_s\zeta'_t) \tag{2.3.4}$$

where

$$\Lambda_t = t^2 + 9\nu^2(\rho_1^2 + \rho_2^2 + \rho_3^2 + 4\rho_1\rho_2 + 4\rho_2\rho_3 + 4\rho_3\rho_1)$$

and

$$\Lambda_s = t^2 + 3\nu^2(-\rho_1^2 + \rho_2^2 + 5\rho_3^2 - 6\rho_1\rho_2 + 12\rho_2\rho_3 + 6\rho_3\rho_1).$$

Denoting $a = \frac{\Lambda_t - \Lambda_s}{2}$ and $b = \frac{\Lambda_t + \Lambda_s}{2}$, we diagonalize (2.3.4) by the symplectic change of variables [8]

$$\begin{cases} \zeta_{t^-} &= \frac{1}{\sqrt{1-\alpha^2}}(\zeta_s' - i\alpha\zeta_t') \quad \eta_{t^-} = \frac{1}{\sqrt{1-\alpha^2}}(\eta_s' - i\alpha\eta_t') \\ \zeta_{t^+} &= \frac{1}{\sqrt{1-\alpha^2}}(\zeta_t' + i\alpha\zeta_s') \quad \eta_{t^+} = \frac{1}{\sqrt{1-\alpha^2}}(\eta_t' + i\alpha\eta_s') \end{cases}$$

where

$$\alpha = -\frac{a - \sqrt{a^2 - 18^2\nu^4\rho_1^2\rho_2\rho_3}}{\nu^2\rho_1\sqrt{\rho_2\rho_3}}.$$

Then (2.3.4) becomes

$$\Lambda_{t^+}|\zeta_{t^+}|^2 + \Lambda_{t^-}|\zeta_{t^-}|^2$$

where $\Lambda_{t^\pm} = b \pm \sqrt{a^2 - 18^2\nu^4\rho_1^2\rho_2\rho_3}$. We see that two modes t^+, t^- correspond to hyperbolic direction if and only if $a^2 - 18^2\nu^4\rho_1^2\rho_2\rho_3 < 0$, a condition related to the choice of ρ. Precisely, for $\rho \in \mathcal{D}_1 = [1,2]^3$, we have $\Lambda_{t^\pm} \in \mathbb{R}$ while for $\rho = (2,1,9)$ we have $a = 0$ and $a^2 - 18^2\nu^4\rho_1^2\rho_2\rho_3 = -18^2\nu^4\rho_1^2\rho_2\rho_3 < 0$. Hence, there exist $\epsilon > 0$(choose $\epsilon = 10^{-2}$) such that for $\rho \in \mathcal{D}_2 = \mathcal{D}_\epsilon = [2-\epsilon, 2+\epsilon] \times [1-\epsilon, 1+\epsilon] \times [9-\epsilon, 9+\epsilon]$ we have $|\mathrm{Im}\,\Lambda_{t^\pm}| > \nu^2$. We call Φ_2 the composition of changes of variables related to $\mathcal{B}$.

For the set $\mathcal{E}$, without loss of generality, assume that

$$\begin{cases} 2p + q &= m + 2s \\ 2p^2 + q^2 &= m^2 + 2s^2. \end{cases} \tag{2.3.5}$$

Then, using the symplectic change of variables $\Psi_{ang,2} : \mathcal{O}^s(\sigma,\mu) \to \mathcal{O}^s(\sigma, 2\mu)$; $(r,\theta,\zeta) \mapsto (r',\theta,\zeta')$ defined by

8. $\sqrt{-1} = i$

$$\begin{cases} \zeta'_s &= e^{i\alpha/2}\zeta_s \quad \eta'_s = e^{-i\alpha/2}\eta_s \\ \zeta'_j &= \zeta_j, \qquad\quad \eta'_j = \eta_j \quad j \neq s, p, q \\ r'_1 &= r_1 + |\zeta_s|^2 \\ r'_2 &= r_2 + \frac{1}{2}|\zeta_s|^2 \\ r'_3 &= r_3 - \frac{1}{2}|\zeta_s|^2. \end{cases}$$

The effective part related to s becomes

$$\Lambda_s|\zeta'_s|^2 + \nu^2\rho_1\sqrt{\rho_2\rho_3}(\zeta'^2_s + \eta'^2_s) \tag{2.3.6}$$

where

$$\Lambda_s = 3\nu^2(2\rho_1^2 + \rho_2^2 - \rho_3^2 + 9\rho_1\rho_2 + 3\rho_3\rho_1).$$

If $\Lambda_s \neq 0$, we can diagonalize (2.3.6) into $\frac{1-\beta^2}{1+\beta^2}\Lambda_s|\frac{\zeta'_s+\beta\eta'_s}{\sqrt{1-\beta^2}}|^2$ with β satisfying $\Lambda_s\beta = (1-\beta^2)\nu^2\rho_1\sqrt{\rho_2\rho_3}$, otherwise we rewrite it into $i\nu^2\rho_1\sqrt{\rho_2\rho_3}(\frac{\zeta'_s+i\eta'_s}{\sqrt{2}}\frac{\eta'_s+i\zeta'_s}{\sqrt{2}})$, however $meas\{\rho \in \mathbb{R}^3 : \Lambda_s = 0\} = 0$. We call Φ_3 the composition of all changes of variables related to $\mathcal{E}$.

By construction of Φ_i and definition of $\mathcal{O}^s(\sigma,\nu)$, the composition $\Phi_3 \circ \Phi_2 \circ \Phi_1$ maps $\mathcal{O}^s(\sigma,\nu)$ into $\mathcal{O}^s(\sigma,3\nu)$. Using the rescaling Ψ introduced in (2.3.2), as the previous case we get

Theorem 2.3.4. Assume that the equation (2.1.7) with $j_1, j_2, j_3 \in \{p, q, m\}$ has no solution in $\mathbb{Z}$ and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{E}$ are disjoint. The change of variables $\Phi_\rho := \Psi\circ\Phi_3\circ\Phi_2\circ\Phi_1\circ\tau$ is a holomorphic, symplectic transformation, and analytically depending on $\rho \in \mathcal{D}$, satisfying

— $\Phi_\rho : \mathcal{O}^s(\sigma,\mu) \to \mathbf{T}_\rho(\nu, 2\sigma, 4\mu, s)$;
— Φ_ρ puts the Hamiltonian h in normal form in the following sense :

$$\frac{1}{\nu}(h \circ \Phi_\rho - C) = h_0 + f$$

where C is a constant and the effective part h_0 of the Hamiltonian reads

$$h_0 = \Omega(\rho)\cdot r + \sum_{a\in\mathcal{Z}} \Lambda_a|\zeta_a|^2$$

where

$$\Omega(\rho) = \begin{pmatrix} p^2 + 3\nu^2\left(\rho_1^2 + 3\rho_2^2 + 3\rho_3^2 + 6\rho_1\rho_2 + 6\rho_1\rho_3 + 12\rho_2\rho_3\right) \\ q^2 + 3\nu^2\left(\rho_2^2 + 3\rho_1^2 + 3\rho_3^2 + 6\rho_1\rho_2 + 6\rho_2\rho_3 + 12\rho_1\rho_3\right) \\ m^2 + 3\nu^2\left(\rho_3^2 + 3\rho_1^2 + 3\rho_2^2 + 6\rho_1\rho_3 + 6\rho_3\rho_2 + 12\rho_2\rho_1\right) \end{pmatrix}$$

— $\mathcal{Z}$ is the disjoint union $\mathcal{L}\cup\mathcal{F}$; $\mathcal{F}$ is consistent with $\mathcal{B}$ and corresponds to hyperbolic part ; $\mathcal{L}$ is consistent with other exterior modes and corresponds to elliptic part ;

— the remainder term f belongs to $\mathcal{T}^s(\sigma,\mu,\mathcal{D})$ and satisfies

$$[f]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^2$$

and

$$[f^T]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^{7/2}.$$

Proof of Theorem 2.1.3. By Theorem 2.3.4, for $(p,q,m) = (-3,10,-6)$ and $\rho \in \mathcal{D}_\nu \subset \mathcal{D}_2$, there exists a symplectic change of variables Φ_ρ on $\mathcal{D}_\nu$ that puts the Hamiltonian $h = N+P$ in normal form h_0+f. In appendix A we verify that h_0+f satisfies assumptions of KAM theorem 2.2.2 for $\delta = \nu^2$, $\varepsilon = \nu^{7/2} = \delta^{7/4}$ and $\Omega_0 = \omega = (3^2,10^2,6^2) + \mathcal{O}(\nu^2)$. Since the hyperbolic set $\mathcal{F}$ is not empty, $\Phi_\rho^{-1} \circ \mathbf{T}_\rho^{lin}$ is an invariant KAM torus that is linearly unstable. □

2.4 Appendix A

In this appendix, we will verify the hypothesis A0, A1, A2 of Theorem 2.2.2 for the Hamiltonian in our applications. The hypothesis A0, A1 is trivial, so we focus on A2.

2.4.1 Two-modes case

The first case In this case, we have $\mathcal{F} = \emptyset$ and the other estimates are trivial. For the hypothesis A2, we recall that

$$\Omega(\rho) = \begin{pmatrix} p^2 + 3\nu^2\left(\rho_1^2 + 3\rho_2^2 + 6\rho_1\rho_2\right) \\ q^2 + 3\nu^2\left(\rho_2^2 + 3\rho_1^2 + 6\rho_1\rho_2\right) \end{pmatrix}$$

and

$$\Lambda_j = j^2 + 9\nu^2\left(\rho_1^2 + \rho_2^2 + 4\rho_1\rho_2\right).$$

Let $k = (k_1, k_2) \in \mathbb{Z}^2/\{0\}$ and $z = z(k) = \frac{(k_2,k_1)}{|k|}$, then we have

$$\begin{aligned}(\nabla_\rho \cdot z)(\Omega(\rho)\cdot k) &= 6\nu^2\left(3(\rho_1+\rho_2)k_2^2 + 3(\rho_2+3\rho_1)k_1^2 + 4(\rho_1+\rho_2)k_1k_2\right)|k|^{-1}\\ &\geq \frac{6}{\sqrt{2}}\nu^2|k|\end{aligned}$$

and

$$(\nabla_\rho \cdot z)\Lambda_j = 18\nu^2((\rho_1+2\rho_2)k_2 + (\rho_2+2\rho_1)k_1)|k|^{-1}.$$

Choosing $\delta = 4\nu^2$, we get the hypothesis A2 (1). Since $(\nabla_\rho \cdot z)(\Lambda_j - \Lambda_\ell) = 0$, the estimate of small divisor $\Omega\cdot k + \Lambda_j - \Lambda_\ell$ follows. To estimate the small divisors $\Omega \cdot k + \Lambda_j$ and $\Omega\cdot k + \Lambda_j + \Lambda_\ell$ we use the fact that f commutes with both the mass $\mathbb{L}$ and momentum $\mathbb{M}$. We just need to control small divisors $\Omega\cdot k + \Lambda_j$ and $\Omega\cdot k + \Lambda_j + \Lambda_\ell$ whenever $e^{ik\cdot\theta}\eta_j \in f$ and $e^{ik\cdot\theta}\eta_j\eta_\ell \in f$, respectively. We have for the mass and momentum :

$$\mathbb{L} = \nu(\rho_1+\rho_2) + r_1 + r_2 + \sum_j |\zeta_j|^2$$

and

$$\mathbb{M} = \nu(p\rho_1 + q\rho_2) + pr_1 + qr_2 + \sum_j j|\zeta_j|^2.$$

By conservation of $\mathbb{L}$, we have

$$\{e^{ik\cdot\theta}\eta_j, \mathbb{L}\} = ie^{ik\cdot\theta}\eta_j(k_1+k_2+1) = 0.$$

Therefore, for A2 (2) we just have to study the case $k_1 + k_2 = -1$. In this situation

$$\begin{aligned}(\nabla_\rho \cdot z)(\Omega(\rho)\cdot k + \Lambda_j) &= 6\nu^2|k|^{-1}\left(3(\rho_1+\rho_2)k_2^2 + 3(\rho_2+\rho_1)k_1^2 + 4(\rho_1+\rho_2)k_1k_2\right)\\ &\quad + 6\nu^2|k|^{-1}\left(3(\rho_1+2\rho_2)k_2 + 3(\rho_2+2\rho_1)k_1\right)\\ &= 6\nu^2|k|^{-1}\left((\rho_1+\rho_2)k_2^2 + (\rho_2+\rho_1)k_1^2 + 2(\rho_1+\rho_2)\right)\\ &\quad + 6\nu^2|k|^{-1}\left(3\rho_2k_2 + 3\rho_1k_1 - 3(\rho_1+\rho_2)\right)\\ &= 6\nu^2|k|^{-1}\left(2(\rho_1+\rho_2)k_1^2 + (5\rho_1-\rho_2)k_1 - 3\rho_2\right).\end{aligned}$$

This term is greater than δ except in the cases $k = (-1, 0)$ and $(0, -1)$. The conservation of $\mathbb{M}$ gives us

$$\{e^{ik\cdot\theta}\eta_j, \mathbb{M}\} = ie^{ik\cdot\theta}\eta_j(pk_1 + qk_2 + j) = 0.$$

For $k \in \{(-1,0),\ (0,-1)\}$, this implies $j \in \{p,q\}$, which is excluded.
We consider the small divisor $\Omega \cdot k + \Lambda_j + \Lambda_\ell$ in the same way. The conservation of the mass $\mathbb{L}$ gives us $k_1 + k_2 = -2$ and then by computation we get

$$k \in \{(0,-2),(-2,0),(-1,-1),(-3,1),(1,-3)\}.$$

The conservation of the momentum gives us $pk_1 + qk_2 + j + \ell = 0$. We have

$$\Omega \cdot k + \Lambda_j + \Lambda_\ell = N(p,q,j,\ell) + \mu(\rho,k)$$

where $N(p,q,j,\ell) = p^2k_1 + q^2k_2 + j^2 + \ell^2$ and $\mu(\rho,k)$ very small for $|k| \leq 4$. We see that $N(p,q,j,\ell) \in \mathbb{Z}$, so $N(p,q,j,\ell) \leq \delta$ if and only if $p^2k_1 + q^2k_2 + j^2 + \ell^2 = 0$. Combined with conservation of the momentum, this gives
for the case $k = (-1,-1)$

$$p + q = j + \ell \quad \text{and} \quad p^2 + q^2 = j^2 + \ell^2$$

for the case $k = (-2,0)$

$$2p = j + \ell \quad \text{and} \quad 2p^2 = j^2 + \ell^2$$

for the case $k = (0,-2)$

$$2q = j + \ell \quad \text{and} \quad 2q^2 = j^2 + \ell^2$$

for the case $k = (-3,1)$

$$3p = q + j + \ell \quad \text{and} \quad 3p^2 = q^2 + j^2 + \ell^2$$

for the case $k = (1,-3)$

$$3q = p + j + \ell \quad \text{and} \quad 3q^2 = p^2 + j^2 + \ell^2.$$

In all these cases, we get $j, \ell \in \{p,q\}$ which is excluded.

The second case We see that Ω and $\{\Lambda_j\}_{j \neq p,q,s,t}$ are all the same as the previous case except Λ_{t^+} and Λ_{t^-}.We recall that

$$\begin{cases} 2p + s &= 2q + t \\ 2p^2 + s^2 &= 2q^2 + t^2. \end{cases}$$

Thank to Lemma 2.2 in [GT12], $\{p,q,s,t\}$ is in form of $\{p,\ p+2n,\ p+3n,\ p-n\}$. Without loss of generality, we can assume that[9] $p = 0$, so we have $q = -2t$. For $\Omega \cdot k + \Lambda_{t^+}$ and $\Omega \cdot k + \Lambda_{t^-}$, by conservation the momentum, we just need to consider the case when k satisfies $pk_1 + qk_2 + t = 0$ i.e. $k_2 = 1/2$, which is not an integer. For $\Omega \cdot k + \Lambda_{t^\pm} \pm \Lambda_j$, again by conservation of the momentum, we have

$$\begin{cases} pk_1 + qk_2 + t \pm j & = 0 \\ p^2k_1 + q^2k_2 + t^2 \pm j^2 & = 0 \end{cases}$$

i.e.

$$\begin{cases} j & = \mp(2k_2 - 1)n \\ j^2 & = \mp(4k_2 + 1)n^2. \end{cases}$$

This system has two solutions for j, either $j = 0 = p$ or $j = 3m = s$, which are both excluded.

2.4.2 Three modes case.

It is too complicated to verify all the possibility, in this appendix we just consider the example $(p, q, m) = (-3, 10, -6)$, which we are interesting in Theorem 2.1.3. In this situation, we have that $\mathcal{C}$, $\mathcal{E}$ are empty, $\mathcal{A} = \{-14, 2\}$ and $\mathcal{B} = \{9, 1\}$. Recall that

$$\Omega(\rho) = \begin{pmatrix} p^2 + 3\nu^2\left(\rho_1^2 + 3\rho_2^2 + 3\rho_3^2 + 6\rho_1\rho_2 + 6\rho_1\rho_3 + 12\rho_2\rho_3\right) \\ q^2 + 3\nu^2\left(\rho_2^2 + 3\rho_1^2 + 3\rho_3^2 + 6\rho_1\rho_2 + 6\rho_2\rho_3 + 12\rho_1\rho_3\right) \\ m^2 + 3\nu^2\left(\rho_3^2 + 3\rho_1^2 + 3\rho_2^2 + 6\rho_1\rho_3 + 6\rho_3\rho_2 + 12\rho_2\rho_1\right) \end{pmatrix}$$

and

$$\Lambda_j = j^2 + 9\nu^2(\rho_1^2 + \rho_2^2 + \rho_3^2 + 4\rho_1\rho_2 + 4\rho_2\rho_3 + 4\rho_3\rho_1) \qquad j \neq -14, -6, -3, 2, 1, 9, 10.$$

The hypothesis $A0$ and $A1$ are trivial. For hypothesis A2 (1), let $k = (k_1, k_2, k_3) \in \mathbb{Z}^3/\{0\}$, $k' = (k_2 + k_3, k_1 + k_3, k_2 + k_1)$ and $z = z(k) = \frac{k'}{|k'|}$, then we have

$$\begin{aligned} (\nabla_\rho \cdot z)(\Omega(\rho) \cdot k) =& 6\nu^2|k'|^{-1}[\rho_1(3k_2^2 + 3k_3^2 + k_1k_2 + k_1k_3 + 6(k_1 + k_2 + k_3)^2) \\ &+ \rho_2(3k_1^2 + 3k_3^2 + k_1k_2 + k_2k_3 + 6(k_1 + k_2 + k_3)^2) \\ &+ \rho_3(3k_2^2 + 3k_1^2 + k_3k_2 + k_1k_3 + 6(k_1 + k_2 + k_3)^2)]. \end{aligned}$$

9. using the change of variables $j = j - p$

This term is greater than $\delta = \nu^2$. Since $(\nabla_\rho \cdot z)(\Lambda_j - \Lambda_\ell) = 0$, the estimate of small divisor $\Omega \cdot k + \Lambda_j - \Lambda_\ell$ follows.
For hypothesis A2 (2), (3), choose $z = z(k) = -\frac{k}{|k|}$, then we have

$$\begin{aligned}(\nabla_\rho \cdot z)(\Omega(\rho) \cdot k) = &- 6\nu^2|k|^{-1}[\rho_1(k_1^2 + 3k_2^2 + 3k_3^2 + 6k_1k_2 + 6k_1k_3 + 12k_2k_3)\\ &+ \rho_2(k_2^2 + 3k_1^2 + 3k_3^2 + 6k_1k_2 + 6k_2k_3 + 12k_1k_3)\\ &+ \rho_3(k_3^2 + 3k_2^2 + 3k_1^2 + 6k_3k_2 + 6k_1k_3 + 12k_2k_1)]\end{aligned}$$

and

$$(\nabla_\rho \cdot z)\Lambda_j = -18\nu^2|k|^{-1}[\rho_1(k_1 + 2k_2 + 2k_3) + \rho_2(k_2 + 2k_1 + 2k_3) + \rho_3(k_3 + 2k_2 + 2k_1)].$$

For $\Omega \cdot k + \Lambda_j$, by conservation of the mass, we just need to estimate this divisor in the case $k_1 + k_2 + k_3 = -1$, then by computation we have

$$\begin{aligned}|(\nabla_\rho \cdot z)(\Omega(\rho) \cdot k + \Lambda_j) &= 6\nu^2|k|^{-1}[\rho_1(2k_1^2 - 6k_2k_3 + 3k_1 + 3) + \rho_2(2k_2^2 - 6k_1k_3 + 3k_2 + 3)\\ &+ \rho_3(2k_3^2 - 6k_2k_1 + 3k_3 + 3)]\\ &\geq 6\nu^2|k|^{-1}[\rho_1(2k_1^2 - \frac{3}{2}(k_1+1)^2 + 3k_1 + 3) + \rho_2(2k_2^2\\ &- \frac{3}{2}(k_2+1)^2 + 3k_2 + 3) + \rho_3(2k_3^2 - \frac{3}{2}(k_3+1)^2 + 3k_3 + 3)]\\ &= 3\nu^2|k|^{-1}[\rho_1(k_1^2 + 3) + \rho_2(k_2^2 + 3) + \rho_3(k_3^2 + 3)]\\ &\geq \nu^2.\end{aligned}$$

For $\Omega \cdot k + \Lambda_j + \Lambda_\ell$, again we have $k_1 + k_2 + k_3 = -2$ by conservation of the mass, hence

$$\begin{aligned}|(\nabla_\rho \cdot z)(\Omega(\rho) \cdot k + \Lambda_j) =&6\nu^2|k|^{-1}[\rho_1(2k_1^2 - 6k_2k_3 + 6k_1 + 12) + \rho_2(2k_2^2 - 6k_1k_3 + 6k_2 + 12)\\ &+ \rho_3(2k_3^2 - 6k_2k_1 + 6k_3 + 12)]\\ &\geq 6\nu^2|k|^{-1}[\rho_1(2k_1^2 - \frac{3}{2}(k_1+1)^2 + 6k_1 + 12) + \rho_2(2k_2^2\\ &- \frac{3}{2}(k_2+2)^2 + 6k_2 + 12) + \rho_3(2k_3^2 - \frac{3}{2}(k_3+2)^2 + 6k_3 + 12)]\\ &= 3\nu^2|k|^{-1}[\rho_1(k_1^2 + 12) + \rho_2(k_2^2 + 12) + \rho_3(k_3^2 + 12)]\\ &\geq \nu^2.\end{aligned}$$

The set $\mathcal{B}$ For $\rho \in \mathcal{D}_2$: we have

$$|\mathrm{Im}\,\Lambda_{1\pm}| > \nu^2 = \delta$$

so that

$$|\Omega \cdot k + \Lambda_{1^+} - \Lambda_{1^-}| \geq 2\nu^2 > \delta.$$

For $\Omega \cdot k + \Lambda_{1^+} + \Lambda_{1^-}$, by conservation of the mass and the momentum, we just need to estimate this small divisor if

$$\begin{cases} k_1 + k_2 + k_3 + 2 & = 0 \\ -3k_1 + 10k_2 - 6k_3 + 2 & = 0 \\ 9k_1 + 100k_2 + 36k_3 + 2 & = 0 \\ k_1, k_2, k_3 \in \mathbb{Z} \end{cases}$$

This equation system has no solution [10].

The set $\mathcal{A}$ For $\Omega \cdot k + \Lambda_{2^\pm}$ and $\Omega \cdot k + \Lambda_{2^\pm} + \Lambda_j$ again the conservation of the mass and the momentum give

$$(*) \begin{cases} k_1 + k_2 + k_3 + 1 & = 0 \\ -3k_1 + 10k_2 - 6k_3 + 2 & = 0 \\ 9k_1 + 100k_2 + 36k_3 + 4 & = 0 \end{cases} \qquad (**) \begin{cases} k_1 + k_2 + k_3 + 2 & = 0 \\ -3k_1 + 10k_2 - 6k_3 + 2 + j & = 0 \\ 9k_1 + 100k_2 + 36k_3 + 4 + j^2 & = 0. \end{cases}$$

It is easy to see that $(*)$ has no solution in $\mathbb{Z}^3$. For $(**)$ we have $j \equiv -k_2 - 2 \pmod 3$ and $j^2 \equiv -k_2 - 4 \pmod 9$. If $j \equiv \pm 1 \pmod 3$ then we have $k_2 \equiv 0, 2 \pmod 4$ and $k_2 = 4 \pmod 9$, which can not both happen. If $j \equiv 0 \pmod 3$ then we have $k_2 \equiv 1 \pmod 4$ and $k_2 = 5 \pmod 9$, which again can not happen. For $\Omega \cdot k + \Lambda_{2^\pm} - \Lambda_j$, because of changes of variables, we have

$$\begin{aligned} \Lambda_{2^+} &= \Lambda_2 - g(\rho_1, \rho_2, \rho_3) \\ \Lambda_{2^-} &= \Lambda_2 - g(\rho_1, \rho_2, \rho_3) + 12(\rho_3^2 - \rho_2^2 + 3\rho_1\rho_3 - 3\rho_2\rho_1) \end{aligned}$$

with

$$g(x, y, z) = \mu^2 \sqrt{81y^2z^2 + (-18xy + 18xz - 6y^2 + 6z^2)^2} - \mu^2(-18xy + 18xz - 6y^2 + 6z^2).$$

10. with the implicit form of $\{p, q, m, s, t\}$ in appendix B, we can solve for general p, q, m

By conservation of the mass we just need to consider the case $k_1 + k_2 + k_3 = 0$, then

$$\begin{aligned}(\nabla_\rho \cdot z)(\Omega \cdot k + \Lambda_{2^\pm} - \Lambda_j) =& 12\mu^2|k|^{-1}[\rho_1(k_2^2 + k_3^2 - k_2k_3 - 2k_2 + k_3) \\ &+ \rho_2(k_1^2 + k_3^2 - k_1k_3 + 2k_1 - k_3) \\ &+ \rho_3(k_2^2 + k_1^2 - k_2k_1 + 3k_1 - 3k_2)] \pm (\nabla_\rho \cdot z)g \\ &\approx 12|k|\mu^2\rho \pm |(\nabla_\rho \cdot z)g|.\end{aligned}$$

The conservation of the momentum implies

$$\begin{cases} -3k_1 + 10k_2 - 6k_3 + 2 - j &= 0 \\ 9k_1 + 100k_2 + 36k_3 + 4 - j^2 &= 0. \end{cases}$$

The solution k of this equation system that closet to the origin is $k = (-975, 195, 780)$ and with such a big k, $(\nabla_\rho \cdot z)(\Omega \cdot k + \Lambda_{2^\pm} - \Lambda_j)$ is far greater than δ.

2.5 Appendix B

In this appendix, we solve the set $\mathcal{B}$ in general

$$\begin{cases} 2p + q &= m + s + t \\ 2p^2 + q^2 &= m^2 + s^2 + t^2. \end{cases}$$

Let $q_1 = q - p$, $m_1 = m - p$, $s_1 = s - p$, $t_1 = t - p$, it becomes

$$\begin{cases} q_1 &= m_1 + s_1 + t_1 \\ q_1^2 &= m_1^2 + s_1^2 + t_1^2. \end{cases}$$

This give us $m_1s_1 + t_1s_1 + t_1m_1 = 0$, hence $s_1 = -\frac{m_1t_1}{m_1+t_1}$. Assume more that s_1, t_1, m_1 have no common divisor except ±1. Let k is a prime common divisor of t_1 and m_1, i.e. $t_1 = t_2k$, $m_1 = m_2k$, then $s_1 = -\frac{km_2t_2}{m_2+t_2}$. Since $k \nmid s_1$, we have $k \mid t_2 + m_2$, i.e. $t_2 = kh - m_2$, hence $s_1 = -\frac{m_2(kh-m_2)}{h} = -km_2 + \frac{{m_2}^2}{h} \in \mathbb{Z}$. Let $h = (-1)^{sgn(h)}\Pi p_i^{k_i}$, $x = \Pi p_i^{[\frac{k_i}{2}]}$ and $y = (-1)^{sgn(h)}\Pi p_i^{k_i - 2[\frac{k_i}{2}]}$, with p_i is prime divisor of h. Then, $h = x^2y$ and we need $xy \mid m^{\mathfrak{tt}}$, i.e. $m^{\mathfrak{tt}} = ryx$. By this, $s_1 = -kxyr + r^2y$, $m_1 = kryx$, $t_1 = k^2x^2y - ryx$. Since s_1, t_1, m_1 have no common divisor except ±1, we have $y = \pm1$. Assume that $y = 1$, and $kx = n$, then $s_1 = r^2 - nr$, $m_1 = nr$, $t_1 = n^2 - nr$ and $q_1 = n^2 - nr + r^2$. In general, we have $\{p, q, m, s, t\} = \{p, p + k(n^2 - nr + r^2), p + knr, p + k(r^2 - nr), p + k(n^2 - nr)\}$.

3

Reducibility of Schrödinger equation on a Zoll Manifold with unbounded potential

3.1 Introduction

The reducibility problem for Schrödinger equations with quasi-periodic in time perturbation has been intensively studied in recent years. The first results adapting the KAM technics were due to Kuksin [Kuk93] followed by many results in one dimensional context (see in particular [BG01 ; LY10 ; GT11]). More recently the techniques were adapted to the higher dimensional case [EK09 ; EGK16 ; GP19]. To consider unbounded perturbations, a new strategy has been developed in [BBM14 ; BBM16] using the pseudo-differential calculus. Without trying to be exhaustive we quote also [**FGP** ; FP14 ; BM20 ; BBHM18] regarding KAM theory for quasi-linear PDEs in one space dimension. This technics were successfully applied for reducibility problems in various cases. For one dimensional linear equations with unbounded potential we quote [Bam17 ; BM18 ; FGP18]. In higher space dimensions with unbounded perturbations only few results exist, one concerning the quantum harmonic oscillator on $\mathbb{R}^n$ with polynomial time dependent perturbation [BGMR17] and some special examples on the torus $\mathbb{T}^n$ [Mon19 ; FGMP19 ; BLM19]. In all these multi-dimensional examples the unperturbed linear system were integrable in the classical sense (for instance on the torus $\mathbb{T}^n$, the Laplacian operator commutes with ∂_j, $j = 1, \cdots, n$), a fact that will be crucial in the control of the perturbed spectrum (see (3.1.11) below). In this article we consider a Schrödinger equation on a Zoll manifold on which the Laplace Beltrami operator Δ_g has, in general, no other first integral than energy (in particular Δ_g doesn't commute with ∂_j) .

We first recall that a Zoll manifold of dimension $n \in \mathbb{N}$ is a compact Riemannian manifold $(\mathtt{M}^n, g)$ such that all the geodesic curves have all the same period T. In this paper we assume $T := 2\pi$. For example the n-dimensional sphere $\mathbb{S}^n$ is a Zoll mani-

fold. We denote by Δ_g the positive Laplace-Beltrami operator on $(\mathtt{M}^n, g)$ and we define $H^s(\mathtt{M}^n) := \mathrm{dom}(\sqrt{1+\Delta_g})^s$ with $s \in \mathbb{R}$ the usual scale of Sobolev spaces.
We denote by $S^m_{\mathrm{cl}}(\mathtt{M}^n)$ the space of classical real valued symbols of order $m \in \mathbb{R}$ on the cotangent bundle $T^*(\mathtt{M}^n)$ and we define $\mathcal{A}_m$ the associated class of pseudo-differential operators (see for instance Hörmander [Hor85] for a definition of pseudo-differential operators on a manifold see also [BGMR2] in the case of a Zoll manifold).

We consider the following linear Schrödinger equation

$$\mathrm{i}\partial_t u = \Delta_g u + \varepsilon W(\omega t) u\,, \qquad u = u(t,x)\,, \quad t \in \mathbb{R}\,, \quad x \in \mathtt{M}^n\,, \tag{3.1.1}$$

where $\varepsilon > 0$ is a small parameter and $W(\omega t)$ is a time dependent unbounded operator from $H^s(\mathtt{M}^n) \to H^{s-\delta}(\mathtt{M}^n)$ for some $\delta \leq 1/2$. More precisely we assume that $W \in C^\infty(\mathbb{T}^d; \mathcal{A}_\delta)$ with $\delta \leq 1/2$, $d \geq 1$ and where $\mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$. So the potential $t \mapsto W(\omega t)$ depends on time quasi-periodically with frequency vector $\omega \in \mathbb{R}^d$ and for any $\varphi \in \mathbb{T}^d$ the linear operator $W(\varphi)$ is a pseudo-differential operator of order δ, i.e. belongs to $\mathcal{A}_\delta$.

The purpose of this article is to construct a change of variables that transforms the non-autonomous equation (3.1.1) into an autonomous equation.

Our main result is the following.

Theorem 3.1.1. *Let $0 < \alpha < 1$ and $\delta \in \mathbb{R}$, $\delta \leq 1/2$. Assume that the map $\varphi \mapsto W(\varphi, \cdot) \in \mathcal{A}_\delta$ is C^∞ in $\varphi \in \mathbb{T}^d$. Then for any $s \in \mathbb{R}$, $s > n/2$ there exists $\varepsilon_0 > 0$ and $C > 0$ such that, for any $0 < \varepsilon \leq \varepsilon_0$ there is a set $\mathcal{O}_\varepsilon \subset [1/2, 3/2]^d \subset \mathbb{R}^d$ with*

$$\mathrm{meas}([1/2, 3/2]^d \setminus \mathcal{O}_\varepsilon) \leq C\varepsilon^\alpha \tag{3.1.2}$$

such that the following holds. For any $\omega \in \mathcal{O}_\varepsilon$ there exists a family of linear isomorphisms $\Psi(\varphi) \in \mathcal{L}(H^s(\mathtt{M}^n))$ and a Hermitian operator $Z \in \mathcal{A}_\delta$ commuting with the Laplacian[1] and satisfying

$$\|Z\|_{\mathcal{L}(H^s(\mathtt{M}^n), H^{s-\delta}(\mathtt{M}^n))} \leq C\varepsilon\,. \tag{3.1.3}$$

Furthermore

- *$\Psi(\varphi)$ is unitary on $L^2(\mathtt{M}^n)$;*

1. actually $[\Delta_g, Z] = 0$ on sphere while on Zoll manifold Z and Δ_g can be diagonalized in the same basis of $L^2(\mathtt{M}^n)$.

• *for any* $\frac{n}{2} < s' \leq s$ *and any* $\omega \in \mathcal{O}_\varepsilon$

$$\begin{aligned} \|\Psi(\varphi) - \mathrm{Id}\|_{\mathcal{L}(H^{s'}(\mathtt{M}^n), H^{s'-\delta}(\mathtt{M}^n))} \\ + \|\Psi(\varphi)^{-1} - \mathrm{Id}\|_{\mathcal{L}(H^{s'}(\mathtt{M}^n), H^{s'-\delta}(\mathtt{M}^n))} &\leq C\varepsilon^{1-\alpha}, \\ \|\Psi(\varphi)\|_{\mathcal{L}(H^{s'}(\mathtt{M}^n))} + \|\Psi(\varphi)^{-1}\|_{\mathcal{L}(H^{s'}(\mathtt{M}^n))} &\leq 1 + C\varepsilon^{1-\alpha}, \end{aligned} \tag{3.1.4}$$

• *for any* $\frac{n}{2} < s' \leq s$ *and any* $\omega \in \mathcal{O}_\varepsilon$ *the map* $t \mapsto u(t, \cdot) \in H^{s'}(\mathtt{M}^n)$ *solves* (3.1.1) *if and only if the map* $t \mapsto v(t, \cdot) := \Psi(\omega t) u(t, \cdot)$ *solves the autonomous equation*

$$\mathrm{i}\partial_t v = \Delta_g v + \varepsilon Z(v)\,. \tag{3.1.5}$$

As a consequence of our reducibility result, we get a control of the flow generated by the (3.1.1) equation in the scale of Sobolev spaces :

Corollary 3.1.2. *Let* $W \in C^\infty(\mathbb{T}^d; \mathcal{A}_\delta)$ *with* $\delta \leq 1/2$. *Then for any* $s \in \mathbb{R}$, $s > n/2$ *there exists* $\varepsilon_0 > 0$ *and* $C > 0$ *such that, for any* $0 < \varepsilon \leq \varepsilon_0$ *there is a set* $\mathcal{O}_\varepsilon \subset [1/2, 3/2]^d \subset \mathbb{R}^d$ *satisfying* (3.1.2) *such that for any* $\omega \in \mathcal{O}_\varepsilon$ *the flow generated by the* (3.1.1) *equation is bounded in* $H^s(\mathtt{M}^n)$.
More precisely if $u_0 \in H^s(\mathtt{M}^n)$ *then there exists a unique solution* $u \in C^1\big(\mathbb{R}\,;\, H^s(\mathtt{M}^n)\big)$ *of* (3.1.1) *such that* $u(0) = u_0$. *Moreover, u is almost-periodic in time and satisfies*

$$(1 - \varepsilon C)\|u_0\|_{H^s} \leq \|u(t)\|_{H^s} \leq (1 + \varepsilon C)\|u_0\|_{H^s}, \quad \forall\, t \in \mathbb{R}\,, \tag{3.1.6}$$

for some $C = C(s) > 0$.

Following the pioneering work [BBM14] we prove Theorem 3.1.1 in two steps :

— *The regularization step* where we use the pseudo-differential calculus (and in particular the technics developed in [BGMR2]) to transform equation (3.1.1) in a system with a smoothing perturbation, still depending on time ;

— *The KAM step* where we use a KAM procedure (going back to [Kuk93] but using recent development in [BBHM18]) on infinite dimensional matrices to eliminate the time in the new system.

The same strategy was recently successfully applied in [BLM19] to prove the reducibility of non-resonant transport equation on the torus $\mathbb{T}^n$. Our main contribution consists in merging these two recent technics in the context of linear Schrödinger equation on Zoll manifold which, in contrast to the transport equation on the torus, is not an integrable system.

Our result has to be compared with the recent work of two of us [FG19] where we consider a Schrödinger equation on the sphere $\mathbb{S}^n$ with a quasi-periodic in time odd perturbations of order $< 1/2$. In that case a more standard approach following [GP16] was possible, in particular our analysis did not require the pseudo-differential calculus. Of course this new paper is a generalization in the sense that we replaced the sphere by a Zoll manifold and we are able to treat perturbations of order $1/2$. But we want to stress that the elimination of the potential symmetry hypothesis may be even more important if we look at generalization to the non-linear case. Actually a natural strategy to solve the non linear Schrödinger eqation

$$\mathrm{i}\partial_t u = \Delta_g u + mu + \varepsilon |u|^2 u\,, \qquad u = u(t,x)\,, \quad t \in \mathbb{R}\,, \quad x \in \mathtt{M}^n\,, \tag{3.1.7}$$

consists in a Newton scheme : we linearize the equation (3.1.7) around an approximate solution u_0, we solve this linear equation to obtain u_1 and we linearize (3.1.7) around u_1 and we iterate. Doing so we have to solve linear Schrödinger equation of the kind [2] (3.1.1) where $W(\omega t) = |u_0(t)|^2$ which is clearly not an odd function.
As a matter of fact the existence of quasi-periodic solutions of the forced non linear Schrödinger equation

$$\mathrm{i}\partial_t u = \Delta_g u + mu = \varepsilon f(\omega t, x, u)\,, \qquad u = u(t,x)\,, \quad t \in \mathbb{R}\,, \quad x \in \mathbb{S}^n \tag{3.1.8}$$

were already addressed by Berti-Corsi-Procesi in [BCP15]. They proved that for ω in a large Cantor's set and for a Hamiltonian and smooth forced nonlinear perturbation f and ε small enough, there is a smooth quasi-periodic solution of (3.1.8). See also [BP11] by Berti-Bolle-Procesi in which the authors prove existence of *periodic* solutions on Zoll manifolds, we also mention [CHP15] where a KAM approach was considered in the context of Lie groups with symmetries. An adaptation of our work to the context of systems of linear Schrödinger equations (see previous footnote) could prove that this solution is linearly stable. We shall remark that such an adaptation is not trivial at all, and some new ideas are required.

We list here the main issues :

• as already said the linearization of (3.1.8) give rise to a linear operators acting on the couple $\left[\begin{smallmatrix} u \\ \overline{u} \end{smallmatrix}\right]$, i.e. and equation of the form

$$\mathrm{i}\partial_t u = \Delta_g u + \varepsilon W_1(\omega t) u + \varepsilon W_2(\omega t)\overline{u}\,, \tag{3.1.9}$$

2. In fact the linearization of (3.1.7) gives rise to a system of linear equations for u and $\overline{u}$.

with W_i, $i = 1, 2$ operators similar to W in (3.1.1). As first step of the regularization procedure, one needs to block diagonalize, up to smoothing remainders, the operator in the r.h.s. of (3.1.9) in the spirit of [FI18], [FI19], i.e. one needs to "eliminate" W_2. This is just a technical point which can be addressed by extending the arguments of section 3.3.

• A second difficulty concerns the pseudo-differential calculus on Zoll manifold which is more difficult and more implicit than the pseudo-differential calculus on the torus. In particular the estimates on the semi-norms of the symbols (see the estimates under Definition 3.2.1) are not sharp in terms of the regularity required on symbols. Actually on tori one can define, explicitly a semi-norm satisfying "tame" estimates. We refer, for instance, to section 2 in [FGP18].

• A major difficulty (which is linked to the item above) regards the minimal regularity one needs on the potential $W(\omega t)$ in (3.1.1) (or W_1, W_2 in (3.1.9)). Theorem 3.3.1 shows that, in order to prove reducibility in $H^s(\mathtt{M}^n)$, $s > n/2$, the potential $W(\omega t)$ must be in H^p, both in time and space, for some $p \gg 1$ depending on s. This is a consequence of the use of pseudo-differential calculus for the regularization. In the proof of Theorem 3.1.1 in section 3.5 it turns out that $p \gg 2s$. In the case of this paper this is not a problem since the potential is C^∞. In the non linear case, the regularity of the potential depends on the approximate solution of the previous step and hence have only finite regularity. Moreover the requirement $p \gg 2s$ is not compatible with the convergence of the Nash-Moser scheme. We notice that in 1d this problem can be overcome (see for instance [BBHM18] or [FGP18]) since we need only few regularization steps and furthermore we can use the so called Poschel's Lemma (see Lemma A.1 in [Posc96]). It is not clear at the moment how to overcome this problem in a multi-dimensionnal context.

• Another serious problem is about the small divisors. It is know that a KAM reducibility scheme requires some non resonance condition on the eigenvalues of the linear operator. We prove such non resonance conditions, for many frequency vector ω, in section 3.4.1. It turns out that the eigenvalues of the operator in the right hand side of (3.1.1) have the expansion

$$\mu_{k,j} = \Lambda_{k,j} + \mu^{(1)}_{k,j} + \mu^{(2)}_{k,j}\,, \quad k \in \mathbb{N}\,, \ \ j = 1, \ldots, d_k\,,$$

where $\Lambda_{k,j}$ are the eigenvalues of Δ_g (see (3.2.5)), d_k the dimension of the corresponding eigenspace, and

$$\mu^{(1)}_{k,j} \sim \varepsilon |k|^{\delta}\,, \qquad \mu^{(2)}_{k,j} \sim \varepsilon\,,$$

with δ the order of the pseudo-differential operator $W(\omega t)$. In the measure estimate Lemma 3.4.3 it is fundamental that the *unbounded* corrections $\mu_{k,j}^{(1)}$ do not depend on the parameters ω ; as a consequence the Lipschitz norm of the eigenvalues $\mu_{k,j}$ is bounded in k. In our case this fact is true since the corrections $\mu_{k,j}^{(1)}$ are obtained form an averaging procedure on the potential W, which, at the beginning, does not depend explicitly on ω. More precisely it depends on ω only through the variable $\varphi = \omega t$. In the non linear case this is no more true, since W would depend on ω also through the function on which we linearized the equation (3.1.8). This problem could be overcome by considering non linear equation like (3.1.8) with *bounded* non linearity, in order to obtain a linear *bounded* operator when linearizing (i.e. $\delta \leq 0$).

In view of the issues discussed above the KAM result for a non linear Schrödinger equation with unbounded non linearities on Zoll manifold is out of reach for the moment. Nevertheless we believe that this paper represent an important milestone in that direction.

Scheme of the proof.

As said above the proof consists in a regularization step (section 3.3) and a KAM step (section 3.4). In section 3.5 we merge the two procedures to prove Theorem 3.1.1.

In the regularization step we prove that we can transform (by using a symplectic map : $u = \Phi(v)$) the original Schrödinger equation (3.1.1) in a new one

$$\mathrm{i}\partial_t v = \Delta_g v + \varepsilon(Z + R(\omega t))v\,, \tag{3.1.10}$$

where Z is a pseudo-differential operator of order δ independent on time and commuting with Δ_g and R is a ρ-regularizing operator in $\mathcal{L}(H^s(\mathtt{M}^n), H^{s+\rho}(\mathtt{M}^n))$ with ρ arbitrary large. It is based on a normal form procedure developed in [BGMR2]. The crucial fact is that we can write $\sqrt{\Delta_g} = K_0 - Q$ where Q is a pseudo-differential operator of order -1 chosen (following [Col79]) in such a way the spectrum of K_0 is included in $\mathbb{N}+\lambda$ for some constant $\lambda \in \mathbb{R}^+$. This property makes the K_0 flow periodic and motivates us to use it to average the original Schrödinger operator : if A is a pseudo-differential operator then its average with respect to the flow of K_0 is given by $\langle A \rangle = \int_0^{2\pi} e^{-\mathrm{i}\tau K_0} A e^{\mathrm{i}\tau K_0} d\tau$. This idea was already used in a pioneering work of Weinstein [Wein77].

Let us sketch the procedure. Let us write $H = H_0 + V(t)$ where $H_0 = \Delta_g$ and $V(t) = \varepsilon W(\omega t)$ is a pseudo-differential operator of order δ (for this averaging procedure we do not need to assume that V depends quasi-periodically on time neither than V is small). We conjugate the flow of H by a Lie transform $e^{\mathrm{i}X(t)}$ where $X(t)$ is a pseudo-differential operator of order $\delta - 1$: if $\mathrm{i}\dot{u} = H(t)u$, in the new variable $v = e^{\mathrm{i}X(t)}u$ we get $\mathrm{i}\dot{v} = H^+(t)v$ with (see subsection 3.2.4)

$$H^+ = H_0 + \mathrm{i}[X, H_0] + V + \mathrm{i}[X, V] - \frac{1}{2}[X, [X, H_0]] + R\,.$$

Using the pseudo-differential calculus we get that $\mathrm{i}[X, V] - \frac{1}{2}[X, [X, H_0]]$ is of order $2\delta - 2$ and that the remainder term R is still of higher order. So if we are able to solve the following homological equation (and we show in Lemma 3.3.2 how to do it)

$$\mathrm{i}[X, H_0] + V = \langle V \rangle + \text{ order } \delta - 1\,,$$

we conclude that $H^+ = H_0 + \langle V(t) \rangle + $ order $\delta - \nu$ with $\nu = \min(1, 2 - \delta)$. Thus if $\delta < 2$ we have a better equation.

In [BGMR2] such a procedure was iterated to obtain an equivalent equation like (3.1.10) but with Z still depending on time and this was used to prove that we can control the Sobolev norms of the solutions of (3.1.1) as follows [3]

$$\forall s > n/2\,,\ \forall \nu > 0\,,\ \exists\, C_{s,\nu} \text{ such that } \|u(t)\|_{H^s} \leq C_{s,\nu}(1 + t)^\nu\,.$$

In this paper, we want more : we want to eliminate totally the time in order to obtain (3.1.6). So we alternate the averaging procedure with a time elimination procedure based on the use of the operator (3.3.27) which solves the homological equation (3.3.28) and thus the Lie transform $\Phi_T = e^{\mathrm{i}T}$ will kill the dependence on time in $Z = \langle V \rangle$ (see Lemma 3.3.4). This time elimination procedure requires a non resonance hypothesis on the frequency vector ω (see (3.3.2)) and requires $\delta < 1$.

Throughout section 3.3 we work at the pseudo-differential level and the main difficulty is to precisely control the flow generated by pseudo-differential operator of positive order (see Appendix 3.6.3 and in particular hypothesis (3.6.13)). We notice that all this section holds true upon the hypothesis $\delta < 1$.

In the KAM step we kill the remainder term R in (3.1.10) which still depends on time but is now a regularizing operator. As in [BBHM18] (see also [Mon19] and [BLM19]) we use a reducibility scheme where the regularizing property of the perturbation compensates the bad non resonance estimates satisfied by the eigenvalues of $\Delta_g + \varepsilon Z$ (see (3.4.13)). The condition $\delta \leq 1/2$ is used to ensure that condition (3.4.13) is preserved during the KAM iteration as long as a small part of the parameters ω are excised (see Lemma (3.4.3) where

3. Notice that this result holds true for any pseudo-differential perturbation of order $\delta < 2$ depending smoothly on time.

$\kappa = 1 - 2\delta$). This constraint in the KAM procedure was not necessary in [BLM19] (they obtain the reducibility for perturbation of order $1 - \mathtt{e}$ for any $\mathtt{e} > 0$ when the transport operator is of order 1) essentially because the unperturbed system is integrable : in the context of the transport equation on $\mathbb{T}^n$, $H_0 = \nu \cdot \nabla$ with $\nu \in \mathbb{R}^n$ and thus H_0 commutes with ∂_m, $m = 1, \cdots, n$ and the same is true for $H_0 + \epsilon Z$ obtained after the regularization procedure. So Z is not depending on x and $H_0 + \epsilon Z$ is still diagonal in Fourier variables. Thus the perturbed eigenvalues have the form,

$$\lambda_j = \lambda_j^{(0)} + z(j) + \text{remainder}\,, \tag{3.1.11}$$

where z is the symbol of Z (see formula (4.13) in [BLM19]). In our case we just know that Z commutes with Δ_g and thus we can just prove that the spectrum of $\Delta_g + V$ preserves the cluster structure inherited from Δ_g on a Zoll manifold. That means that, once written in the laplacian diagonalization basis, the matrix of Z is block-diagonal but not diagonal as in [BLM19]. By the way throughout section 3.4 we work at the matrix level.
As usual the homological equation (3.4.16) is solved blockwise and it is well known that the increasing size of the blocks may generate loss of regularity. In [EK10] Eliasson-Kuksin used geometrical arguments (related to a Bourgain's Lemma, see Lemma 8.1 in [Bou99]) to control the size of the blocks, in [GP16] or [FG19] authors used a different argument introduced by Delort-Szeftel in [DS04] (see Lemma 4.3 in [GP16]). In this paper, as a consequence of the regularization step, we can solve the homological equation with loss of regularity and thus this step is simplified.
On the other hand the KAM procedure of [BBHM18] requires a tame property to deal with product of matrices. This motivates the definition of the space $\mathcal{M}_s$ of matrices with s-decay norm (see Definition 3.2.8) which was first introduced in [BCP15] (see also [BP11]). The tame property for the s-decay norm is stated in Lemma 3.2.11. It is crucial to obtain (3.4.38) and (3.4.39) which express the control of the new remainder R_+ after one KAM step in two different norms, a low s-decay norm and a high $s + \mathtt{b}$-decay norm. The parameter N measures the troncature in the Fourier variable associate to the angle $\varphi = \omega t$ and in the off-diagonal distance in the matrix (see (3.4.20)). When iterating the procedure, this special form of estimates (3.4.38)-(3.4.39) allows to obtain a convergent scheme for the sequence of remainders R_k when choosing conveniently the sequence of troncature parameter N_k.

Section 3.3 and section 3.4 are independent and in fact are at different levels : while all section 3.3 takes place in the context of pseudo-differential operators, all section 3.4 takes place at matrix level. In section 3.5 we merge the two sections and for that we need the Lemma 3.2.14 which makes the link between ρ-smoothing operators and β- regularizing matrices.

3.2 Functional setting

In this section we introduce the space of functions, sequences, linear operators and pseudo differential operators we shall use along the paper.

3.2.1 Spectral decomposition

Following Theorem 1 of Colin de Verdière [Col79], we introduce Q the pseudo-differential operator of order -1, commuting with Δ_g such that, setting

$$K_0 := \sqrt{\Delta_g} + Q\,, \tag{3.2.1}$$

we have $\text{spec}(K_0) \subset \mathbb{N} + \lambda$ for some constant $\lambda \in \mathbb{R}^+$. We notice that our original Schrödinger operator $H(t) := \Delta_g + \varepsilon W(\omega t)$ reads

$$H(t) = \Delta_g + \varepsilon W(\omega t) = K_0^2 + Q_0 + \varepsilon W(\omega t) \tag{3.2.2}$$

where $Q_0 = -2Q\sqrt{\Delta_g} - Q^2$ is a pseudo differential operators of order 0.
Let us denote by λ_k the eigenvalue of K_0 and by E_k be the eigenspace associated to λ_k. We have

$$\lambda_k \sim k\,, \qquad \text{dim} E_k := d_k \leq k^{n-1}\,. \tag{3.2.3}$$

We denote by

$$\Phi_{[k]}(x) := \{\Phi_{k,m}(x)\,, m = 1, \dots, d_k\} \tag{3.2.4}$$

an orthonormal basis of E_k. By formula (3.2.1) we also deduce that $\Delta_g := K_0^2 + Q_0$ where Q_0 is a pseudo differential operator commuting both with the Laplacian Δ_g and K_0. For this reason K_0 and Δ_g diagonalize simultaneously, hence

$$\Delta_g \Phi_{k,j} = \Lambda_{k,j} \Phi_{k,j}\,, \qquad k \in \mathbb{N}\,, \quad j = 1, \dots, d_k\,, \tag{3.2.5}$$

with

$$\Lambda_{k,j} = \lambda_k^2 + \eta_{k,j}\,, \qquad |\eta_{k,j}| \lesssim 1\,.$$

In particular there exists $c_0 > 0$ such that

$$\Lambda_{k,j} \geq c_0 k^2\,, \qquad |\Lambda_{k,j} - \Lambda_{k',j'}| \geq c_0(k+k')\,, \qquad \forall\, k \neq k'\,, \tag{3.2.6}$$

and for any $j = 1, \dots, d_k$, $j' = 1, \dots, d_{k'}$.

3.2.2 Space of functions and sequences

Using the spectral decomposition of the space $L^2(\mathtt{M}^n) = \oplus_{k\in\mathbb{N}} E_k$, any function $u \in L^2(\mathtt{M}^n)$ can be written as

$$\begin{aligned} u(x) &= \sum_{k\in\mathbb{N}} \sum_{m=1}^{d_k} z_{k,m} \Phi_{k,m}(x) = \sum_{k\in\mathbb{N}} z_{[k]} \cdot \Phi_{[k]}(x)\,, \\ z_{[k]} &= (z_{k,1}, \cdots, z_{k,d_k}) \in \mathbb{C}^{d_k}\,, \end{aligned} \tag{3.2.7}$$

where $''\cdot''$ denotes the usual scalar product in $\mathbb{R}^{d_k}$. We denote by Π_{E_k} the L^2-projector on the eigenspace E_k, i.e., for any $k \in \mathbb{N}$,

$$(\Pi_{E_k} u)(x) = z_{[k]} \cdot \Phi_{[k]}(x) \qquad \Rightarrow \qquad (\sqrt{-\Delta} + Q)\Pi_{E_k} u = \lambda_k \Pi_{E_k} u\,. \tag{3.2.8}$$

For $s \geq 0$, we define the (Sobolev) scale of Hilbert sequence spaces

$$h_s := \Big\{ z = \{z_{[k]}\}_{k\in\mathbb{N}}\,, z_{[k]} \in \mathbb{C}^{d_k} \;:\; \|z\|_{h_s}^2 := \sum_{k\in\mathbb{N}} \langle k\rangle^{2s} \|z_{[k]}\|^2 < +\infty \Big\}\,, \tag{3.2.9}$$

where $\langle k\rangle := \sqrt{1+|k|^2}$ and $\|\cdot\|$ denotes the $L^2(\mathbb{C}^{d_k})$-norm. By a slight abuse of notation we define the operator Π_{E_k} on sequences as $\Pi_{E_k} z = z_{[k]}$ for any $z \in h^s$ and $k \in \mathbb{N}$.
We notice that the weight $\langle k\rangle$ we use in the norm in (3.2.9) is related to the eigenvalues of K_0, indeed

$$c|k| \leq \lambda_k \leq C|k| \tag{3.2.10}$$

for some suitable constants $0 < c \leq C$.
As a consequence the space

$$H^s = H^s(\mathtt{M}^n) := \{u(x) = \sum_{k\in\mathbb{N}} z_{[k]} \cdot \Phi_{[k]}(x) \mid z \in h^s\}\,, \tag{3.2.11}$$

is the usual Sobolev space $H^s = \mathrm{dom}((K_0)^s) = \mathrm{dom}(\sqrt{1+\Delta_g})^s$ and $\|u\|_{H^s} := \|z\|_{h_s}$ is equivalent to the standard Sobolev norm $\|u\|_{H^s} \sim \|K_0^s u\|_{L^2(\mathtt{M}^n)}$. Along the paper we shall write $\|\cdot\|_{H^s}$ instead of $\|\cdot\|_{h_s}$. Given $s, s' \in \mathbb{R}$ we denote by $\mathcal{L}(H^s, H^{s'})$ the space of linear bounded operators form H^s to $H^{s'}$ endowed with the standard operator norm $\|\cdot\|_{\mathcal{L}(H^s,H^{s'})}$.

In the paper we shall also deal with quasi periodic in time functions $\mathbb{R}\times\mathtt{M}^n \ni (t,x) \mapsto u(\omega t, x)$ where $\omega \in \mathbb{R}^d$ is a frequency vector and u is periodic in its first variable. To this end we introduce the space $H^r(\mathbb{T}^d; H^s(\mathtt{M}^n))$ defined as the set of functions $u : \mathbb{T}^d \ni \varphi \mapsto H^s(\mathtt{M}^n)$ which are Sobolev in $\varphi \in \mathbb{T}^d$ with values in $H^s(\mathtt{M}^n)$.

Functions in $H^r(\mathbb{T}^d; H^s(\mathtt{M}^n))$ can be expanded, using the standard Fourier theory, as

$$u(\varphi, x) = \sum_{\ell\in\mathbb{Z}^d, k\in\mathbb{N}} z_{[k]}(l)\cdot\Phi_{[k]}(x)e^{\mathrm{i}l\cdot\varphi}\,, \qquad z_{[k]}(l) \in \mathbb{C}^{d_k}\,, \tag{3.2.12}$$

where $e^{\mathrm{i}l\cdot\varphi}\Phi_{k,m}(x)$, $l \in \mathbb{Z}^d$, $k \in \mathbb{N}$, $m = 1, \ldots, d_k$, is an orthogonal basis of $L^2(\mathbb{T}^d \times \mathtt{M}^n; \mathbb{C})$. We define space of sequence (recall (3.2.9))

$$h_{s,r} := \Big\{ z = \{z_{[k]}(l)\}_{l\in\mathbb{Z}^d, k\in\mathbb{N}}\,,\ z_{[k]} \in \mathbb{C}^{d_k} : \ \|z\|_{h_{s,r}}^2 := \sum_{l\in\mathbb{Z}^d} \langle l\rangle^{2r} \|z(l)\|_{h_s}^2 < +\infty \Big\}\,. \tag{3.2.13}$$

Along the paper we shall also consider the space, for $p \in \mathbb{N}$ with $p > (d+n)/2$,

$$\ell_p := \bigcap_{\substack{r>d/2, s>n/2\\ s+r=p}} h_{s,r}\,. \tag{3.2.14}$$

We endow the space ℓ_p with the norm

$$\|z\|_{\ell_p}^2 := \sum_{l\in\mathbb{Z}^d, k\in\mathbb{N}} \langle l, k\rangle^{2p} \|z_{[k]}(l)\|^2\,. \tag{3.2.15}$$

Lipschitz norm. Consider a compact subset $\mathcal{O}$ of $\mathbb{R}^d$, $d \geq 1$. For functions $f : \mathcal{O} \to E$, with $(E, \|\cdot\|_E)$ some Banach space, we define the sup norm and the lipschitz semi-norm as

$$\begin{aligned} \|f\|_E^{sup} &:= \|f\|_E^{sup,\mathcal{O}} := \sup_{\omega\in\mathcal{O}} \|f(\omega)\|_E\,, \\ \|f\|_E^{lip} &:= \|f\|_E^{lip,\mathcal{O}} := \sup_{\substack{\omega_1,\omega_2\in\mathcal{O}\\ \omega_1\neq\omega_2}} \frac{\|f(\omega_1)-f(\omega_2)\|_E}{|\omega_1-\omega_2|}\,. \end{aligned} \tag{3.2.16}$$

For any $\gamma > 0$ we introduce the weighted Lipschitz norms

$$\|f\|_E^{\gamma,\mathcal{O}} := \|f\|_E^{sup,\mathcal{O}} + \gamma\|f\|_E^{lip,\mathcal{O}}\,. \tag{3.2.17}$$

We finally define the space of sequences

$$h_{s,r}^{\gamma,\mathcal{O}} := \left\{ \mathcal{O} \ni \omega \mapsto z(\omega) \in h_{s,r} \ : \ \|z\|_{h_{s,r}}^{\gamma,\mathcal{O}} < +\infty \right\}, \tag{3.2.18}$$

and consequently the space (recall (3.2.14))

$$\ell_p^{\gamma,\mathcal{O}} := \bigcap_{s+r=p} h_{s,r}^{\gamma,\mathcal{O}}\,, \tag{3.2.19}$$

endowed with the norm

$$\|z\|_{\ell_p}^{\gamma,\mathcal{O}} := \|z\|_{\ell_p}^{sup,\mathcal{O}} + \gamma\|z\|_{\ell_p}^{lip,\mathcal{O}}\,. \tag{3.2.20}$$

Notation. We shall use the notation $A \lesssim B$ to denote $A \leq CB$ where C is a positive constant depending on parameters fixed once for all : d, n, δ. We shall use the notation $A \leq_s B$ to denote $A \leq C(s)B$ where $C(s) > 0$ is a constant depending also on s.

3.2.3 Pseudo-differential operators

In this paper we consider operators which are *pseudo-differential*. Here we recall some fundamental properties of operators in $\mathcal{A}_m$ which are collected in [BGMR2]. First $\mathcal{A}_m$ is a Fréchet space for a family of filtering semi-norms $\{\mathcal{N}_{m,p}\}_{p\geq 1}$ such that the embedding (recall (3.2.11)) $\mathcal{A}_m \hookrightarrow \bigcap_{s\in\mathbb{R}} \mathcal{L}(H^s, H^{s-m})$ is continuous. We can also choose the semi-norms in an increasing way, i.e. $\mathcal{N}_{m,p}(A) \leq \mathcal{N}_{m,p+1}(A)$ for $p \geq 1$ and $A \in \mathcal{A}_m$. To state the other properties we need to introduce the following definition.

Definition 3.2.1. Let $S \in \mathcal{L}(H^0)$. We say that S is ρ-smoothing, and we will write $S \in \mathcal{R}_\rho$, if S can be extended to an operator in $\mathcal{L}(H^s, H^{s+\rho})$ for any $s \in \mathbb{R}$. When this is true for every $\rho \geq 0$, we say that S is a smoothing operator.

Then we have the following properties concerning the class $\mathcal{A}_m$ equipped with the semi-norms $\{\mathcal{N}_{m,p}\}_{p\geq 1}$:

(i) let $A \in \mathcal{A}_m$, for any $s \in \mathbb{R}$ there exist constants $C = C(m, s) > 0$, $p = p(m, s) \geq 1$ which are increasing functions[4] of s such that

$$\|A\|_{\mathcal{L}(H^s,H^{s-m})} \leq C\mathcal{N}_{m,p}(A)\,. \tag{3.2.21}$$

4. This fact is quite evident in the case of pseudo-differential operators on $\mathbb{R}^n$ and thus extends to pseudo-differential operators on $\mathtt{M}^n$ by passing to local charts.

(ii) Let $A \in \mathcal{A}_m$, $B \in \mathcal{A}_n$ then $AB \in \mathcal{A}_{m+n}$. Furthermore for any $\rho \geq 0$ there exists S a ρ-smoothing operator such that for any $p \geq 1$ for any $s \in \mathbb{R}$ there are constants $C = C(m, n, p, s, \rho) > 0$, $q = q(m, n, p, s, \rho) \geq p$ such that

$$\mathcal{N}_{m+n,p}(AB - S) \leq C\mathcal{N}_{m,q}(A)\mathcal{N}_{n,q}(B)\,, \tag{3.2.22}$$

$$\|S\|_{\mathcal{L}(H^s,H^{s+\rho})} \leq C\mathcal{N}_{m,q}(A)\mathcal{N}_{n,q}(B)\,. \tag{3.2.23}$$

(iii) Let $A \in \mathcal{A}_m$, $B \in \mathcal{A}_n$ then $[A, B] \in \mathcal{A}_{m+n-1}$. Furthermore for any $\rho \geq 0$ there exists S a ρ-smoothing operator such that for any $p \geq 1$ for any $s \in \mathbb{R}$ there are constants $C = C(m, n, p, s, \rho) > 0$, $q = q(m, n, p, s, \rho) \geq p$ such that

$$\mathcal{N}_{m+n-1,p}([A, B] - S) \leq C\mathcal{N}_{m,q}(A)\mathcal{N}_{n,q}(B)\,, \tag{3.2.24}$$

$$\|S\|_{\mathcal{L}(H^s,H^{s+\rho})} \leq C\mathcal{N}_{m,q}(A)\mathcal{N}_{n,q}(B)\,. \tag{3.2.25}$$

(iv) The map $\tau \to A(\tau) := e^{-\mathrm{i}\tau K_0} A e^{\mathrm{i}\tau K_0} \in C^0_b(\mathbb{R}, \mathcal{A}_m)$. Furthermore for any $\rho \geq 0$ there exists S a ρ-smoothing operator such that for any $p \geq 1$ for any $s \in \mathbb{R}$ there are constants $C = C(m, n, p, s, \rho) > 0$, $q = q(m, n, p, s, \rho) \geq p$ such that

$$\mathcal{N}_{m+n-1,p}(e^{-\mathrm{i}\tau K_0} A e^{\mathrm{i}\tau K_0} - S) \leq C\mathcal{N}_{m,q}(A)\,, \tag{3.2.26}$$

$$\|S\|_{\mathcal{L}(H^s,H^{s+\rho})} \leq C\mathcal{N}_{m,q}(A)\,. \tag{3.2.27}$$

Remark 3.2.2. In (ii), (iii) and (iv) the smoothing correction does not play an important role since it can be chosen as regularizing as one want. In the KAM scheme the level of regularization will be fix once for all. Thus, by a slight abuse of notation, we will often omit in the following the smoothing correction and will just write

$$\mathcal{N}_{m+n,p}(AB) \leq C\mathcal{N}_{m,q}(A)\mathcal{N}_{n,q}(B)\,, \tag{3.2.28}$$

$$\mathcal{N}_{m+n-1,p}([A, B]) \leq C\mathcal{N}_{m,q}(A)\mathcal{N}_{n,q}(B)\,. \tag{3.2.29}$$

We shall also consider H^r-mappings $\mathbb{T}^d \ni \varphi \mapsto A(\varphi)$ with $A(\varphi)$ a symmetric pseudo-differential operators of order m in $\mathcal{A}_m$. We can then decompose A in Fourier writing

$$A(\varphi) = \sum_{l \in \mathbb{Z}^d} A(l) e^{\mathrm{i} l \cdot \varphi} \tag{3.2.30}$$

with $A(l)$ a pseudo-differential operators of order m in $\mathcal{A}_m$. We give the following definition.

Definition 3.2.3. Let $m \in \mathbb{R}$, $r > d/2$. We denote by $\mathcal{A}_{m,s}$ the Fréchet space of mapping $\mathbb{T}^d \ni \varphi \mapsto A = A(\varphi) \in \mathcal{A}_m$ that are H^r on $\mathbb{T}^d$. We endow $\mathcal{A}_{m,r}$ with the family of semi-norms

$$\Big(\mathcal{N}_{m,r,p}(A)\Big)^2 := \sum_{l\in\mathbb{Z}^d} \langle l\rangle^{2r} \mathcal{N}^2_{m,p}(A(l)) , \quad p \geq 1 . \tag{3.2.31}$$

Consider a Lipschitz family $\mathcal{O} \ni \omega \mapsto A(\omega) \in \mathcal{A}_{m,r}$ where $\mathcal{O}$ is a compact subset of $\mathbb{R}^d$, $d \geq 1$. For $\gamma > 0$ we define the Lipschitz semi-norms (recall (3.2.16)) as

$$\mathcal{N}^{\gamma,\mathcal{O}}_{m,r,p}(A) := \mathcal{N}^{sup,\mathcal{O}}_{m,r,p}(A) + \gamma \mathcal{N}^{lip,\mathcal{O}}_{m,r,p}(A) \tag{3.2.32}$$

We denote by $\mathcal{A}^{\gamma,\mathcal{O}}_{m,r}$ the Fréchet space of families of pseudo differential operators $A(\omega) \in \mathcal{A}_{m,r}$ endowed with with the family of semi-norms $\{\mathcal{N}^{\gamma,\mathcal{O}}_{m,r,p}\}_{p\geq1}$.

Similarly we define the corresponding class of ρ-smoothing operators $R(\omega,\varphi)$, H^r in φ and Lipschitz in ω.

Definition 3.2.4. Let $\rho \in \mathbb{R}$ and $r > d/2$. We denote by $\mathcal{R}_{\rho,r}$ the Fréchet space of ρ-smoothing H^r-mapping $\mathbb{T}^d \ni \varphi \mapsto R(\varphi) \in \mathcal{L}(H^s, H^{s+\rho})$ for all $s \in \mathbb{R}$ endowed with the family of semi-norms

$$|R|^2_{\rho,r,s} := \sum_{l\in\mathbb{Z}^d} \langle l\rangle^{2r} \|R(l)\|^2_{\mathcal{L}(H^s,H^{s+\rho})} , \quad s \in \mathbb{R} . \tag{3.2.33}$$

Consider a family $\mathcal{O} \ni \omega \mapsto R(\omega) \in \mathcal{R}_{\rho,r}$ where $\mathcal{O}$ is a compact subset of $\mathbb{R}^d$, $d \geq 1$. For $\gamma > 0$ we denote by $\mathcal{R}^{\gamma,\mathcal{O}}_{\rho,r}$ the Fréchet space of families of pseudo differential operators $R(\omega) \in \mathcal{R}_{\rho,r}$ endowed with with the family of semi-norms $\{\mathcal{N}^{\gamma,\mathcal{O}}_{\rho,r,p}\}_{p\in\mathbb{N}}$ defined by (recall (3.2.16))

$$|R|^{\gamma,\mathcal{O}}_{\rho,r,p} := |R|^{sup,\mathcal{O}}_{\rho,r,p} + \gamma |R|^{lip,\mathcal{O}}_{\rho,r,p} . \tag{3.2.34}$$

We notice that by (3.2.21) we have $\mathcal{A}_{m,r} \subset \mathcal{R}_{-m,r}$.

Lemma 3.2.5. *Let $r > d/2$, $m, \rho \in \mathbb{R}$ and consider $R \in \mathcal{R}^{\gamma,\mathcal{O}}_{\rho,r}$ and $A \in \mathcal{A}^{\gamma,\mathcal{O}}_{m,r}$. Then, for any $s \in \mathbb{R}$, there are $C = C(s,r) > 0$, $p(s,m) > 0$ such that*

$$\|Ah\|_{h^{\gamma,\mathcal{O}}_{s-m,r}} \leq C \mathcal{N}^{\gamma,\mathcal{O}}_{m,r,p}(A) \|h\|_{h^{\gamma,\mathcal{O}}_{s,r}} , \tag{3.2.35}$$

$$\|Rh\|_{h^{\gamma,\mathcal{O}}_{s+\rho,r}} \leq C |R|^{\gamma,\mathcal{O}}_{\rho,r,s} \|h\|_{h^{\gamma,\mathcal{O}}_{s,r}} , \tag{3.2.36}$$

for any $h \in h_{s,r}^{\gamma,\mathcal{O}}$.

Proof. We start by proving the (3.2.36) for the norm $\|\cdot\|_{h_{s+\rho,r}}$. Recalling (3.2.13) we have

$$
\begin{aligned}
\|Rh\|_{h_{s+\rho,r}} &\le \sum_{l\in\mathbb{Z}^d} \langle l\rangle^{2r}\Big(\sum_{l'\in\mathbb{Z}^d} \|R(l-l')h(l')\|_{H^{s+\rho}}\Big)^2 \\
&\le \sum_{l\in\mathbb{Z}^d} \langle l\rangle^{2r}\Big(\sum_{l'\in\mathbb{Z}^d} \|R(l-l')\|_{\mathcal{L}(H^s;H^{s+\rho})}\|h(l')\|_{H^s}\Big)^2 \\
&\le \sum_{l\in\mathbb{Z}^d}\Big(\sum_{|l'|>\frac{1}{2}|l|} \langle l-l'\rangle^r\|R(l-l')\|_{\mathcal{L}(H^s;H^{s+\rho})}\langle l'\rangle^r\|h(l')\|_{H^s}\frac{\langle l\rangle^r}{\langle l'\rangle^r}\Big)^2 \\
&+ \sum_{l\in\mathbb{Z}^d}\Big(\sum_{|l'|\le\frac{1}{2}|l|} \langle l-l'\rangle^r\|R(l-l')\|_{\mathcal{L}(H^s;H^{s+\rho})}\langle l'\rangle^r\|h(l')\|_{H^s}\frac{\langle l\rangle^r}{\langle l-l'\rangle^r}\Big)^2 .
\end{aligned}
$$

Hence, by using the Cauchy-Schwartz inequality, we get

$$
\|Rh\|_{h_{s+\rho,r}} \le C\sum_{l,l'\in\mathbb{Z}^d}\langle l-l'\rangle^{2r}\|R(l-l')\|^2_{\mathcal{L}(H^s;H^{s+\rho})}\langle l'\rangle^{2r}\|h(l')\|^2_{H^s} \le C\|h\|^2_{h_{s,r}}|R|^2_{\rho,r,s}\,,
$$

which implies the (3.2.36) for the norm $\|\cdot\|_{h_{s+\rho,r}}$. The Lipschitz bound on the norm $\|\cdot\|^{\gamma,\mathcal{O}}_{h_{s+\rho,r}}$ and the (3.2.35) follows similarly. □

In the following Lemma we state some properties and estimates [5] that will be proved in Appendix 3.6.1.

Lemma 3.2.6. *Let* A, B *are pseudo-differential operators in* $\mathcal{A}^{\gamma,\mathcal{O}}_{m,r}$ *and* $\mathcal{A}^{\gamma,\mathcal{O}}_{n,r}$. *For any* $p\ge 1$ *there exist constants* $C=C(r,m,n,p)$ *and* $q=q(r,m,n,p)$ *which are increasing in* p *such that*
(i) $AB, BA \in \mathcal{A}^{\gamma,\mathcal{O}}_{m+n,r}$ *and*

$$
\mathcal{N}^{\gamma,\mathcal{O}}_{m+n,r,p}(AB)\,,\ \mathcal{N}^{\gamma,\mathcal{O}}_{m+n,r,p}(BA) \le C\mathcal{N}^{\gamma,\mathcal{O}}_{m,r,q}(A)\mathcal{N}^{\gamma,\mathcal{O}}_{n,r,q}(B)\,. \tag{3.2.37}
$$

(ii) $[A,B]\in\mathcal{A}^{\gamma,\mathcal{O}}_{m+n-1,r}$ *and*

$$
\mathcal{N}^{\gamma,\mathcal{O}}_{m+n-1,r,p}([A,B]) \le C\mathcal{N}^{\gamma,\mathcal{O}}_{m,r,q}(A)\mathcal{N}^{\gamma,\mathcal{O}}_{n,r,q}(B)\,. \tag{3.2.38}
$$

5. Estimates (3.2.37), (3.2.38) and (3.2.42) are written taking into account Remark 3.2.2. For instance (3.2.38) should be interpreted as : for any $\rho\ge 0$ there exists S a ρ-smoothing operator such that for any $p\ge 1$ for any $s\in\mathbb{R}$ there are constants $C=C(m,n,p,r,\rho)>0$, $q=q(m,n,p,r,\rho)\ge 1$ such that

$$
\mathcal{N}^{\gamma,\mathcal{O}}_{m+n-1,r,p}([A,B]-S)\le C\mathcal{N}^{\gamma,\mathcal{O}}_{m,r,q}(A)\mathcal{N}^{\gamma,\mathcal{O}}_{n,r,q}(B)\,,\quad |S|^{\gamma,\mathcal{O}}_{\rho,s,p}\le C\mathcal{N}^{\gamma,\mathcal{O}}_{m,r,q}(A)\mathcal{N}^{\gamma,\mathcal{O}}_{n,r,q}(B)\,.
$$

(iii) Let $\omega \in \mathbb{R}^d$*, then* $\omega \cdot \partial_\varphi A \in \mathcal{A}_{m,r-1}$ *and*

$$\mathcal{N}^{\gamma,\mathcal{O}}_{m,r-1,p}(\omega \cdot \partial_\varphi A) \le C\mathcal{N}^{\gamma,\mathcal{O}}_{m,r,p}(A)\,. \tag{3.2.39}$$

If furthermore ω *satisfies, for some* $\alpha > d-1$,

$$|\omega \cdot l| > \gamma |l|^{-\alpha}\,, \qquad \forall\, l \in \mathbb{Z}^d \setminus \{0\}\,, \tag{3.2.40}$$

and $r - 2\alpha - 1 > d/2$ *then* $(\omega \cdot \partial_\varphi)^{-1}A \in \mathcal{A}_{m,r-(2\alpha+1)}$ *and*

$$\mathcal{N}^{\gamma,\mathcal{O}}_{m,r-(2\alpha+1),p}((\omega\cdot\partial_\varphi)^{-1}A) \le C\gamma^{-1}\mathcal{N}^{\gamma,\mathcal{O}}_{m,r,p}(A)\,. \tag{3.2.41}$$

(iv) For any $\tau \in [0, 2\pi]$ *we have* $e^{-\mathrm{i}\tau K_0} A e^{\mathrm{i}\tau K_0} \in \mathcal{A}^{\gamma,\mathcal{O}}_{m,r}$ *and*

$$\mathcal{N}^{\gamma,\mathcal{O}}_{m,r,p}(e^{-\mathrm{i}\tau K_0} A e^{\mathrm{i}\tau K_0}) \le C\mathcal{N}^{\gamma,\mathcal{O}}_{m,r,q}(A)\,. \tag{3.2.42}$$

3.2.4 Conjugation rules

Let $\omega \cdot \partial_\varphi$ be the diagonal operator acting on sequences $z \in \ell_{s,r}$ (see (3.2.13)) defined by

$$\omega \cdot \partial_\varphi z := \mathrm{diag}_{l\in\mathbb{Z}^d, k\in\mathbb{N}}(\mathrm{i}\omega \cdot l) z = (\mathrm{i}\omega \cdot l z_{[k]}(l))_{l\in\mathbb{Z}^d, k\in\mathbb{N}}\,. \tag{3.2.43}$$

Consider an operator of the form

$$L := L(\varphi, \omega) := \omega \cdot \partial_\varphi + \mathrm{i}M(\varphi)\,, \tag{3.2.44}$$

where $M(\varphi)$ is some map $\mathbb{T}^d \ni \varphi \mapsto M = M(\varphi) \in \mathcal{L}(H^s; H^{s+m})$, for some $m \in \mathbb{R}$. We shall study how the operator L in (3.2.44) conjugates under the map Φ_S defined as

$$\Phi_S := (\Phi_S^\tau)_{|\tau=1}\,, \qquad \Phi_S^\tau := e^{\mathrm{i}\tau S} = \sum_{p=0}^{\infty} \frac{1}{p!}(\mathrm{i}\tau S)^p\,, \tag{3.2.45}$$

where $S(\varphi)$ is some map $\mathbb{T}^d \ni \varphi \mapsto S = S(\varphi) \in \mathcal{L}(H^s; H^{s+m'})$, for some $m' \in \mathbb{R}$. For the well-posedness of a map of the form (3.2.45) we refer to Lemma 3.6.6 in Appendix 3.6.3.

By using the Lie series expansions we have

$$L^+ = L^+(\varphi) := \Phi_S \circ L \circ \Phi_S^{-1} = \omega \cdot \partial_\varphi + \mathrm{i}M^+(\varphi)\,, \tag{3.2.46}$$

where $M^+(\varphi) = M_1^+(\varphi) + M_2^+(\varphi)$ with, for any $q \in \mathbb{N}$,

$$\mathrm{i}M_1^+(\varphi) := \Phi_S \circ \mathrm{i}M \circ \Phi_S^{-1} = \mathrm{i}M + \sum_{p=1}^{q} \frac{1}{p!}\mathrm{ad}_{\mathrm{i}S}^p(\mathrm{i}M) + \frac{1}{q!}\int_0^1 (1-\tau)^q \Phi_S^\tau \mathrm{ad}_{\mathrm{i}S}^{q+1}(\mathrm{i}M)\Phi_S^{-\tau} d\tau\,, \tag{3.2.47}$$

and

$$\begin{aligned}\mathrm{i}M_2^+(\varphi) &:= \Phi_S \circ \omega\cdot\partial_\varphi \circ \Phi_S^{-1} - \omega\cdot\partial_\varphi \\ &= -\mathrm{i}\omega\cdot\partial_\varphi S - \sum_{p=2}^{q}\frac{1}{p!}\mathrm{ad}_{\mathrm{i}S}^{p-1}(\mathrm{i}\omega\cdot\partial_\varphi S) + \frac{1}{q!}\int_0^1 (1-\tau)^q \Phi_S^\tau \mathrm{ad}_{\mathrm{i}S}^q(\mathrm{i}\omega\cdot\partial_\varphi S)\Phi_S^{-\tau} d\tau\,,\end{aligned} \tag{3.2.48}$$

where we defined $\mathrm{ad}_S^0(M) = M$ and

$$\mathrm{ad}_S^p(M) = \mathrm{ad}_S^{p-1}([S,M])\,, \quad [S,M] = SM - MS\,. \tag{3.2.49}$$

Remark 3.2.7. (Hamiltonian structure) We remark that, if the operator S in and M are *Hermitian*, then by Lemma 2.9 in [FG19], we have that also the operator M^+ in (3.2.46) is Hermitian.

3.2.5 Linear operators and matrices.

According to the orthogonal splitting

$$L^2(\mathtt{M}^n) = \bigoplus_{k\in\mathbb{N}} E_k,$$

we identify a linear operator acting on $L^2(\mathtt{M}^n)$ with its matrix representation $A := \left(A_{[k]}^{[k']}\right)_{k,k'\in\mathbb{N}}$ in $\mathcal{L}(h^0)$ (recall (3.2.9)) with blocks $A_{[k]}^{[k']} \in \mathcal{L}(E_{k'}; E_k)$. Notice that each block $A_{[k]}^{[k']}$ is a $d_k \times d_{k'}$:

$$A_{[k]}^{[k']} := \left(A_{k,j}^{k',j'}\right)_{\substack{j=1,\dots,d_k,\\ j'=1,\dots,d_{k'}}}\,. \tag{3.2.50}$$

The action of the operator A on functions $u(x)$ as in (3.2.7) of the space variable in $L^2(\mathtt{M}^n)$ is given by

$$(Au)(x) = \sum_{k\in\mathbb{N}} (Az)_{[k]}\cdot\Phi_{[k]}(x)\,, \qquad z_{[k]} \in \mathbb{C}^{d_k}\,, \qquad (Az)_{[k]} = \sum_{j\in\mathbb{N}} A_{[k]}^{[j]} z_{[j]}\,. \tag{3.2.51}$$

In this paper we also consider regular φ-dependent families of linear operators

$$\mathbb{T}^d \ni \varphi \mapsto A = A(\varphi) = \sum_{l\in\mathbb{Z}^d} A(l) e^{\mathrm{i} l\cdot\varphi} \tag{3.2.52}$$

where $A(l)$ are linear operators in $\mathcal{L}(H^s, H^{s'})$, for any $l \in \mathbb{Z}^d$. We also regard A as an operator acting on functions $u(\varphi, x)$ of space-time as $(Au)(\varphi, x) = (A(\varphi)u(\varphi, \cdot))(x)$. More precisely, expanding u as in (3.2.12), we have

$$\begin{aligned} (Au)(\varphi, x) &= \sum_{l\in\mathbb{Z}^d, k\in\mathbb{N}} (Az)_{[k]}(l) e^{\mathrm{i} l\cdot\varphi} \Phi_{[k]}(x)\,, \\ (Az)_{[k]}(l) &= \sum_{p\in\mathbb{Z}^d, k'\in N} A_{[k]}^{[k']}(l-p) z_{[k']}(p)\,. \end{aligned} \tag{3.2.53}$$

Relation (3.2.51) shows that, in order to define operators that conserve the H^s regularity in space we need to assume some decay of $\|A_{[k]}^{[k']}\|^2_{\mathcal{L}(L^2)}$ with respect to $|k-k'|$. That the reason for the following definition first introduced in [BP11] for (i) and in [BCP15] for (ii).

Definition 3.2.8. (s-decay norm).
(i) We define the s-decay norm of a matrix $A \in \mathcal{L}(H^s; H^s)$ as

$$|A|_s^2 := \sum_{h\in\mathbb{N}} \langle h\rangle^{2s} \sup_{|k-k'|=h} \|A_{[k]}^{[k']}\|^2_{\mathcal{L}(L^2)} \tag{3.2.54}$$

where $\|\cdot\|_{\mathcal{L}(L^2)}$ is the L^2-operator norm in $\mathcal{L}(E_{k'}, E_k)$.
(ii) Consider a map $\mathbb{T}^d \ni \varphi \mapsto A = A(\varphi) \in \mathcal{L}(H^s; H^s)$. We define its decay norm as

$$[\![A]\!]_s^2 := \sum_{l\in\mathbb{Z}^d, h\in\mathbb{N}} \langle l, h\rangle^{2s} \sup_{|k-k'|=h} \|A_{[k]}^{[k']}(l)\|^2_{\mathcal{L}(L^2)}\,. \tag{3.2.55}$$

We denote by $\mathcal{M}_s$ the space matrices with finite s-decay norm $[\![\cdot]\!]_s$.
(iii) Consider a Lipschitz family $\mathcal{O} \ni \omega \mapsto A(\omega) \in \mathcal{M}_s$ where $\mathcal{O}$ is a compact subset of $\mathbb{R}^d$, $d \geq 1$. For $\gamma > 0$ we define the Lipschitz decay norm as

$$\begin{aligned} [\![A]\!]_s^{\gamma,\mathcal{O}} &:= [\![A]\!]_s^{sup,\mathcal{O}} + \gamma [\![A]\!]_s^{lip,\mathcal{O}} \\ &= \sup_{\omega\in\mathcal{O}} [\![A(\omega)]\!]_s + \gamma \sup_{\substack{\omega_1,\omega_2\in\mathcal{O} \\ \omega_1\neq\omega_2}} \frac{[\![A(\omega_1) - A(\omega_2)]\!]_s}{|\omega_1 - \omega_2|}\,. \end{aligned} \tag{3.2.56}$$

We denote by $\mathcal{M}_s^{\gamma,\mathcal{O}}$ the space of families of Lipschitz mapping $\omega \mapsto A(\omega) \in \mathcal{M}_s$ with finite $|\cdot|_s^{\gamma,\mathcal{O}}$-norm.

Remark 3.2.9. The s-decay norm (3.2.55) link the regularity in space and the regularity in φ (i.e. in time). In fact for s integer we have

$$\mathcal{M}_s = \cap_{p+q\leq s} H^p(\mathbb{T}^d, \mathcal{L}_{dec}(H^q, H^q)) \,,$$

where $\mathcal{L}_{dec}(H^q, H^q)$ are bounded operator from H^q to H^q with finite $|\cdot|_s$-norm (see (3.2.54)).

Remark 3.2.10. Notice that, if the s-decay norm of a matrix A is finite, then

$$\|A_{[k]}^{[k']}\|_{\mathcal{L}(L^2)} \leq C(s) [\![A]\!]_s \langle k - k' \rangle^{-s} \,.$$

We have the following fundamental lemma stating in particular that the s-decay norm is tame (see (3.2.58)). This tame property will be crucial in the KAM procedure.

Lemma 3.2.11. *For any $s > (d+n)/2$ the following holds :*
(i) there is $C = C(s) > 0$ such that (recall (3.2.14),(3.2.15)*)*

$$\|Az\|_{\ell_s} \leq C [\![A]\!]_s \|z\|_{\ell_{s_0}} + C [\![A]\!]_{s_0} \|z\|_{\ell_s} \,, \tag{3.2.57}$$

for any $h \in \ell_s$;
(ii) there is $C = C(s) > 0$ such that

$$[\![AB]\!]_s \leq C [\![A]\!]_s [\![B]\!]_{s_0} + C [\![A]\!]_{s_0} [\![B]\!]_s \,; \tag{3.2.58}$$

(iii) for $N > 0$ we define (recall (3.2.52)*) the matrix $\Pi_N A$ as*

$$(\Pi_N A)_{[k]}^{[k']}(l) := \begin{cases} A_{[k]}^{[k']}(l) \,, \quad l \in \mathbb{Z}^d \,, k, k' \in \mathbb{N} \,, & \begin{array}{l} |l| \leq N \,, \\ |k - k'| \leq N \,, \end{array} \\ 0 \,, \qquad \text{otherwise} & \end{cases} \tag{3.2.59}$$

One has

$$[\![(\mathrm{Id} - \Pi_N) A]\!]_s \leq C N^{-\beta} [\![A]\!]_{s+\beta} \,, \quad \beta \geq 0 \,, \tag{3.2.60}$$

for some $C = C(s) > 0$.

Similar bounds holds also replacing $\|\cdot\|_{\ell_s}$, $[\![\cdot]\!]_s$ with the norms $\|\cdot\|_s^{\gamma,\mathcal{O}}$, $[\![\cdot]\!]_s^{\gamma,\mathcal{O}}$ respectively (see (3.2.20), (3.2.56)*).*

Proof. Items (i) and (ii) follow by lemmata 2.6, 2.7 in [BCP15]. Item (iii) follows by the definition of the norm in (3.2.55). □

We will also need a class of matrices that take into account a notion of regularization.

Definition 3.2.12. Define the diagonal φ-independent operator $\mathcal{D}$, acting on $z \in \ell_s$ (see (3.2.14)), as

$$\mathcal{D}z := \mathrm{diag}_{l\in\mathbb{Z}^d,k\in\mathbb{N}}\Big(\lambda_k\Big)z = \Big(\lambda_k z_{[k]}(l)\Big)_{l\in\mathbb{Z}^d,k\in\mathbb{N}} . \tag{3.2.61}$$

For $\beta \in \mathbb{R}$ we define the norm $[\![\cdot]\!]_{\beta,s}$ of a matrix A in (3.2.52) as

$$[\![A]\!]_{\beta,s} := [\![\mathcal{D}^\beta A]\!]_s + [\![A\mathcal{D}^\beta]\!]_s . \tag{3.2.62}$$

We denote by $\mathcal{M}_{\beta,s}$ the space of maps $\mathbb{T}^d \ni \varphi \mapsto A = A(\varphi) \in \mathcal{L}(L^2)$ with finite $[\![\cdot]\!]_{\beta,s}$-norm.
Consider a family $\mathcal{O} \ni \omega \mapsto A(\omega) \in \mathcal{M}_{\beta,s}$ where $\mathcal{O}$ is a compact subset of $\mathbb{R}^d$, $d \geq 1$. For $\gamma > 0$ we define the Lipschitz norm as

$$\begin{aligned} [\![A]\!]^{\gamma,\mathcal{O}}_{\beta,s} &:= [\![A]\!]^{sup,\mathcal{O}}_{\beta,s} + \gamma[\![A]\!]^{lip,\mathcal{O}}_{\beta,s} \\ &= \sup_{\omega\in\mathcal{O}}[\![A(\omega)]\!]_{\beta,s} + \gamma \sup_{\substack{\omega_1,\omega_2\in\mathcal{O}\\ \omega_1\neq\omega_2}} \frac{[\![A(\omega_1) - A(\omega_2)]\!]_{\beta,s}}{|\omega_1 - \omega_2|} . \end{aligned} \tag{3.2.63}$$

We denote by $\mathcal{M}^{\gamma,\mathcal{O}}_{\beta,s}$ the space of families of matrices $A(\omega)$ with finite $[\![\cdot]\!]^{\gamma,\mathcal{O}}_{\beta,s}$-norm.

For properties of matrices in $\mathcal{M}^{\gamma,\mathcal{O}}_{\beta,s}$ we refer to Appendix 3.6.2 and in particular Lemma 3.6.4 stating a tame property for the norm given by (3.2.63).
We end this section with the following definition :

Definition 3.2.13. (Block-diagonal matrices). We say that $A(\varphi)$ is *block-diagonal* if and only if $A^{[k']}_{[k]}(\varphi) = 0$ for any $k \neq k'$ and any $\varphi \in \mathbb{T}^d$.

We notice that operators commuting with K_0 have matrices that are block-diagonal : let Z be such that

$$[K_0, Z] = 0 . \tag{3.2.64}$$

Since

$$[H_0, Z]^{[k']}_{[k]} = (\lambda_{k'} - \lambda_k)Z^{[k']}_{[k]} \quad \forall k, k' ,$$

condition (3.2.64) implies that the matrix $(Z^{[k']}_{[k]}))_{k,k'\in\mathbb{N}}$ representing the operator Z is block-diagonal according to Definition 3.2.13.

3.2.6 Link between pseudo-differential operators and matrices

To a linear operator R we associate its matrix representation still denoted R through the formula

$$R_{[k]}^{[k']} = \int_{\mathtt{M}^n} R\Phi_{[k]}\Phi_{[k']}dx. \tag{3.2.65}$$

In the following we show that the decay norm $[\![\cdot]\!]_{\beta,s}$ (see Definitions 3.2.8 and 3.2.12) is well designed to capture the smoothing property.

Lemma 3.2.14. *Fix $s > (d+n)/2$ and $\beta \geq 0$. Assume that $R \in \mathcal{R}_{\rho,s}$ with $\rho \geq s+\beta+1/2$ and that R is symmetric then $R \in \mathcal{M}_{\beta,s}$. Moreover, there exists a constant $C = C(s,\rho,\beta)$ such that*

$$[\![R]\!]_{\beta,s} \leq C|R|_{\rho,s,s} \tag{3.2.66}$$

If $R \in \mathcal{R}^{\gamma,\mathcal{O}}_{\rho,s}$ then the bound (3.2.66) *holds with the norms $[\![\cdot]\!]_{\beta,s,}$, $|\cdot|_{\rho,s,s}$ replaced by the norms $[\![\cdot]\!]^{\gamma,\mathcal{O}}_{\beta,s}$, $|\cdot|^{\gamma,\mathcal{O}}_{\rho,s,s}$.*

Proof. We have for $l \in \mathbb{Z}^d$

$$\begin{aligned}
||R_{[k]}^{[k']}(l)||_{\mathcal{L}(L^2)} &= |\langle D^{\rho+s}R(l)\Phi_{[k]}, D^{-\rho-s}\Phi_{[k']}\rangle| \\
&\leq ||D^{\rho+s}R(l)\Phi_{[k]}||_{L^2}||\Phi_{[k']}||_{L^2}\langle k'\rangle^{-\rho-s} \\
&\leq \|R(l)\|_{\mathcal{L}(H^s,H^{s+\rho})}\|\Phi_{[k]}\|_{H^s}\langle k'\rangle^{-\rho-s} \\
&\leq \|R(l)\|_{\mathcal{L}(H^s,H^{s+\rho})}\langle k\rangle^s\langle k'\rangle^{-\rho-s},
\end{aligned}$$

where we used that, for $s \in \mathbb{R}$ (recall (3.2.10)),

$$\|\Phi_{k,j}\|_{H^s} \sim \|K_0^s\Phi_{k,j}\|_{L^2} = \lambda_k^s \sim \langle k\rangle^s.$$

Similarly, since R is symmetric,

$$||R_{[k]}^{[k']}(l)||_{\mathcal{L}(L^2)} \leq \|R(l)\|_{\mathcal{L}(H^s,H^{s+\rho})}\langle k'\rangle^s\langle k\rangle^{-\rho-s},$$

therefore we get

$$||R_{[k]}^{[k']}(l)||_{\mathcal{L}(L^2)} \leq \min\Big(\langle k'\rangle^s\langle k\rangle^{-\rho-s}, \langle k\rangle^s\langle k'\rangle^{-\rho-s}\Big)\|R(l)\|_{\mathcal{L}(H^s,H^{s+\rho})}.$$

So, by definition, we get using that $\langle h,l\rangle \leq \langle l\rangle\langle h\rangle$,

$$
\begin{aligned}
[\![\mathcal{D}^\beta R]\!]_s^2 &= \sum_{h\in\mathbb{N},l\in\mathbb{Z}^d} \langle h,l\rangle^{2s} \sup_{|k-k'|=h} \|(\mathcal{D}^\beta R)_{[k]}^{[k']}(l)\|_{\mathcal{L}(L^2)}^2 \\
&\leq \sum_{l\in\mathbb{Z}^d} \langle l\rangle^{2s}\|R(l)\|_{\mathcal{L}(H^s,H^{s+\rho})} \sum_{h\in\mathbb{N}} \langle h\rangle^{2s} \sup_{|k-k'|=h} \langle k\rangle^{2\beta} \min\Big(\langle k'\rangle^s\langle k\rangle^{-\rho-s}, \langle k\rangle^s\langle k'\rangle^{-\rho-s}\Big) \\
&\leq 2^{2\rho-2\beta}|R|_{\rho,s,s}^2 \sum_{h\in\mathbb{N}} \langle h\rangle^{2s+2\beta-2\rho}
\end{aligned}
$$

where we used that if $|k-k'|=h$ then $\max(|k|,|k'|)\geq h/2$. A similar estimates holds true for $[\![R\mathcal{D}^\beta]\!]_s$ and thus for $[\![R]\!]_{\beta,s} = [\![\mathcal{D}^\beta R]\!]_s + [\![R\mathcal{D}^\beta]\!]_s$. Following a similar reasoning one gets the Lipschitz bounds. □

3.3 Regularization procedure

Let us consider $0<\delta<1$, $r>d/2$ and the operator

$$
\mathcal{F} = \mathcal{F}(\omega) := \omega\cdot\partial_\varphi + \mathrm{i}(\Delta_g + V(\varphi)), \qquad V\in\mathcal{A}_{\delta,r}. \tag{3.3.1}
$$

We also assume that the operator V is *self-adjoint*. Let us define the diophantine set $\mathcal{O}_0 \subseteq [1/2,3/2]^d$ by

$$
\mathcal{O}_0 := \Big\{\omega\in[1/2,3/2]^d \,:\, |\omega\cdot l| \geq 4\gamma|l|^{-\tau},\ \forall\, l\in\mathbb{Z}^d\Big\}, \qquad \tau := d+1. \tag{3.3.2}
$$

The aim of this section is to prove the following result.

Theorem 3.3.1. (Regularization). *Let $\rho_0\geq 0$, $0<\delta<1$ and $r_0>d/2$. There is $r_*=r_*(\delta,\rho_0,r_0)$ such that, for $r>r_*$ and $S\geq s_0>n/2$, there exist $p=p(S,\rho_0)\geq 1$ and $0<\varepsilon_*=\varepsilon_*(S,\rho_0)$ such that the following holds. If*

$$
\gamma^{-1}\mathcal{N}_{\delta,r,p}(V) \leq \varepsilon_*. \tag{3.3.3}
$$

then there is, for any $\varphi\in\mathbb{T}^d$, for any $\omega\in\mathcal{O}_0$, a bounded and invertible map $\Phi\in\mathcal{L}(H^s,H^s)$ for any $s\in[s_0,S]$ such that

$$
\mathcal{F}_+ := \Phi\circ\mathcal{F}\circ\Phi^{-1} := \omega\cdot\partial_\varphi + \mathrm{i}(\Delta_g + Z + R), \tag{3.3.4}
$$

where $Z\in\mathcal{A}_\delta^{\gamma,\mathcal{O}_0}$ is independent of φ, Z is Hermitian and

$$
[Z,K_0] = 0, \tag{3.3.5}
$$

$R(\varphi)$ is a Hermitian ρ_0-smoothing operator in $\mathcal{R}^{\gamma,\mathcal{O}_0}_{\rho_0,r_0}$.
Furthermore $Z = Z_1 + Z_2$ with $Z_1 \in \mathcal{A}_\delta$ is independent of $\omega \in \mathcal{O}_0$, and $Z_2 \in \mathcal{A}^{\gamma,\mathcal{O}_0}_{2\delta-1}$.
Moreover the following estimates holds : for any $s \in [s_0, S]$ there exit constants $q = q(s,\rho_0) \geq p$ and $C = C(s,\rho_0) > 0$ such that

$$\mathcal{N}_{\delta,s}(Z_1) + \mathcal{N}^{\gamma,\mathcal{O}_0}_{2\delta-1,s}(Z_2) \leq C\mathcal{N}_{\delta,r,q}(V)\,, \tag{3.3.6}$$

$$|R|^{\gamma,\mathcal{O}_0}_{\rho_0,r_0,s} \leq C\mathcal{N}_{\delta,r,q}(V)\,, \tag{3.3.7}$$

$$\sup_{\varphi\in\mathbb{T}^d} \|\Phi^{\pm 1}(\varphi) - \mathrm{Id}\|_{\mathcal{L}(H^s,H^{s-\delta})} \leq C\mathcal{N}_{\delta,r,q}(V)\,, \tag{3.3.8}$$

$$\sup_{\varphi\in\mathbb{T}^d} \|\Phi^{\pm 1}(\varphi)\|_{\mathcal{L}(H^s,H^s)} \leq 1 + C\mathcal{N}_{\delta,r,q}(V)\,. \tag{3.3.9}$$

As explained in the introduction this Theorem will be demonstrated by an iterative procedure alternating an averaging step according to the periodic flow of K_0 (section 3.3.1) and a step of eliminating the time dependence of the averaged term (section 3.3.2). The iteration is detailed in section 3.3.3.

3.3.1 Averaging procedure

For $A \in \mathcal{A}_m$, $m \in \mathbb{R}$, we denote for $\tau \in [0, 2\pi]$

$$A(\tau) := e^{-\mathrm{i}\tau K_0} A e^{\mathrm{i}\tau K_0} \tag{3.3.10}$$

and

$$\langle A\rangle := \int_0^{2\pi} A(\tau)d\tau\,, \tag{3.3.11}$$

the average of A along the flow of K_0.
We notice that $\langle A\rangle$ belongs to $\mathcal{A}_m$, commutes with K_0 and that if A is Hermitian then $\langle A\rangle$ is Hermitian. Let $\mathcal{O} \subset \mathcal{O}_0$ (see (3.3.2)) and consider the operator

$$G = \omega\cdot\partial_\varphi + \mathrm{i}M(\varphi) \qquad M(\varphi) := \Delta_g + W + A(\varphi) + R(\varphi) \tag{3.3.12}$$

where $W \in \mathcal{A}^{\gamma,\mathcal{O}}_\delta$, $0 < \delta < 1$, is independent of time and commutes with K_0, $A \in \mathcal{A}^{\gamma,\mathcal{O}}_{\delta',r}$ for some $\delta' \leq \delta$ and $R(\varphi) \in \mathcal{R}^{\gamma,\mathcal{O}}_{\rho,r}$ (see Def. 3.2.4). We also assume that $M(\varphi)$ is Hermitian $\forall\,\varphi \in \mathbb{T}^d$.

Lemma 3.3.2. *Let $r > d/2$, $0 < \delta < 1$, $\delta' \leq \delta$ there exists $S \in \mathcal{A}^{\gamma,\mathcal{O}}_{\delta'-1,r}$ such that for any $s > n/2$ and $\rho \geq 0$ there exists $p = p(s,\rho) \geq 1$, an increasing function of s, and*

$0 < \varepsilon_0 = \varepsilon_0(s, \rho)$ such that if

$$\gamma^{-1}\mathcal{N}^{\gamma,\mathcal{O}}_{\delta',r,p}(A) \leq \varepsilon_0 \,, \qquad \mathcal{N}^{\gamma,\mathcal{O}}_{\delta,r,p}(W) \leq 1 \tag{3.3.13}$$

the symplectic change of variable $\Phi_S = e^{\mathrm{i}S(\varphi)}$ belongs to $\mathcal{L}(H^s, H^s)$ and we have

$$G^+ := \Phi_S \circ G \circ \Phi_S^{-1} = \omega \cdot \partial_\varphi + \mathrm{i}M^+(\varphi) \tag{3.3.14}$$

$$M^+(\varphi) := \Delta_g + W + \langle A(\varphi)\rangle + A^+(\varphi) + R^+(\varphi) \tag{3.3.15}$$

where $\langle A(\varphi)\rangle$ is defined as in (3.3.11), $A^+ \in \mathcal{A}^{\gamma,\mathcal{O}}_{\delta'-1,r-1}$ and $R^+ \in \mathcal{R}^{\gamma,\mathcal{O}}_{\rho,r-1}$. The operator $M^+(\varphi)$ is Hermitian $\forall\, \varphi \in \mathbb{T}^d$.
Moreover there exists $C = C(s,\rho)$ such that

$$\mathcal{N}^{\gamma,\mathcal{O}}_{\delta'-1,r,s}(S) \leq C\mathcal{N}^{\gamma,\mathcal{O}}_{\delta',r,p}(A) \tag{3.3.16}$$

$$\sup_{\varphi\in\mathbb{T}^d} \|\Phi_S^\tau(\varphi)\|^{\gamma,\mathcal{O}}_{\mathcal{L}(H^s,H^s)} \leq 1 + C\mathcal{N}^{\gamma,\mathcal{O}}_{\delta',r,p}(A) \tag{3.3.17}$$

$$\sup_{\varphi\in\mathbb{T}^d} \|\Phi_S^\tau(\varphi) - \mathrm{Id}\|^{\gamma,\mathcal{O}}_{\mathcal{L}(H^s,H^s)} \leq C\mathcal{N}^{\gamma,\mathcal{O}}_{\delta',r,p}(A) \quad \forall \tau \in [0,1]\,, \tag{3.3.18}$$

$$\mathcal{N}^{\gamma,\mathcal{O}}_{\delta'-1,r-1,s}(A^+) \leq C\mathcal{N}^{\gamma,\mathcal{O}}_{\delta',r,p}(A) \tag{3.3.19}$$

$$|R^+|^{\gamma,\mathcal{O}}_{\rho,r-1,s} \leq C|R|^{\gamma,\mathcal{O}}_{\rho,r,s} + C\mathcal{N}^{\gamma,\mathcal{O}}_{\sigma,\delta',p}(A)\,. \tag{3.3.20}$$

Proof. The idea comes from [Wein77], [Col79] and was extensively used in [BGMR2]. It consists to average with respect to the flow of K_0 (see (3.3.11)) which is periodic since its spectrum is included in $\mathbb{N} + \lambda$ (see (3.2.1)).
Let us define $Y = \frac{1}{2\pi}\int_0^{2\pi} \tau(A - \langle A\rangle)(\tau)d\tau$. Then $Y \in \mathcal{A}^{\gamma,\mathcal{O}}_{\delta',r}$ and by integration by parts we verify that Y solves the homological equation

$$i[K_0, Y] = A - \langle A\rangle\,.$$

Then we define

$$S = \frac{1}{4}(YK_0^{-1} + K_0^{-1}Y) \tag{3.3.21}$$

and we note that $S \in \mathcal{A}^{\gamma,\mathcal{O}}_{\delta'-1,r}$ is a pseudo-differential operator of order $\delta'-1 \leq 0$. Moreover, by using Lemma 3.2.6, we deduce the estimate (3.3.16). By applying Lemma 3.6.6 we obtain estimates (3.3.17) and (3.3.18) (see (3.6.16), (3.6.17)). By an explicit computation we also get

$$i[K_0^2, S] = A - \langle A\rangle - \frac{1}{4}\Big[[A, K_0], K_0^{-1}\Big]\,. \tag{3.3.22}$$

To study the conjugate of G in (3.3.12) under the map Φ_S defined as in (3.2.45) with S in (3.3.21) we use the Lie expansions (3.2.47) and (3.2.48) for some $q \in \mathbb{N}$ large to be chosen later. Recalling the splitting (3.2.1)-(3.2.2) we have by (3.2.47)

$$\begin{aligned}\Phi_S \circ \mathrm{i}M \circ \Phi_S^{-1} &\overset{(3.3.22)}{=} \mathrm{i}K_0^2 + \mathrm{i}Q_0 + \mathrm{i}W + \mathrm{i}\langle A\rangle + \frac{\mathrm{i}}{4}\Big[[A, K_0], K_0^{-1}\Big] \\ &- \mathrm{i}[Q_0 + W + A, \mathrm{i}S] + \sum_{j=2}^{q} \frac{1}{j!}\mathrm{ad}_{\mathrm{i}S}^{j}(\mathrm{i}\Delta_g + \mathrm{i}W + \mathrm{i}A) \\ &+ \frac{1}{q!}\int_0^1 (1-\tau)^q e^{\mathrm{i}\tau S}\mathrm{ad}_{\mathrm{i}S}^{q+1}(\mathrm{i}\Delta_g + \mathrm{i}W + \mathrm{i}A)e^{-\mathrm{i}\tau S}d\tau \\ &+ \mathrm{i}\Phi_S \circ R \circ \Phi_S^{-1}\,.\end{aligned}$$

Taking into account the time contribution given by (3.2.48) we obtain that the conjugate $\Phi_S \circ G \circ \Phi_S^{-1}$ has the form (3.3.14)-(3.3.15) where

$$\begin{aligned}\mathrm{i}A^+ &= \frac{\mathrm{i}}{4}\Big[[A, K_0], K_0^{-1}\Big] - \mathrm{i}[Q_0 + W + A, \mathrm{i}S] \\ &+ \sum_{j=2}^{q} \frac{1}{j!}\mathrm{ad}_{\mathrm{i}S}^{j}(\mathrm{i}\Delta_g + \mathrm{i}W + \mathrm{i}A) - \sum_{p=1}^{q} \frac{1}{p!}\mathrm{ad}_{\mathrm{i}S}^{p-1}(\mathrm{i}\omega\cdot\partial_\varphi S)\end{aligned} \tag{3.3.23}$$

and

$$\begin{aligned}\mathrm{i}R^+ &= \frac{1}{q!}\int_0^1 (1-\tau)^q e^{\mathrm{i}\tau S}\mathrm{ad}_{\mathrm{i}S}^{q+1}(\mathrm{i}\Delta_g + \mathrm{i}W + \mathrm{i}A)e^{-\mathrm{i}\tau S}d\tau \\ &+ \frac{1}{q!}\int_0^1 (1-\tau)^q \Phi_S^{\tau}\mathrm{ad}_{\mathrm{i}S}^{q}(\mathrm{i}\omega\cdot\partial_\varphi S)\Phi_S^{-\tau}d\tau\,, \\ &+ \mathrm{i}\Phi_S \circ R \circ \Phi_S^{-1}\end{aligned} \tag{3.3.24}$$

We need to prove the bounds (3.3.19)-(3.3.20). We start by studying the remainder R^+ in (3.3.24). To simplify the notation we shall write $a \lesssim b$ to denote $a \le Cb$ for some constant $C = C(s,\rho)$.

Using the smallness condition (3.3.13), we have that the third summand in (3.3.24) is a ρ-smoothing operator satisfying (3.3.20) by Lemma 3.6.8.

By items $(ii), (iii)$ of Lemma 3.2.6 we have (up to smoothing remainder and for some p depending on s and ρ)

$$\begin{aligned}&\mathcal{N}_{\delta',r,s}^{\gamma,\mathcal{O}}\Big(\mathrm{ad}_{\mathrm{i}S}(\mathrm{i}\Delta_g + \mathrm{i}W + \mathrm{i}A)\Big) + \mathcal{N}_{\delta'-1,r-1,s}^{\gamma,\mathcal{O}}(\omega\cdot\partial_\varphi S) \\ &\qquad\overset{(3.3.13),(3.3.16)}{\lesssim} \mathcal{N}_{\delta',r,p}^{\gamma,\mathcal{O}}(A)\,.\end{aligned} \tag{3.3.25}$$

By iterating the estimate above and using the smallness condition (3.3.13) we deduce, for

$1 \le j \le q$ and for some p depending on s, ρ, q,

$$\mathcal{N}^{\gamma,\mathcal{O}}_{j\delta'-2(j-1),r,s}\Big(\mathrm{ad}^j_{\mathrm{i}S}(\mathrm{i}\Delta_g + \mathrm{i}W + \mathrm{i}A)\Big) + \mathcal{N}^{\gamma,\mathcal{O}}_{(j+1)\delta'-1-2j,r-1,s}(\mathrm{ad}^j_{\mathrm{i}S}(\omega\cdot\partial_\varphi S)) \lesssim \mathcal{N}^{\gamma,\mathcal{O}}_{\delta',r,p}(A)\,. \tag{3.3.26}$$

The sequences $j\delta' - 2(j-1)$ and $(j+1)\delta' - 1 - 2j$ are decreasing since $\delta' \le 1$. Hence, by choosing q large enough, the integrands in (3.3.24) are ρ-smoothing operator (with arbitrary ρ) conjugated by the flow $e^{\mathrm{i}\tau S}$. Therefore by Lemma 3.6.8 all the expressions in (3.3.24) are smoothing remainders satisfying (3.3.20) for some p depending on s and ρ.

Let us now consider the terms in (3.3.23). First of all we have

$$\begin{aligned}\mathcal{N}^{\gamma,\mathcal{O}}_{\delta'-2,r,s}\big(\big[[A,K_0],K_0^{-1}\big]\big) &\lesssim \mathcal{N}^{\gamma,\mathcal{O}}_{\delta',r,N_1}([A,K_0])\mathcal{N}^{\gamma,\mathcal{O}}_{-1,N_1}(K_0^{-1})\\ &\lesssim \mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta',r,N}(A)\mathcal{N}^{\gamma,\mathcal{O}}_{1,N}(K_0)\mathcal{N}^{\gamma,\mathcal{O}}_{-1,N}(K_0^{-1})\\ &\lesssim \mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta',r,N}(A)\,,\end{aligned}$$

for some constant $N_1 \le N \le p$ depending only on s, ρ. In the same way (recalling also (3.3.13)) we have

$$\begin{aligned}\mathcal{N}^{\gamma,\mathcal{O}}_{\delta+\delta'-2,r,s}([Q_0+W+A,\mathrm{i}S]) &\lesssim \mathcal{N}^{\gamma,\mathcal{O}}_{\delta'-1,r,p}(S)\Big(\mathcal{N}_{0,p}(Q_0) + \mathcal{N}^{\gamma,\mathcal{O}}_{\delta,p}(W)\Big)\\ &\quad + \mathcal{N}^{\gamma,\mathcal{O}}_{\delta'-1,r,p}(S)\mathcal{N}^{\gamma,\mathcal{O}}_{\delta',r,p}(A)\\ &\overset{(3.3.13),(3.3.16)}{\lesssim} \mathcal{N}^{\gamma,\mathcal{O}}_{\delta',r,p}(A)\,.\end{aligned}$$

The other summands in (3.3.23) can be estimated by using (3.3.25) and (3.3.26). This proves the (3.3.19). □

3.3.2 Time elimination

Let us consider the operator G^+ in (3.3.14)-(3.3.15) obtained after an average step (see Lemma 3.3.2). The aim of this section is to eliminate the time dependence (i.e. the dependence with respect to φ) in the term $\langle A(\varphi)\rangle$ in (3.3.15). First we introduce the pseudo-differential operator $T = T(\varphi)$ defined as

$$T(\varphi) = \sum_{0\neq l\in\mathbb{Z}^d} \frac{e^{\mathrm{i}l\cdot\varphi}}{\mathrm{i}\omega\cdot l}\langle A(l)\rangle\,. \tag{3.3.27}$$

We have the following Lemma.

Lemma 3.3.3. *Let $r \geq 5d/2 + 9/2$ and $\omega \in \mathcal{O}_0$ (see* (3.3.2)*). Then the operator T in* (3.3.27) *belongs to $\mathcal{A}^{\gamma,\mathcal{O}_0}_{\delta',r-(2\tau+1)}$ is Hermitian, commutes with the operator K_0. Moreover it solves the equation*

$$\langle A(\varphi)\rangle - \omega\cdot\partial_\varphi T = \langle A(0)\rangle\,, \tag{3.3.28}$$

and satisfies

$$\mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta',r-(2\tau+1),s}(T) \leq C\mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta',r,p}(A)\,. \tag{3.3.29}$$

Furthermore, setting $\Phi^\tau_T := e^{i\tau T(\varphi)}$, we have that for any $s > d/2$ there are constants C, p (depending only on s and ρ) such that if (3.3.13) *holds then*

$$\sup_{\varphi\in\mathbb{T}^d}\|\Phi^\tau_T\|_{\mathcal{L}(H^s,H^s)} \leq 1 + C\mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta',r,p}(A) \tag{3.3.30}$$

$$\sup_{\varphi\in\mathbb{T}^d}\|\Phi^\tau_T - \mathrm{Id}\|_{\mathcal{L}(H^s,H^{s-\delta'})} \leq C\mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta',r,p}(A) \quad \forall\tau\in[0,1]\,. \tag{3.3.31}$$

Proof. The operator T is Hermitian and commutes with K_0 thanks to the properties of $\langle A\rangle$. The fact that T solves (3.3.28) is obtained by an explicit computation. The bound (3.3.29) follows by item (*iii*) of Lemma 3.2.6. Finally applying Lemma 3.6.6 we obtain the estimates (3.3.30)-(3.3.31) (see (3.6.18) and (3.6.19)). □

In the following lemma we study how the operator G^+ in (3.3.14)-(3.3.15) changes under the map Φ^τ_T defined by Lemma 3.3.3. We have to distinguish the cases δ' strictly positive or δ' less or equal zero.

Lemma 3.3.4. *Let $\delta' \leq 0$ and $r > 2\tau + 2 + d/2$. Let us define $\delta_1 := \delta + \delta' - 1$ and $\Phi_T := \Phi^1_T$. Then the conjugated operator $G_1 := \Phi_T \circ G^+ \circ \Phi_T^{-1}$ has the form*

$$G_1 = \omega\cdot\partial_\varphi + iM_1(\varphi) \tag{3.3.32}$$

$$M_1(\varphi) := \Delta_g + W_1 + A_1(\varphi) + R_1(\varphi) \tag{3.3.33}$$

where

$$W_1 = W + \int_{\mathbb{T}^d}\langle A(\varphi)\rangle d\varphi\,, \tag{3.3.34}$$

is independent of $\varphi \in \mathbb{T}^d$, $A_1 \in \mathcal{A}^{\gamma,\mathcal{O}}_{\delta_1,r-2\tau-2}$ and $R_1 \in \mathcal{R}^{\gamma,\mathcal{O}}_{\rho,r-2\tau-2}$. The operator $M_1(\varphi)$ is Hermitian $\forall\,\varphi\in\mathbb{T}^d$.
Moreover for any $s > d/2$ there exist $p = p(s,\rho)$ and $C = C(s,\rho)$ such that if (3.3.13)

holds then

$$\mathcal{N}^{\gamma,\mathcal{O}}_{\delta_1,r-2\tau-2,s}(A_1) \leq C\mathcal{N}^{\gamma,\mathcal{O}}_{\delta',r,p}(A) \tag{3.3.35}$$

$$|R_1|^{\gamma,\mathcal{O}}_{\rho,r-2\tau-2,s} \leq C|R|^{\gamma,\mathcal{O}}_{\rho,r,s} + \mathcal{N}^{\gamma,\mathcal{O}}_{\delta',r,p}(A)\,. \tag{3.3.36}$$

Proof. Notice that, since T in (3.3.27) commutes with K_0, then $\Phi_T \circ K_0^2 \circ \Phi_T^{-1} = K_0^2$. By using the expansions (3.2.47)-(3.2.48) and since T solves (3.3.28) we have that the conjugate L_1 has the form (3.3.32)-(3.3.33) with W_1 as in (3.3.34) and

$$\mathrm{i}A_1 := \mathrm{i}A^+ + \sum_{j=1}^{q} \frac{1}{j!}\mathrm{ad}^j_{\mathrm{i}T}\Big(\mathrm{i}Q_0 + \mathrm{i}W + \mathrm{i}\langle A(\varphi)\rangle + \mathrm{i}A^+\Big) - \sum_{j=2}^{q} \frac{1}{j!}\mathrm{ad}^{j-1}_{\mathrm{i}T}(\mathrm{i}\omega\cdot\partial_\varphi T)\,, \tag{3.3.37}$$

$$\begin{aligned}\mathrm{i}R_1 &:= \mathrm{i}\Phi_T \circ R^+ \circ \Phi_T^{-1} \\ &+ \frac{1}{q!}\int_0^1 (1-\tau)^q \Phi_T^\tau \mathrm{ad}^{q+1}_{\mathrm{i}T}\Big(\mathrm{i}Q_0 + \mathrm{i}W + \mathrm{i}\langle A(\varphi)\rangle + \mathrm{i}A^+\Big)\Phi_T^{-\tau} d\tau \\ &+ \frac{1}{q!}\int_0^1 (1-\tau)^q \Phi_T^\tau \mathrm{ad}^{q}_{\mathrm{i}T}(\mathrm{i}\omega\cdot\partial_\varphi T)\Phi_T^{-\tau} d\tau\,,\end{aligned} \tag{3.3.38}$$

and where $q \in \mathbb{N}$ is a large constant to be chosen later. We now estimate the different terms in (3.3.37), (3.3.38).

By (3.2.38) we have, for some $p' = p'(s,\rho)$,

$$\mathcal{N}^{\gamma,\mathcal{O}_0}_{2\delta'-1,r-2\tau-2,s}(\mathrm{ad}_{\mathrm{i}T}(\mathrm{i}\omega\cdot\partial_\varphi T)) \lesssim \mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta',r-2\tau-2,p'}(T)\mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta',r-2\tau-2,p'}(\omega\cdot\partial_\varphi T)\,.$$

On the other hand we have by (3.2.39)

$$\mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta',r-2\tau-2,p'}(\omega\cdot\partial_\varphi T) \lesssim \mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta',r-2\tau-1,p'}(T)\,,$$

thus using (3.3.29) we deduce

$$\mathcal{N}^{\gamma,\mathcal{O}_0}_{2\delta'-1,r-2\tau-2,s}(\mathrm{ad}_{\mathrm{i}T}(\mathrm{i}\omega\cdot\partial_\varphi T)) \lesssim \Big(\mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta',r,p'}(A)\Big)^2 \overset{(3.3.13)}{\lesssim} \mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta',r,p'}(A)\,. \tag{3.3.39}$$

Similarly we prove

$$\begin{aligned}\mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta'-1,r-2\tau-2,s}(\mathrm{ad}_{\mathrm{i}T}(Q_0)) + \mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta+\delta',r-2\tau-2,s}(\mathrm{ad}_{\mathrm{i}T}(W + \langle A(\varphi)\rangle)) \\ + \mathcal{N}^{\gamma,\mathcal{O}_0}_{2\delta'-2,r-2\tau-2,s}(\mathrm{ad}_{\mathrm{i}T}(A^+)) \lesssim \mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta',r,p'}(A)\,.\end{aligned} \tag{3.3.40}$$

Notice that, since $0 < \delta < 1$, the highest order pseudo-differential operator among the ones estimated in (3.3.39), (3.3.40) is the one of order $\delta + \delta' - 1 < \delta'$. By the estimates above, by choosing the constant $q \in \mathbb{N}$ large enough with respect to ρ and by reasoning as in the proof of Lemma 3.3.2 one gets the estimates (3.3.35), (3.3.36). In particular, since $\delta' \leq 0$ we shall use Lemma 3.6.8 in order to estimate the conjugates of smoothing operator under the flow Φ_T^τ. □

In the next Lemma we study the case in which the generator T of Lemma 3.3.3 has order $\delta' > 0$.

Lemma 3.3.5. *Let $0 < \delta' \leq \delta$. Let us define $\delta_1 := \delta + \delta' - 1$ and $\Phi_T := \Phi_T^1$. Fix moreover $r_1 > d/2$ and $\rho_1 \geq 0$ and assume $r > \max(r_1 + d/2, 2\tau + 2 + d/2)$ and $\rho \geq \rho_1 + \delta' r_1 + 1$. Then the conjugated operator $G_1 := \Phi_T \circ G^+ \circ \Phi_T^{-1}$ (see (3.3.14)) has the form (3.3.32), (3.3.33), (3.3.34), is independent of $\varphi \in \mathbb{T}^d$, $A_1 \in \mathcal{A}^{\gamma,\mathcal{O}}_{\delta_1, r-2\tau-2}$ and $R_1 \in \mathcal{R}^{\gamma,\mathcal{O}}_{\rho_1, r_1}$. The operator $M_1(\varphi)$ is Hermitian $\forall\, \varphi \in \mathbb{T}^d$.*
Moreover for any $s \in \mathbb{R}$ there exist $p = p(s, \rho)$ and if (3.3.13) holds then $C = C(s, \rho)$ such that

$$\mathcal{N}^{\gamma,\mathcal{O}}_{\delta_1, r-2\tau-2, s}(A_1) \leq C \mathcal{N}^{\gamma,\mathcal{O}}_{\delta', r, p}(A) \tag{3.3.41}$$

$$|R_1|^{\gamma,\mathcal{O}}_{\rho_1, r_1, s} \leq C |R|^{\gamma,\mathcal{O}}_{\rho, r, s} + \mathcal{N}^{\gamma,\mathcal{O}}_{\delta', r, p}(A)\,. \tag{3.3.42}$$

Proof. One reasons as in the proof of Lemma 3.3.4. The difference is in estimating the remainder R_1 in (3.3.38). Since the generator T is of order $\delta' > 0$ one has to apply Lemma 3.6.7 (instead of Lemma 3.6.8) which provides estimates (3.3.42) instead of the (3.3.36). □

3.3.3 Proof of Theorem 3.3.1

In this section we give the proof of Theorem 3.3.1 which is based on an iterative application of Lemmata of the previous section. Recalling (3.3.1) we set

$$G_0 := \mathcal{F} = \omega \cdot \partial_\varphi + \mathrm{i}\Delta_g + \mathrm{i}V\,.$$

The operator G_0 above has the form (3.3.12) with

$$\mathcal{O} = \mathcal{O}_0, \quad W \equiv 0\,, \quad R \equiv 0\,, \quad A(\varphi) = V(\varphi)\,, \quad \delta' = \delta\,. \tag{3.3.43}$$

Since V is C^∞, $r > d/2$ can be chosen arbitrary large. We will chose it later in function of the order δ, of the final regularity r_0 and the smoothness ρ_0 prescribed by (3.3.7).
Lemma 3.3.2 provides $p_1(S)$ such that if $p \geq p_1(S)$ in (3.3.3) then (3.3.13) holds for any $s \in [s_0, S]$. By applying Lemma 3.3.2 to G_0 we obtain a symplectic map Φ_{S_0} such that

(see (3.3.14))

$$\begin{gathered}\widetilde{G}_0 := \Phi_{S_0} \circ G_0 \circ \Phi_{S_0}^{-1} = \omega \cdot \partial_\varphi + \mathrm{i}\Delta_g + \mathrm{i}\langle V(\varphi)\rangle + \mathrm{i}\widetilde{A}_0 + \mathrm{i}\widetilde{R}_0\,,\\ \widetilde{A}_0 \in \mathcal{A}^{\gamma,\mathcal{O}}_{\delta-1,r-1}\,, \qquad \widetilde{R}_0 \in \mathcal{R}^{\gamma,\mathcal{O}}_{\rho,r-1}\,,\end{gathered} \tag{3.3.44}$$

with $\rho > 0$ arbitrary to be chosen later and where $\langle V(\varphi)\rangle$ is defined as in (3.3.11). We apply Lemma 3.3.5 to the operator given by (3.3.44) with $\rho_1 \rightsquigarrow \rho_0$ of Theorem 3.3.1 and $r_1 > d/2$ (to be chosen later) provided that ρ and r are sufficiently large ($\rho > \rho_0 + \delta r_1 + 1$ and $r > \max(r_1 + d/2, 2\tau + 2 + d/2)$). Hence we obtain a symplectic map Φ_{T_0} such that

$$\begin{aligned}G_1 &:= \Phi_{T_0} \circ \widetilde{G}_0 \circ \Phi_{T_0}^{-1} = \Phi_{T_0} \circ \Phi_{S_0} \circ G_0 \circ \Phi_{S_0}^{-1} \circ \Phi_{T_0}^{-1}\\ &= \omega \cdot \partial_\varphi + \mathrm{i}\Delta_g + \mathrm{i}W_1 + \mathrm{i}A_1 + \mathrm{i}R_1\end{aligned}$$

with $W_1 := \fint_{\mathbb{T}^d}\langle V(\varphi)\rangle d\varphi$, $A_1 \in \mathcal{A}^{\gamma,\mathcal{O}_0}_{2\delta-1,r_1}$, $R_1 \in \mathcal{R}^{\gamma,\mathcal{O}_0}_{\rho_0,r_1}$ and estimates (3.3.41) (3.3.42) are satisfied for all $s \in [s_0, S]$ provided $p \geq p_2(p_1, S)$ depending on p_1 and S (and still increasing in S).
We notice that W_1 is independent of φ and of the parameters $\omega \in \mathcal{O}_0$.

Now we want to iterate this procedure.

Let us first consider the case[6] $0 < \delta \leq 1/2$. Then $2\delta - 1 \leq 0$ and hence, form now on, we will apply iteratively Lemmata 3.3.2 and 3.3.4 (instead of Lemma 3.3.5).

We introduce the following parameters : for $n \geq 1$ we set

$$\delta_n = (n+1)\delta - n\,, \qquad r_n = r_1 - n(2\tau + 2)\,, \qquad q_n = q_0 \circ q_{n-1} \tag{3.3.45}$$

where $q_0(\cdot) = q_1(\cdot, S)$ is the composition of the two function $s \mapsto p(s)$ given by Lemmata 3.3.2 and 3.3.4 and $q_1 = p_2 \circ p_1$. We notice that q_n is an increasing function of S.
Then applying Lemmata 3.3.2 and 3.3.4 iteratively, there exist symplectic changes of variables $\{\Phi_{S_n}\}_n$ and $\{\Phi_{T_n}\}_n$ such that, setting $\Phi_n := \Phi_{T_n} \circ \Phi_{S_n}$, we have

$$G_{n+1} := \Phi_n \circ G_n \circ \Phi_n^{-1} = \omega \cdot \partial_\varphi + \mathrm{i}\Delta_g + \mathrm{i}W_{n+1} + A_{n+1} + R_{n+1} \tag{3.3.46}$$

where W_n is pseudo-differential operator independent of φ of order δ commuting with K_0; A_n is pseudo-differential operator of order δ_n; R_n is ρ_0-smoothing operator. Moreover, by estimates (3.3.19), (3.3.20) and (3.3.35), (3.3.36) we get

$$\mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta,s}(W_n) + \mathcal{N}^{\gamma,\mathcal{O}_0}_{\delta_n,r_n,s}(A_n) + |R_n|^{\gamma,\mathcal{O}_0}_{\rho_0,r_n,s} \leq C\mathcal{N}_{\delta,r,q_n}(V) \quad \text{for all } s \in [s_0, S]\,. \tag{3.3.47}$$

6. Actually Theorem 3.3.1 will be applied only in this case.

We perform $N = N(\rho_0, \delta)$ steps of this procedure in order to get $\delta_N = \delta - N(1-\delta) \leq -\rho_0$. This require to choose r_1 (and hence r) sufficiently large. More precisely, we want $r_N \geq r_0$, the prescribed regularity, and thus in view of (3.3.45) $r_1 \geq N(2\tau+2)+r_0$. Then recalling that we need $r > \max(r_1 + d/2, 2\tau + 2 + d/2)$ we have to chose

$$r > \max(N(2\tau + 2) + r_0 + d/2, 2\tau + 2 + d/2) := r_*(\delta, \rho_0, r_0)\,.$$

Moreover the constant ρ appearing in (3.3.44) should be chosen in such a way

$$\rho \geq \rho_0 + \delta r_1 + 1 \geq \rho_0 + 2N(\tau + 1) + r_0\,.$$

Therefore the operator G_N, defined as in (3.3.46), has the form (3.3.4) with $Z := W_N$. We notice that $W_1 := \fint_{\mathbb{T}^d} \langle V(\varphi)\rangle d\varphi \in \mathcal{A}_\delta$ is independent of ω and that $W_N - W_1 \in \mathcal{A}^{\mathcal{O}_0}_{2\delta-1,r_1}$ which leads to the desired splitting $Z = Z_1 + Z_2$. The bounds (3.3.6), (3.3.7) follows by (3.3.47) with $q = q_N$. The estimates (3.3.8), (3.3.9) follows by composition and estimates (3.3.17), (3.3.18), (3.3.30) and (3.3.31).

The case $1/2 \leq \delta < 1$ requires to apply Lemmata 3.3.2 and 3.3.5 iteratively to construct $\tilde{A}_n \in \mathcal{A}_{\tilde{\delta}_n, \tilde{r}_n}$ with $\delta_n = 2\delta_{n-1} - 1$ and $\delta_0 = \delta$, until $\tilde{\delta}_n$ became negative. Then we can apply the second procedure using Lemmata 3.3.2 and 3.3.4 as in the previous case.

3.4 KAM reducibility

In this section we will prove an abstract KAM Theorem for a matrix operator of the form

$$L_0 = L_0(\omega; \varphi) := \omega \cdot \partial_\varphi + \mathrm{i}(\Delta_g + Z_0 + R_0(\varphi))\,. \tag{3.4.1}$$

To precise our hypothesis on L_0 we define the following constants

$$\begin{aligned} &\mathtt{b} := 6d + 15n + 23\,, \qquad \tau = d + 1\,, \qquad \rho = 5n + 3\,, \\ &\gamma \in (0,1)\,, \quad 0 \leq \kappa \leq 1\,, \qquad s_0 > \frac{d+n}{2}\,, \quad S \geq s_0 + \mathtt{b}\,. \end{aligned} \tag{3.4.2}$$

In this section we assume :

(**A1**) the matrix Z_0 is Hermitian, block diagonal, independent of φ and Lipschitz in $\omega \in \mathcal{O} \subseteq \mathcal{O}_0 \equiv \mathcal{O}_0(\gamma, \tau)$ (see (3.3.2)). Furthermore, denoting $(\mu^{(0)}_{k,j})_{j=1,\cdots,d_k}$ the eigenvalues of the block $(Z_0)^{[k]}_{[k]}$, we assume that there exists $\kappa \geq 0$ such that (recall that c_0 is defined in (3.2.6))

$$|\mu^{(0)}_{k,j}(\omega)| \leq \frac{c_0}{2}|k|\,, \quad \omega \in \mathcal{O},\ k \in \mathbb{N},\ j = 1, \cdots, d_k\,, \tag{3.4.3}$$

$$\|(Z_0)^{[k]}_{[k]}\|^{lip,\mathcal{O}}_{\mathcal{L}(L^2)} \leq \frac{1}{4}\langle k\rangle^{-\kappa}\,, \quad k \in \mathbb{N}\,. \tag{3.4.4}$$

(**A2**) the operator R_0 is in $\mathcal{M}^{\gamma,\mathcal{O}}_{\rho,S}$ (see Def. 3.2.12) and is Hermitian.

Let us define

$$\epsilon := \gamma^{-1}[\![R_0]\!]^{\gamma,\mathcal{O}_0}_{\rho,s_0+\mathtt{b}}\,. \tag{3.4.5}$$

We shall prove the following.

Theorem 3.4.1. (Reducibility). *Let $s \in [s_0, S - \mathtt{b}]$. There exist positive constants $\epsilon_0 = \epsilon_0(s), C = C(s)$ such that, if*

$$\epsilon \leq \epsilon_0\,, \tag{3.4.6}$$

then there is a set $\mathcal{O}_\epsilon \subseteq \mathcal{O}$ with

$$\text{meas}(\mathcal{O} \setminus \mathcal{O}_\epsilon) \leq C\gamma \tag{3.4.7}$$

such that the following holds. For any $\omega \in \mathcal{O}_\epsilon$ there are

- (i) **(Normal form)** *a matrix $Z_\infty = Z_0 + \tilde{Z}_\infty$ with $\tilde{Z}_\infty \in \mathcal{M}^{\gamma,\mathcal{O}_\epsilon}_{\rho,s}$ which is φ-independent, Hermitian and block-diagonal;*
- (ii) **(Conjugacy)** *a bounded and invertible map $\Phi_\infty = \Phi_\infty(\omega,\varphi) : H^s \to H^s$ such that for all $\varphi \in \mathbb{T}^d$, for all $\omega \in \mathcal{O}_\epsilon$,*

$$L_\infty := \Phi_\infty \circ L_0 \circ \Phi_\infty^{-1} := \omega \cdot \partial_\varphi + \mathrm{i}(\Delta_g + Z_\infty)\,. \tag{3.4.8}$$

Moreover we have

$$[\![\Phi^{\pm 1}_\infty(\varphi) - \mathrm{Id}]\!]^{\gamma,\mathcal{O}_\infty}_{\rho,s} \leq C\gamma^{-1}[\![R_0]\!]^{\gamma,\mathcal{O}}_{\rho,s+\mathtt{b}}\,, \quad \forall\, \omega \in \mathcal{O}_\epsilon\,, \tag{3.4.9}$$

$$[\![\tilde{Z}_\infty]\!]^{\gamma,\mathcal{O}_\epsilon}_{\rho,s} \leq C[\![R_0]\!]^{\gamma,\mathcal{O}}_{\rho,s+\mathtt{b}}\,. \tag{3.4.10}$$

3.4.1 The KAM step

The proof of Theorem 3.4.1 is based on an iterative scheme. In this section we show how to perform one step of the iteration. We consider an operator

$$L := \omega \cdot \partial_\varphi + \mathrm{i}(\Delta_g + Z + R)\,, \tag{3.4.11}$$

where $Z = Z_0 + Z_2$ is Hermitian with Z_0 satisfying (**A1**) and $Z_2 \in \mathcal{M}^{\gamma,\mathcal{O}}_{\rho,s}$ for all $s \in [s_0, S]$ and for some $\mathcal{O} \subseteq \mathcal{O}_0$ (see (3.3.2)). The remainder R satisfies (**A2**), i.e. belongs to $\mathcal{M}^{\gamma,\mathcal{O}}_{\rho,s}$ for all $s \in [s_0, S]$ and is Hermitian.

Control of the small divisors

Let us denote by $\mu_{k,j}$, $k \in \mathbb{N}$ and $j = 1, \dots, d_k$ (see (3.2.3)), the eigenvalues of the block $(\Delta_g + Z)^{[k]}_{[k]}$. First of all we prove the following.

Lemma 3.4.2. *One has*

$$\sup_{k\in\mathbb{N}} \langle k\rangle^{\kappa} |\mu_{[k]}|^{lip,\mathcal{O}} \le \frac{1}{4} + [\![Z_2]\!]^{lip,\mathcal{O}}_{\kappa,s_0}\,. \tag{3.4.12}$$

Proof. By Corollary $A.7$ in [FG19] the Lipschitz variation of the eigenvalues of an Hermitian matrix is controlled by the Lipschitz variation of the matrix. Then, in view of hypothesis (**A1**), we get

$$|\mu_{[k]}|^{lip,\mathcal{O}_0} \le \|(Z_0)^{[k]}_{[k]}\|^{lip,\mathcal{O}}_{\mathcal{L}(L^2)} + \|(Z_2)^{[k]}_{[k]}\|^{lip,\mathcal{O}_0}_{\mathcal{L}(L^2)} \le \langle k\rangle^{-\kappa}\big(\frac{1}{4} + [\![Z_2]\!]^{lip,\mathcal{O}_0}_{\kappa,s_0}\big)$$

and the (3.4.12) follows. $\square$

We define the set $\mathcal{O}_+ \subseteq \mathcal{O}$ of parameters ω for which we have a good control of the small divisors. We set, for $N \ge 1$,

$$\begin{aligned}\mathcal{O}_+ \equiv \mathcal{O}_+(\gamma, N) := \Big\{\omega \in \mathcal{O} \,:\, |\omega\cdot l + \mu_{k,j} - \mu_{k',j'}| \ge \frac{2\gamma}{N^\tau \langle k,k'\rangle^{2n+2}}\,,\\ |l| \le N, \quad k,k' \in \mathbb{N}\,, \quad j = 1,\dots,d_k\,,\\ j' = 1,\dots,d'_k\,, \quad (l,k,k') \ne (0,k,k)\Big\}\,.\end{aligned} \tag{3.4.13}$$

We have the following.

Lemma 3.4.3. *Assume that*

$$[\![Z_2]\!]^{\gamma,\mathcal{O}}_{\kappa,s_0+\mathtt{b}} \le \gamma/8 \tag{3.4.14}$$

for some $0 < \gamma \leq \frac{c_0}{5}$ *(see* (3.2.6)*) then we have*

$$\text{meas}\Big(\mathcal{O} \setminus \mathcal{O}_+(\gamma, N)\Big) \leq C\gamma N^{-1} \tag{3.4.15}$$

for some constant $C > 0$ *depending only on* d*.*

Proof. We write

$$\mathcal{O} \setminus \mathcal{O}_+ = \bigcup_{\substack{l\in\mathbb{Z}^d, |l|\leq N \\ k,k'\in\mathbb{N} \\ (\ell,k,k')\neq(0,k,k)}} \bigcup_{\substack{j=1,\dots,d_k \\ j'=1,\dots,d_{k'}}} R^{j,j'}_{l,k,k'}$$

where

$$R^{j,j'}_{l,k,k'} := \Big\{\omega \in \mathcal{O} \,:\, |\omega \cdot l + \mu_{k,j} - \mu_{k',j'}| \leq \frac{2\gamma}{N^\tau \langle k, k'\rangle^{2n+2}}\Big\}.$$

Notice that when $l = 0$ and $k \neq k'$ then $R^{j,j'}_{l,k,k'} = \emptyset$ for all j, j'. Indeed in such case we get using (3.4.3), (3.2.6) and (3.4.14)

$$|\mu_{k,j} - \mu_{k',j'}| \geq \frac{c_0}{2}(k + k') - 2[\![Z_2]\!]^{\infty,\mathcal{O}}_{\kappa,s_0+\mathtt{b}} \geq \frac{c_0}{2} - \frac{\gamma}{4} \geq 2\gamma.$$

Let us now consider the case $l \neq 0$. We give the estimate of the measure of a single *bad* set $R^{j,j'}_{l,k,k'}$. Let us consider the Lipschitz function

$$f(\omega) = \omega \cdot l + \mu_{k,j}(\omega) - \mu_{k',j'}(\omega) = \omega \cdot l + g(\omega)\,.$$

Using condition (3.4.14) we have that Lemma 3.4.2 implies that (recall that $l \neq 0$)

$$|g|^{lip,\mathcal{O}} \leq \frac{1}{2}\,.$$

Then Lemma 5.2 in [FG19] implies that $\text{meas}(R^{j,j'}_{l,k,k'}) \leq \frac{C\gamma}{N^\tau\langle k,k'\rangle^{2n+2}}$ for some constant $C > 0$ depending only on d. Finally by (3.2.3) we have that

$$d_k d_{k'} \leq \langle k, k'\rangle^{2(n-1)}\,.$$

Hence

$$\begin{aligned}\text{meas}\Big(\mathcal{O} \setminus \mathcal{O}_+\Big) &\leq C \sum_{\substack{l\in\mathbb{Z}^d, 0<|l|\leq N \\ k,k'\in\mathbb{N}}} \sum_{\substack{j=1,\dots,d_k \\ j'=1,\dots,d_{k'}}} R^{j,j'}_{l,k,k'} \\ &\leq C \sum_{\substack{l\in\mathbb{Z}^d, 0<|l|\leq N \\ k,k'\in\mathbb{N}}} 2\frac{\gamma}{N^\tau\langle k, k'\rangle^4} \leq CN^{-1}\gamma\end{aligned}$$

since $\tau = d+1$. □

Resolution of the Homological equation

In this section we solve the following homological equation

$$-\mathrm{i}\omega \cdot \partial_\varphi S + \Big[\mathrm{i}S, \Delta_g + Z\Big] + R = \mathrm{Diag}R + Q \tag{3.4.16}$$

where Q is some remainder to be determined and

$$\begin{aligned} &\mathrm{Diag}R = \Big((\mathrm{Diag}R)^{[k']}_{[k]}(l)\Big)_{l\in\mathbb{Z}^d, k,k'\in\mathbb{N}}\,, \\ &(\mathrm{Diag}R)^{[k']}_{[k]}(l) := 0 \quad \text{for} \quad l \neq 0\,, k,k' \in \mathbb{N} \ \text{ or } \ l = 0\,, k \neq k'\,, \\ &(\mathrm{Diag}R)^{[k]}_{[k]}(0) := A^{[k]}_{[k]}(0)\,, \qquad \text{otherwise}\,. \end{aligned} \tag{3.4.17}$$

Lemma 3.4.4. (Homological equation). *Let* $R \in \mathcal{M}^{\gamma,\mathcal{O}}_{\rho,s}$ *for* $s \in [s_0, S]$, ρ *in* (3.4.2). *For any* $\omega \in \mathcal{O}_+ \equiv \mathcal{O}_+(\gamma, N)$ *(defined in* (3.4.13)*) there exist Hermitian matrices* S, Q *solving equation* (3.4.16) *and satisfying*

$$\begin{aligned} [\![S]\!]^{\gamma,\mathcal{O}_+}_{s} &\leq_s \frac{N^{2\tau+1}}{\gamma}[\![R]\!]^{\gamma,\mathcal{O}}_{\rho,s}\,, \\ [\![\mathcal{D}^{\pm\rho}S\mathcal{D}^{\mp\rho}]\!]^{\gamma,\mathcal{O}_+}_{s} &\leq_s \frac{N^{2\tau+\rho+1}}{\gamma}[\![R]\!]^{\gamma,\mathcal{O}}_{\rho,s}\,, \end{aligned} \qquad s \in [s_0, S]\,, \tag{3.4.18}$$

$$\begin{aligned} [\![Q]\!]^{\gamma,\mathcal{O}_+}_{\rho,s} &\leq_s [\![R]\!]^{\gamma,\mathcal{O}}_{\rho,s+\mathtt{b}}N^{-\mathtt{b}}\,, \\ [\![Q]\!]^{\gamma,\mathcal{O}_+}_{\rho,s+\mathtt{b}} &\leq_s [\![R]\!]^{\gamma,\mathcal{O}}_{\rho,s+\mathtt{b}}\,, \end{aligned} \qquad s \in [s_0, S-\mathtt{b}]\,. \tag{3.4.19}$$

Proof. For $N > 0$ we define (recall (3.2.52)) the matrix $\Pi_N R$ as

$$(\Pi_N R)^{[k']}_{[k]}(l) := \begin{cases} R^{[k']}_{[k]}(l)\,, \quad l \in \mathbb{Z}^d\,, k,k' \in \mathbb{N}\,, & \begin{array}{l} |l| \leq N\,, \\ |k-k'| \leq N\,, \end{array} \\ 0\,, \qquad \text{otherwise} & \end{cases} \tag{3.4.20}$$

Then we set

$$Q = (1 - \Pi_N)R \tag{3.4.21}$$

By Lemma 3.6.4, and since the regularity in φ has been fixed at $r = \mathtt{b}$, one deduces the estimates (3.4.19). Moreover, recalling (3.4.17), we have that equation (3.4.16) is equivalent to

$$\mathcal{G}(l,k,k',\omega)S_{[k]}^{[k']}(l) + (\Pi_N R)_{[k]}^{[k']}(l) = 0 \tag{3.4.22}$$

for any $l \in \mathbb{Z}^d$, $k, k' \in \mathbb{N}$ with $(l,k,k') \neq (0,k,k)$ where the operator $\mathcal{G}(l,k,k',\omega)$ is the linear operator acting on complex $d_k \times d_{k'}$-matrices as

$$\mathcal{G}(l,k,k',\omega)A := -\mathrm{i}\Big[\omega \cdot l + \Big(\Delta_g + Z\Big)_{[k]}^{[k]}\Big]A + \mathrm{i}A\Big(\Delta_g + Z\Big)_{[k']}^{[k']}. \tag{3.4.23}$$

Now, since $\Big(\Delta + Z\Big)_{[k]}^{[k]}$ is Hermitian, there is a orthogonal $d_k \times d_k$-matrix $U_{[k]}$ such that

$$U_{[k]}^T\Big(\Delta_g + Z\Big)_{[k]}^{[k]}U_{[k]} = D_{[k]} := \mathrm{diag}_{j=1,\dots,d_k}\Big(\mu_{k,j}\Big)\,,$$

where $\mu_{k,j}$ are the eigenvalues of the k-th block. By setting

$$\widehat{S}_{[k]}^{[k']}(l) := U_{[k]}^T S_{[k]}^{[k']}(l)U_{[k']}\,, \qquad \widehat{R}_{[k]}^{[k']}(l) := U_{[k]}^T R_{[k]}^{[k']}(l)U_{[k']}$$

equation (3.4.22) reads

$$-\mathrm{i}\Big(\omega \cdot l + D_{[k]}\Big)\widehat{S}_{[k]}^{[k']}(l) + \mathrm{i}\widehat{S}_{[k]}^{[k']}(l)D_{[k']} + (\Pi_N \widehat{R})_{[k]}^{[k']}(l) = 0\,. \tag{3.4.24}$$

For $\omega \in \mathcal{O}_+$ (see (3.4.13)) the solution of (3.4.24) is given by (recalling the notation (3.2.50))

$$\widehat{S}_{k,j}^{k',j'}(l) := \begin{cases} \dfrac{-\mathrm{i}\widehat{R}_{k,j}^{k',j'}(l)}{\omega \cdot l + \mu_{k,j} - \mu_{k',j,}}\,, & \begin{array}{l}|l| \leq N\,,\\ |k-k'| \leq N\,,\end{array} \quad (l,k,k') \neq (0,k,k)\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{3.4.25}$$

Since R is Hermitian it is easy to check that also S is Hermitian. Using the bound on the small divisors in (3.4.13) we have that

$$|\widehat{S}_{k,j}^{k',j'}(l)| \leq \gamma^{-1}|\widehat{R}_{k,j}^{k',j'}(l)|N^\tau\langle k,k'\rangle^{2n+2}\,. \tag{3.4.26}$$

Then, by denoting by $\|\cdot\|_\infty$ the sup-norm of a $d_k \times d_{k'}$-matrix, we deduce

$$\begin{aligned}\|S_{[k]}^{[k']}(l)\|_{\mathcal{L}(L^2)} = \|\widehat{S}_{[k]}^{[k']}(l)\|_{\mathcal{L}(L^2)} &\leq \sqrt{d_k d_{k'}}\|\widehat{S}_{[k]}^{[k']}(l)\|_\infty \\ &\overset{(3.4.26),(3.2.3)}{\leq} \gamma^{-1}\|R_{[k]}^{[k']}(l)\|N^\tau\langle k,k'\rangle^{3n+1}\,.\end{aligned} \tag{3.4.27}$$

We now estimates the decay norm of the matrix S. We have

$$
\begin{aligned}
[\![S]\!]_s^2 &\overset{(3.4.27)}{\leq} \gamma^{-2}N^{2\tau}\sum_{l,h}\langle l,h\rangle^{2s}\sup_{|k-k'|=h}\|R_{[k]}^{[k']}(l)\|^2\langle k,k'\rangle^{6n+2}\\
&\leq \gamma^{-2}N^{2\tau}\sum_{l,h}\langle l,h\rangle^{2s}\sup_{\substack{|k-k'|=h\\ k\geq k'}}\|(\mathcal{D}^{\rho}R)_{[k]}^{[k']}(l)\|^2\langle k\rangle^{(6n+2-2\rho)}\\
&+ \gamma^{-2}N^{2\tau}\sum_{l,h}\langle l,h\rangle^{2s}\sup_{\substack{|k-k'|=h\\ k<k'}}\|(R\mathcal{D}^{\rho})_{[k]}^{[k']}(l)\|^2\langle k'\rangle^{(6n+2-2\rho)}\\
&\leq_s \gamma^{-2}N^{2\tau}[\![R]\!]_{\rho,s}^2\,,
\end{aligned}
\tag{3.4.28}
$$

provided that $\rho \geq 3n+1$ which is true thanks to the choices in (3.4.2). Hence the bound (3.6.5) in Lemma 3.6.1 implies

$$
[\![\mathcal{D}^{\pm\rho}S\mathcal{D}^{\mp\rho}]\!]_s \leq_s \gamma^{-1}N^{\tau+\rho}[\![R]\!]_{\rho,s}\,. \tag{3.4.29}
$$

To obtain (3.4.18), it remains to estimate the Lipschitz variation of the matrix S. We reason as in the proof of item *(iii)* of Lemma 3.2.6. To simplify the notation, for any $l\in\mathbb{Z}^d$, $k,k'\in\mathbb{N}$, $j=1,\ldots,d_k$ and $j'=1,\ldots,d_{k'}$, we set

$$
d(\omega) := \mathrm{i}(\omega\cdot l + \mu_{k,j}(\omega) - \mu_{k',j'}(\omega))\,,\quad \forall\omega\in\mathcal{O}_+\,. \tag{3.4.30}
$$

By (3.4.25) we have that, for any $\omega_1,\omega_2\in\mathcal{O}_+$

$$
\begin{aligned}
\widehat{S}_{k,j}^{k',j'}(\omega_1;l) - \widehat{S}_{k,j}^{k',j'}(\omega_2;l) &= \frac{\widehat{R}_{k,j}^{k'j'}(\omega_1;l) - \widehat{R}_{k,j}^{k'j'}(\omega_2;l)}{d(\omega_1)}\\
&\quad + \frac{d(\omega_1)-d(\omega_2)}{d(\omega_1)d(\omega_2)}\widehat{R}_{k,j}^{k',j'}(\omega_2;l)\,.
\end{aligned}
$$

Using the (3.4.12), (3.4.4) we deduce

$$
\frac{|d(\omega_1)-d(\omega_2)|}{|\omega_1-\omega_2|} \lesssim |l|\,,\qquad \forall\omega_1,\omega_2\in\mathcal{O}_+\,,\ \ \omega_1\neq\omega_2\,.
$$

Therefore, recalling (3.4.13), (3.2.16) and reasoning as in (3.4.26), (3.4.27), we get

$$
\begin{aligned}
\|S_{[k]}^{[k']}(l)\|_{\mathcal{L}(L^2)}^{lip,\mathcal{O}_+} &\lesssim \gamma^{-1}N^{\tau}\langle k,k'\rangle^{3n+1}\|R_{[k]}^{[k']}(l)\|^{lip,\mathcal{O}}\\
&\quad + \gamma^{-2}N^{2\tau+1}\langle k,k'\rangle^{5n+3}\|R_{[k]}^{[k']}(l)\|^{sup,\mathcal{O}}\,.
\end{aligned}
$$

Finally, reasoning as in (3.4.28) and using (3.2.55), we deduce

$$
[\![S]\!]_s^{lip,\mathcal{O}_+} \leq_s \gamma^{-1}N^{\tau}[\![R]\!]_{\rho,s}^{lip,\mathcal{O}} + \gamma^{-2}N^{2\tau+1}[\![R]\!]_{\rho,s}^{sup,\mathcal{O}}\,, \tag{3.4.31}
$$

provided that $\rho \geq 5n+3$, which is true by (3.4.2). Combining (3.4.28) and (3.4.31) (recall (3.2.56)) we get the first bound in (3.4.18). The second one follows by (3.6.5) in Lemma 3.6.1. □

Lemma 3.4.5. *There is $C(s) > 0$ (depending only on $s \geq s_0$) such that, if*

$$\gamma^{-1} C(s) N^{2\tau+1} [\![R]\!]^{\gamma,\mathcal{O}}_{\rho,s_0} \leq \frac{1}{2}\,, \tag{3.4.32}$$

then the map $\Phi = e^{\mathrm{i}S} = \mathrm{Id} + \Psi$, with S given by Lemma 3.4.4, satisfies

$$[\![\Psi]\!]^{\gamma,\mathcal{O}_+}_{s} \leq_s \gamma^{-1} N^{2\tau+1} [\![R]\!]^{\gamma,\mathcal{O}}_{\rho,s}\,. \tag{3.4.33}$$

Proof. By (3.4.18) and (3.4.32) we have that

$$C(s) [\![S]\!]^{\gamma,\mathcal{O}_+}_{s_0} \leq 1/2\,, \tag{3.4.34}$$

which implies the (3.6.6). Hence the (3.4.33) follows by Lemma 3.6.3. □

The new remainder

In this subsection we study the conjugate of the operator L under the map Φ given by Lemma 3.4.5. We first define the new normal form Z_+ as

$$Z_+ := Z + \mathrm{Diag} R\,. \tag{3.4.35}$$

We have the following.

Lemma 3.4.6. (New normal form). *We have that Z_+ in* (3.4.35) *is φ-independent, Hermitian and block-diagonal, and satisfies*

$$[\![Z_+ - Z]\!]^{\gamma,\mathcal{O}}_{\rho,s} \leq_s [\![R]\!]^{\gamma,\mathcal{O}}_{\rho,s}\,. \tag{3.4.36}$$

Proof. It follows by construction. □

Lemma 3.4.7. (The new remainder). *Assume that the smallness condition* (3.4.32) *holds true. Then one has*

$$L_+ := \Phi \circ L \circ \Phi^{-1} := \omega \cdot \partial_\varphi + \mathrm{i}(\Delta_g + Z_+ + R_+) \tag{3.4.37}$$

where Z_+ is the normal form given by (3.4.35) *and the new remainder R_+ is Hermitian and satisfies for all $s \in [s_0, S - \mathtt{b}]$*

$$[\![R_+]\!]^{\gamma,\mathcal{O}_+}_{\rho,s} \leq_s N^{-\mathtt{b}} [\![R]\!]^{\gamma,\mathcal{O}}_{\rho,s+\mathtt{b}} + \gamma^{-1} N^{2\tau+\rho+1} [\![R]\!]^{\gamma,\mathcal{O}}_{\rho,s_0} [\![R]\!]^{\gamma,\mathcal{O}}_{\rho,s}\,, \tag{3.4.38}$$

$$[\![R_+]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}_+} \leq_s [\![R]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}} + \gamma^{-1} N^{2\tau+\rho+1} [\![R]\!]_{\rho,s_0}^{\gamma,\mathcal{O}} [\![R]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}} \,. \tag{3.4.39}$$

Proof. Using the Lie expansions (3.2.47) and (3.2.48) we get

$$\begin{aligned} L_+ := \Phi \circ L \circ \Phi^{-1} &= \omega \cdot \partial_\varphi + \mathrm{i}(\Delta_g + Z) + \mathrm{i}R + \mathrm{i}[\mathrm{i}S, \Delta_g + Z] - \mathrm{i}\omega \cdot \partial_\varphi S \\ &+ \mathrm{i} \sum_{p \geq 1} \frac{\mathrm{i}^p}{p!} \mathrm{ad}_S^p(R) + \mathrm{i} \sum_{p\geq 2} \frac{\mathrm{i}^{p-1}}{p!} \mathrm{ad}_S\Big([\mathrm{i}S, \Delta_g + Z] - \omega \cdot \partial_\varphi S \Big) \,. \end{aligned}$$

Hence, equations (3.4.16), (3.4.35) lead to the following formula :

$$R_+ = Q + \widetilde{R}_+$$

with $Q := (1 - \Pi_N)R$ satisfying (3.4.19) and

$$\widetilde{R}_+ := \sum_{p\geq 2} \frac{\mathrm{i}^{p-1}}{p!} \mathrm{ad}_S^{p-1}\Big(\mathrm{Diag} R + Q - R \Big) + \sum_{p \geq 1} \frac{1}{p!} \mathrm{ad}_S^p\Big(R\Big) \,. \tag{3.4.40}$$

Thus, in order to prove (3.4.38) we need to estimate $\widetilde{R}_+$. Consider (for instance) the composition operator SR. In order to control the $[\![\cdot]\!]_{\rho,s}^{\gamma,\mathcal{O}_+}$-norm we shall bound the decay norm of $\mathcal{D}^\rho SR$. The estimates for for $SR\mathcal{D}^\rho$ is the same. We have that

$$\begin{aligned} [\![D^\rho SR]\!]_s^{\gamma,\mathcal{O}_+} = [\![D^\rho S \mathcal{D}^{-\rho} \mathcal{D}^\rho R]\!]_s^{\gamma,\mathcal{O}_+} &\overset{(3.2.58)}{\leq_s} [\![D^\rho S \mathcal{D}^{-\rho}]\!]_s^{\gamma,\mathcal{O}_+} [\![\mathcal{D}^\rho R]\!]_{s_0}^{\gamma,\mathcal{O}} \\ &+ [\![D^\rho S \mathcal{D}^{-\rho}]\!]_{s_0}^{\gamma,\mathcal{O}_+} [\![\mathcal{D}^\rho R]\!]_s^{\gamma,\mathcal{O}} \\ &\overset{(3.4.18)}{\leq_s} \gamma^{-1} N^{2\tau+\rho+1} [\![R]\!]_{\rho,s}^{\gamma,\mathcal{O}} [\![R]\!]_{\rho,s_0}^{\gamma,\mathcal{O}} \,. \end{aligned} \tag{3.4.41}$$

The commutator $[S, R]$ satisfies the same bound as (3.4.41). Therefore, by (3.4.41), (3.4.19), formula (3.4.40), the smallness assumption (3.4.32), and reasoning as in the proof of Lemma 3.6.3 we get the (3.4.38) and (3.4.39). □

3.4.2 Iteration and Convergence

In this section we introduce a new constant

$$\mathtt{a} := \mathtt{b} - 2 = 6d + 15n + 18 \,. \tag{3.4.42}$$

For $N_0 \geq 1$ we define the sequence $(N_\nu)_{\nu\geq 0}$ by

$$N_\nu := N_0^{\chi^\nu} \,, \ \nu \geq 0$$

with $\chi := 3/2$ and we set $N_{-1} = 1$. The proof of Theorem 3.4.1 is based on the following iterative lemma.

Proposition 3.4.8. (Iteration). *Let $s \in [s_0, S - \mathtt{b}]$. There exist $C(s) > 0$ and $N_0 \equiv N_0(s) \geq 1$ such that, if (recall* (3.4.5)*)*

$$C(s) N_0^{2\tau+1+\rho+\mathtt{a}} \epsilon \leq \frac{1}{2}, \tag{3.4.43}$$

then we may construct recursively sets $\mathcal{O}_\nu \subset \mathcal{O}_0$ and operators, defined for $\omega \in \mathcal{O}_\nu$,

$$L_\nu := L_\nu(\omega) := \omega \cdot \partial_\varphi + \mathrm{i}(\Delta_g + Z_\nu + R_\nu), \tag{3.4.44}$$

so that the following properties are satisfied for all $\nu \in \mathbb{N}$:

$(\mathbf{S1})_\nu$ *There is a Lipschitz family of symplectic maps $\Phi_\nu(\varphi) = \Phi_\nu(\varphi, \omega) := \mathrm{Id} + \Psi_\nu(\varphi) \in \mathcal{L}(H^s, H^s)$ defined on $\mathcal{O}_\nu$ such that, for $\nu \geq 1$,*

$$L_\nu := \Phi_\nu L_{\nu-1} \Phi_\nu^{-1}, \tag{3.4.45}$$

and, for $s \in [s_0, S - \mathtt{b}]$,

$$[\![\Psi_\nu]\!]_s^{\gamma,\mathcal{O}_\nu} \leq \gamma^{-1} [\![R_0]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}_0} N_{\nu-1}^{2\tau+2} N_{\nu-2}^{-\mathtt{a}}. \tag{3.4.46}$$

$(\mathbf{S2})_\nu$ *The operator $Z_\nu = Z_0 + Z_{\nu,2}$ where $Z_{\nu,2}$ is φ-independent, block-diagonal and Hermitian. Moreover it satisfies*

$$[\![Z_\nu - Z_{\nu-1}]\!]_{\rho,s}^{\gamma,\mathcal{O}_\nu} \leq [\![R_0]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}_0} N_{\nu-2}^{-\mathtt{a}}. \tag{3.4.47}$$

Moreover there is a sequence of Lipschitz function

$$\mu_{[k]}^{(\nu)} : \mathcal{O}_0 \to \mathbb{R}^{d_k}, \quad k \in \mathbb{N}$$

such that, for $\omega \in \mathcal{O}_\nu$, the functions $\mu_{k,j}^{(\nu)}$, for $j = 1, \ldots, d_k$, are the eigenvalues of the block

$$(\Delta + Z_\nu)_{[k]}^{[k]},$$

satisfying

$$\sup_{k\in\mathbb{N}} \langle k \rangle^\kappa |\mu_{[k]}^{(\nu)}|^{lip,\mathcal{O}_0} \leq \frac{1}{4} + \epsilon \sum_{j=1}^{\nu-1} 2^{-\nu}, \tag{3.4.48}$$

where ϵ is defined in (3.4.5).

(S3)$_\nu$ *The remainder* R_ν *is Hermitian and satisfies, for any* $s \in [s_0, S - \mathtt{b}]$,

$$[\![R_\nu]\!]^{\gamma,\mathcal{O}_\nu}_{\rho,s} \leq [\![R_0]\!]^{\gamma,\mathcal{O}_0}_{\rho,s+\mathtt{b}} N_{\nu-1}^{-\mathtt{a}}\,, \qquad [\![R_\nu]\!]^{\gamma,\mathcal{O}_\nu}_{\rho,s+\mathtt{b}} \leq [\![R_0]\!]^{\gamma,\mathcal{O}_0}_{\rho,s+\mathtt{b}} N_{\nu-1}\,. \tag{3.4.49}$$

(S4)$_\nu$ *One has* $\mathcal{O}_\nu \subset \mathcal{O}_{\nu-1} \subset \mathcal{O}_0$ *(see* (3.3.2)*) and*

$$\operatorname{meas}\Big(\mathcal{O}_{\nu+1} \setminus \mathcal{O}_\nu\Big) \leq \gamma N_\nu^{-1} \qquad \text{and} \qquad \operatorname{meas}\Big(\mathcal{O}_0 \setminus \mathcal{O}_{\nu+1}\Big) \leq 2\gamma\,. \tag{3.4.50}$$

Proof. We proceed by induction. We first verify the inductive step. So we assume that conditions $(\mathbf{Si})_j$, $i = 1, 2, 3, 4$, hold for $1 \leq j \leq \nu$. We shall prove that they holds for $\nu \rightsquigarrow \nu + 1$.

We define the set $\mathcal{O}_{\nu+1}$ as in (3.4.13) with $\mathcal{O} \rightsquigarrow \mathcal{O}_\nu$, $N \rightsquigarrow N_\nu$, $\mu_{k,j} \rightsquigarrow \mu^{(\nu)}_{k,j}$. Using the (3.4.47) for $s \rightsquigarrow s_0$, we have that

$$\begin{aligned}[\![Z^{(\nu)}_{\nu,2}]\!]^{\gamma,\mathcal{O}_\nu}_{\rho,s_0} &\leq [\![Z_{0,2}]\!]^{\gamma,\mathcal{O}_0}_{\rho,s_0+\mathtt{b}} + \sum_{j=1}^{\nu} [\![Z_{j,2} - Z_{j-1,2}]\!]^{\gamma,\mathcal{O}_j}_{\rho,s_0+\mathtt{b}} \\ &\overset{(3.4.47),(3.4.5)}{\leq} \gamma\epsilon \sum_{j=0}^{\nu-1} 2^{-j}\end{aligned} \tag{3.4.51}$$

for $N_0 \geq 1$ large enough. Hence condition (3.4.14) is satisfied for ϵ small enough, i.e.again N_0 large enough (recall (3.4.43)). Therefore Lemma 3.4.3 implies that (3.4.50) holds for the set $\mathcal{O}_{\nu+1}$ which is the $(\mathbf{S4})_{\nu+1}$.

We define the new normal form $Z_{\nu+1}$ as (recall (3.4.35))

$$Z_{\nu+1} := Z_\nu + \operatorname{Diag}(R_\nu)\,.$$

Lemma 3.4.6, applied with $R \rightsquigarrow R_\nu$, together with the estimates (3.4.49), implies the estimate (3.4.47). Let $\widetilde{\mu}^{(\nu+1)}_{[k]}$ be the eigenvalues of the block $(\Delta + Z_{\nu+1})^{[k]}_{[k]}$ which are defined on the set $\mathcal{O}_{\nu+1}$. The bound (3.4.48) follows by Lemma 3.4.2 and (3.4.51). Moreover, by Kirtzbraun Theorem, there is an extension $\mu^{(\nu+1)}_{[k]}$ of $\widetilde{\mu}^{(\nu+1)}_{[k]}$ to the whole set $\mathcal{O}_0$ with the same Lipschitz norm. This prove the $(\mathbf{S2})_{\nu+1}$.

Then we want to construct a map $\Phi_{\nu+1} = \mathrm{Id} + \Psi_{\nu+1}$. First by the inductive hypothesis (3.4.49) we deduce that (with $C(s)$ given in Lemma 3.4.5)

$$C(s)\gamma^{-1} N_\nu^{2\tau+1} [\![R_\nu]\!]^{\gamma,\mathcal{O}_\nu}_{\rho,s_0} \leq C(s)\gamma^{-1} N_\nu^{2\tau+1} N_{\nu-1}^{-\mathtt{a}} [\![R_0]\!]^{\gamma,\mathcal{O}_0}_{\rho,s_0+\mathtt{b}} \overset{(3.4.5),(3.4.42)}{\leq} C(s)\epsilon N_\nu^{2\tau+1-\frac{2}{3}\mathtt{a}} \leq \frac{1}{2} \tag{3.4.52}$$

for ϵ small enough and since $2\tau + 1 - \frac{2}{3}\mathtt{a} \leq 0$. Hence the smallness condition (3.4.32) holds true. We then apply Lemmata 3.4.4 and 3.4.5 with $R \rightsquigarrow R_\nu$ and $\mathcal{O}_+ \rightsquigarrow \mathcal{O}_{\nu+1}$ and construct a map $\Phi_{\nu+1} = \mathrm{Id} + \Psi_{\nu+1}$.

Furthermore using (3.4.33), (3.4.49) at rank ν, $N_{\nu-1} = N_\nu^{2/3}$ and $2\tau + 1 - \frac{2}{3}\mathtt{a} \leq -1$, we obtain the estimate (3.4.46) at rank $\nu + 1$. This proves the $(\mathbf{S1})_{\nu+1}$.

We finally set

$$L_{\nu+1} := \Phi_{\nu+1} \circ L_\nu \circ \Phi_{\nu+1}^{-1} = \omega \cdot \partial_\varphi + \mathrm{i}(\Delta_g + Z_{\nu+1} + R_{\nu+1}) \tag{3.4.53}$$

where the remainder $R_{\nu+1}$ is given by Lemma 3.4.7. We have

$$\begin{aligned} [\![R_{\nu+1}]\!]_{\rho,s}^{\gamma,\mathcal{O}_{\nu+1}} &\overset{(3.4.38)}{\leq_s} N_\nu^{-\mathtt{b}} [\![R_\nu]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}_\nu} + \gamma^{-1} N_\nu^{2\tau+\rho+1} [\![R_\nu]\!]_{\rho,s_0}^{\gamma,\mathcal{O}_\nu} [\![R_\nu]\!]_{\rho,s}^{\gamma,\mathcal{O}_\nu} \\ &\overset{(3.4.49)}{\leq_s} [\![R_0]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}_0} \Big(N_\nu^{-\mathtt{b}+1} + \gamma^{-1} [\![R_0]\!]_{\rho,s_0}^{\gamma,\mathcal{O}_0} N_\nu^{2\tau+\rho+1-\frac{4}{3}\mathtt{a}} \Big) \\ &\leq N_\nu^{-\mathtt{a}} [\![R_0]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}_0} \end{aligned} \tag{3.4.54}$$

for N_0 large enough where we used that $\gamma^{-1}[\![R_0]\!]_{\rho,s_0}^{\gamma,\mathcal{O}_0} \leq 1$ (thanks to (3.4.6)) and

$$\mathtt{b} \geq \mathtt{a} + 2\,, \qquad 2\tau + \rho + 1 - \frac{1}{3}\mathtt{a} \leq -1.$$

The latter condition is implied by the choice of $\mathtt{a}$ in (3.4.42) recalling the (3.4.2). The (3.4.54) is the first estimate in (3.4.49) at step $\nu + 1$. We now give the estimate in "high" norm. We have

$$\begin{aligned} [\![R_{\nu+1}]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}_{\nu+1}} &\overset{(3.4.39)}{\leq_s} [\![R_\nu]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}_\nu} + \gamma^{-1} N_\nu^{2\tau+\rho+1} [\![R_\nu]\!]_{\rho,s_0}^{\gamma,\mathcal{O}_\nu} [\![R_\nu]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}_\nu} \\ &\overset{(3.4.49)}{\leq_s} [\![R_0]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}_0} N_{\nu-1} \Big(1 + \gamma^{-1} [\![R_0]\!]_{\rho,s_0+\mathtt{b}}^{\gamma,\mathcal{O}_0} N_\nu^{2\tau+\rho+1} N_{\nu-1}^{-\mathtt{a}-1} \Big) \\ &\leq_s N_\nu [\![R_0]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}_0} \end{aligned} \tag{3.4.55}$$

for N_0 large enough depending on s and thanks to fact that $3\tau + \frac{3}{2}\rho + \frac{1}{2} - \mathtt{a} \leq 0$. This is the $(\mathbf{S3})_{\nu+1}$.

Now we have to verify the initial step : $\nu = 1$. $(\mathbf{S2})_1$ and $(\mathbf{S4})_1$ are proved exactly in the way as in the inductive step. Now to proceed we have to construct Φ_1 but now (3.4.52) becomes

$$C(s)\gamma^{-1} N_0^{2\tau+1} [\![R_0]\!]_{\rho,s_0}^{\gamma,\mathcal{O}_0} \overset{(3.4.5)}{\leq} C(s)\epsilon N_0^{2\tau+1} \leq \frac{1}{2} \tag{3.4.56}$$

which is less than $\frac{1}{2}$ for ϵ and N_0 satisfying (3.4.43).

Furthermore using (3.4.33) we obtain

$$[\![\Psi_1]\!]_s^{\gamma,\mathcal{O}_1} \leq C(s)\gamma^{-1} N_0^{2\tau+1} [\![R_0]\!]_{\rho,s}^{\gamma,\mathcal{O}} \leq \gamma^{-1} N_0^{2\tau+2} [\![R_0]\!]_{\rho,s}^{\gamma,\mathcal{O}}$$

for N_0 large enough. This proves the $(\mathbf{S1})_{\nu+1}$.

Then we set

$$L_1 := \Phi_1 \circ L_0 \circ \Phi_1^{-1} = \omega \cdot \partial_\varphi + \mathrm{i}(\Delta_g + Z_1 + R_1) \tag{3.4.57}$$

where the remainder R_1 is given by Lemma 3.4.7. We have

$$\begin{aligned}
[\![R_1]\!]^{\gamma,\mathcal{O}_1}_{\rho,s} &\overset{(3.4.38)}{\leq_s} N_0^{-\mathtt{b}}[\![R_0]\!]^{\gamma,\mathcal{O}_0}_{\rho,s+\mathtt{b}} + \gamma^{-1}N_0^{2\tau+\rho+1}[\![R_0]\!]^{\gamma,\mathcal{O}_0}_{\rho,s_0}[\![R_0]\!]^{\gamma,\mathcal{O}_0}_{\rho,s} \\
&\overset{(3.4.49)}{\leq_s} [\![R_0]\!]^{\gamma,\mathcal{O}_0}_{\rho,s+\mathtt{b}}\Big(N_0^{-\mathtt{b}+1} + \epsilon N_0^{2\tau+\rho+1}\Big) \\
&\leq N_0^{-\mathtt{a}}[\![R_0]\!]^{\gamma,\mathcal{O}_0}_{\rho,s+\mathtt{b}}
\end{aligned} \tag{3.4.58}$$

for N_0 large enough where we used (3.4.43) and $\mathtt{b} \geq \mathtt{a}+2$. The (3.4.58) is the first estimate in (3.4.49) at step 1, the other is proved similarly.

□

Proof of Theorem 3.4.1. Consider the operator L_0 in (3.4.1). The smallness condition (3.4.6) implies the (3.4.43), hence Proposition 3.4.8 applies. We define the set

$$\mathcal{O}_\epsilon \equiv \mathcal{O}_\infty := \cap_{\nu \geq 0}\mathcal{O}_\nu\,. \tag{3.4.59}$$

By the measure estimate (3.4.50) we deduce (3.4.7). For any $\omega \in \mathcal{O}_\infty$, $\nu \geq 0$, we define (see (3.4.45), (3.4.46)) the map

$$\widetilde{\Phi}_{\nu+1} := \Phi_1 \circ \Phi_2 \circ \cdots \Phi_{\nu+1} = \widetilde{\Phi}_\nu \Phi_{\nu+1} = \widetilde{\Phi}_\nu(\mathrm{Id} + \Psi_{\nu+1})\,. \tag{3.4.60}$$

We want to prove that $(\widetilde{\Phi}_\nu)_{\nu\geq 1}$ converges in $\mathcal{M}^{\gamma,\mathcal{O}_\infty}_s$. Let us define

$$\delta^{(\nu)}_s := [\![\widetilde{\Phi}_\nu]\!]^{\gamma,\mathcal{O}_\infty}_s\,. \tag{3.4.61}$$

We have

$$\delta^{(\nu+1)}_{s_0} \overset{(3.2.58)}{\leq} \delta^{(\nu)}_{s_0}(1 + C[\![\Psi_{\nu+1}]\!]^{\gamma,\mathcal{O}_\infty}_{s_0}) \overset{(3.4.46),(3.4.5)}{\leq} \delta^{(\nu)}_{s_0}(1 + C\epsilon N_\nu^{-1})\,. \tag{3.4.62}$$

By iterating the (3.4.62) we get, for any ν,

$$[\![\widetilde{\Phi}_\nu]\!]^{\gamma,\mathcal{O}_\infty}_{s_0} \leq (1 + [\![\Psi_1]\!]^{\gamma,\mathcal{O}_\infty}_{s_0})\Pi_{j\geq 1}(1 + C\epsilon N_\nu^{-1}) \leq 2 \tag{3.4.63}$$

where we used the (3.4.46) to estimate $[\![\Psi_1]\!]^{\gamma,\mathcal{O}_\infty}_{s_0}$ and we take N_0 large enough.

The high norm of $\widetilde{\Phi}_{\nu+1}$ is estimated by

$$\begin{aligned}\delta_s^{(\nu+1)} &\overset{(3.2.58)}{\leq} \delta_s^{(\nu)}(1 + C(s)[\![\Psi_{\nu+1}]\!]_{s_0}^{\gamma,\mathcal{O}_\infty}) + C(s)[\![\Psi_{\nu+1}]\!]_s^{\gamma,\mathcal{O}_\infty}[\![\widetilde{\Phi}_\nu]\!]_{s_0}^{\gamma,\mathcal{O}_\infty} \\ &\overset{(3.4.46),(3.4.63)}{\leq} \delta_s^{(\nu)}(1 + C(s)\epsilon N_\nu^{-1}) + \varepsilon_\nu\end{aligned} \tag{3.4.64}$$

where

$$\varepsilon_\nu := C(s)\gamma^{-1}[\![R_0]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}_0} N_\nu^{-1}\,.$$

By iterating (3.4.64), using $\Pi_{j\geq 0}(1 + C(s)\epsilon N_\nu^{-1}) \leq 2$ for N_0 large enough, we obtain

$$[\![\widetilde{\Phi}_\nu]\!]_s^{\gamma,\mathcal{O}_\infty} \leq [\![\widetilde{\Phi}_1]\!]_s^{\gamma,\mathcal{O}_\infty} + 2\sum_{j\geq 1}\varepsilon_j \leq 1 + C(s)\gamma^{-1}[\![R_0]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}_0}\,. \tag{3.4.65}$$

Then we have

$$\begin{aligned}[\![\widetilde{\Phi}_{\nu+1} - \widetilde{\Phi}_\nu]\!]_s^{\gamma,\mathcal{O}_\infty} = [\![\widetilde{\Phi}_\nu\Psi_{\nu+1}]\!]_s^{\gamma,\mathcal{O}_\infty} &\overset{(3.2.58)}{\leq_s} [\![\widetilde{\Phi}_\nu]\!]_s^{\gamma,\mathcal{O}_\infty}[\![\Psi_{\nu+1}]\!]_{s_0}^{\gamma,\mathcal{O}_\infty} + [\![\widetilde{\Phi}_\nu]\!]_{s_0}^{\gamma,\mathcal{O}_\infty}[\![\Psi_{\nu+1}]\!]_s^{\gamma,\mathcal{O}_\infty} \\ &\overset{(3.4.63),(3.4.65),(3.4.46)}{\leq_s} (1 + \gamma^{-1}[\![R_0]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}_0})\epsilon N_\nu^{-1} + \gamma^{-1}[\![R_0]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}_0} N_\nu^{-1} \\ &\leq_s \gamma^{-1}[\![R_0]\!]_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}_0} N_\nu^{-1}\,.\end{aligned} \tag{3.4.66}$$

Now fix $s \in [s_0, S - \mathtt{b}]$, since by hypothesis (**A2**), $R_0 \in \mathcal{M}_{\rho,s+\mathtt{b}}^{\gamma,\mathcal{O}}$, we deduce from the last estimate that $(\widetilde{\Psi}_\nu)_{\nu\geq 0}$ is a Cauchy sequence in $\mathcal{M}_s^{\gamma,\mathcal{O}_\infty}$. Hence $\widetilde{\Phi}_\nu \to \Phi_\infty \in \mathcal{M}_s^{\gamma,\mathcal{O}_\infty}$. Furthermore by (3.4.66) one deduces the (3.4.9). The estimate on $\Phi_\infty^{-1} - \mathrm{Id}$ follows by using Neumann series and reasoning as in the proof of Lemma 3.6.3. By (3.4.47) we deduce that $Z_{\nu,2}$ is a Cauchy sequence in $\mathcal{M}_{\rho,s}^{\gamma,\mathcal{O}_\infty}$. Hence we set

$$Z_\infty = Z_0 + Z_{\infty,2} := Z_0 + \lim_{\nu\to\infty} Z_{\nu,2}\,, \tag{3.4.67}$$

The (3.4.10) follows again by (3.4.47). We also notice that (3.4.49) implies that $R_\nu \to 0$ in $\mathcal{M}_{\rho,s}^{\gamma,\mathcal{O}_\infty}$. Now by applying iteratively the (3.4.45) we have that $L_\nu = \widetilde{\Phi}_\nu \circ L_0 \circ \widetilde{\Phi}_\nu^{-1}$. Hence, passing to the limit, we get $L_\nu \to_{\nu\to\infty} L_\infty$ of the form (3.4.8) with Z_∞ given by (3.4.67). □

3.5 Proof of Theorem 3.1.1

In this short section we merge the two previous sections to prove the reducibility of the Schrödinger equation (3.1.1) : Theorem 3.1.1.

We recall that equation (3.1.1) has the form

$$\partial_t u = -\mathrm{i}(\Delta_g + \varepsilon W(\omega t))u$$

where $\varphi \mapsto W(\varphi)$ is a C^∞ map from $\mathbb{T}^d$ to $\mathcal{A}_\delta$, $\delta \leq 1/2$, and thus $W \in \mathcal{A}_{\delta,r}$ for any $r > d/2$. Its reducibility rely on the reducibility of the operator $\mathcal{F}$ in (3.3.1) with $V(\varphi) = \varepsilon W(\varphi)$. Rouglhy speaking we want to apply Theorem 3.3.1 to regularize $\mathcal{F}$ in such a way operator $\mathcal{F}$ is transformed into the operator $\mathcal{F}_+$ in (3.3.4). Then we apply Lemma 3.2.14 to control the remainder R in (3.3.4) in s-decay norm. This allows, for ε small enough, to apply the reducibility Theorem 3.4.1 and to conclude.
To justify all these steps we have to carefully follow the parameters and the smallness conditions. First we fix $\alpha \in (0,1)$ and $\gamma = \varepsilon^\alpha$, $\delta \leq \frac{1}{2}$, $s > n/2$ and W belonging to all the $\mathcal{A}_{\delta,r}$ with $r > d/2$. Then we fix ρ, $\mathtt{b}$, τ as in (3.4.2), we set $\kappa = 2\delta - 1$ and we fix $s_0 > n/2$ and S such that s and $s + \mathtt{b}$ belong to $[s_0, S]$ and $S \geq p(\delta, 0)$ (see (3.2.21)). Finally we set

$$\rho_0 = S + \rho + \frac{1}{2}, \quad r_0 = S.$$

With these values of ρ_0, r_0, Theorem 3.3.1 provide us with $\varepsilon_* = \varepsilon_*(S, n, d, \delta)$, $r_* = r_*(S, n, d, \delta)$ and $p = p(S, n, d, \delta)$ such that if $r > r_*$ and

$$\gamma^{-1}\mathcal{N}_{\delta,r,p}(\varepsilon W) < \varepsilon_*(n, d, \delta) \tag{3.5.1}$$

then we can apply Theorem 3.3.1 to $\mathcal{F}$ with $V = \varepsilon W$. Since W belongs to $\mathcal{A}_{\delta,r}$ for any $r > d/2$, (3.5.1) is satisfied for ε small enough. So there exists $\Phi(\varphi) \in \mathcal{L}(H^s, H^s)$ such that (see (3.3.4))

$$\Phi\mathcal{F}\Phi^{-1} = \mathcal{F}_+ = \omega \cdot \partial_\varphi + \mathrm{i}(\Delta_g + Z + R)\,.$$

Further we knows that $R \in \mathcal{R}^{\gamma,\mathcal{O}_0}_{\rho_0,r_0}$ and

$$|R|^{\gamma,\mathcal{O}_0}_{\rho_0,r_0,s} \leq C\mathcal{N}_{\delta,r,p}(\varepsilon W)\,. \tag{3.5.2}$$

Now we apply Lemma 3.2.14 to conclude that $R \in \mathcal{M}^{\gamma,\mathcal{O}_0}_{\rho,S}$ and

$$[\![R]\!]^{\gamma,\mathcal{O}_0}_{\rho,S} \leq C\mathcal{N}_{\delta,r,p}(\varepsilon W)\,. \tag{3.5.3}$$

We notice that the operator $\mathcal{F}_+$ has the same form of the operator L_0 in (3.4.1) with $Z_0 = Z$, $R_0 = R$ and $\mathcal{O} = \mathcal{O}_0$ (see (3.3.2)). The remainder R_0 satisfies the assumption (**A2**) by the discussion above. Notice also that, in view of (3.5.3), the constant ϵ given by (3.4.5) satisfies

$$\epsilon \leq \mathcal{N}_{\delta,r,p}(W)\varepsilon^{1-\alpha} \tag{3.5.4}$$

and thus the smallness condition (3.4.6) is satisfied provided that ε is small enough. We now prove that Z_0 satisfies assumption (**A1**) with $\kappa := 2\delta - 1$. First we note that, since $\delta \leq 1/2$ then $\kappa \leq 0$. Moreover, by Theorem 3.3.1, we have that $Z_0 := Z = Z_1 + Z_2$ with $Z_1 \in \mathcal{A}_\delta$ independent of $\omega \in \mathcal{O}_0$, and $Z_2 \in \mathcal{A}^{\gamma,\mathcal{O}_0}_{2\delta-1}$. Estimate (3.3.6) implies that for all $s \in [s_0, S]$

$$\mathcal{N}_{\delta,s}(Z) \leq C\mathcal{N}_{\delta,r,p}(\varepsilon W) \leq C\mathcal{N}_{\delta,r,p}(W)\varepsilon^{1-\alpha}.$$

Since $S \geq p(\delta, 0)$ we deduce by (3.2.21) that

$$\|Z\|_{\mathcal{L}(L^2,H^{-\delta})} \leq C\mathcal{N}_{\delta,r,p}(W)\varepsilon^{1-\alpha} \leq \frac{c_0}{2}$$

for ε small enough which in turn implies that

$$|\mu^{(0)}_{k,j}(\omega)| \leq \frac{c_0}{2}|k|^\delta$$

and thus (3.4.3) holds true. Furthermore since Z_1 does not depend on ω, we have

$$\|(Z_0)^{[k]}_{[k]}\|^{lip,\mathcal{O}}_{\mathcal{L}(L^2)} = \|(Z_2)^{[k]}_{[k]}\|^{lip,\mathcal{O}}_{\mathcal{L}(L^2)}$$

and thus (3.3.6) implies also (3.4.4) for ε small enough.
Hence all the hypothesis of Theorem 3.4.1 are satisfied for $L_0 = \mathcal{F}_+$ and this theorem provides a set of frequencies $\mathcal{O}_\epsilon$ such that, for $\omega \in \mathcal{O}_\epsilon$, there is a map Φ_∞ satisfying, (see estimates (3.4.9), (3.5.4))

$$[\![\Phi^{\pm 1}_\infty(\varphi) - \mathrm{Id}]\!]^{\gamma,\mathcal{O}_\infty}_{\rho,\tilde{s}} \lesssim_S \mathcal{N}_{\delta,r,p}(W)\varepsilon^{1-\alpha} \quad \forall\, \omega \in \mathcal{O}_\epsilon\,, \tag{3.5.5}$$

for any $(d+n)/2 < \tilde{s} < S - \mathtt{b}$, such that $L_0 \equiv \mathcal{F}_+$ transforms into L_∞ in (3.4.8). By (3.4.7) we have

$$\mathrm{meas}(\mathcal{O}_0 \setminus \mathcal{O}_\epsilon) \leq C\gamma\,,$$

for some constant $C > 0$ depending on s. It is also know that (recall (3.3.2)) $\mathrm{meas}([1/2, 3/2]^d \setminus \mathcal{O}_0) \leq C\gamma$. Therefore, recalling that we set $\gamma = \varepsilon^\alpha$ we have that the (3.1.2) holds. Moreover, by Lemma 3.6.2 and (3.5.5) we have

$$\sup_{\varphi\in\mathbb{T}^d} \|\Phi_\infty^{\pm 1}(\varphi) - \mathrm{Id}\|_{\mathcal{L}(H^s;H^s)} \lesssim_S \mathcal{N}_{\delta,r,p}(W)\varepsilon^{1-\alpha}\,. \tag{3.5.6}$$

For $\omega \in \mathcal{O}_\varepsilon$ we set

$$\Psi(\omega t) := \Phi_\infty(\omega t) \circ \Phi(\omega t)\,.$$

By construction the function $v := \Psi(\omega t)u$ satisfies the equation (3.1.5) with $\varepsilon Z \rightsquigarrow Z_\infty$ in (3.4.8). Moreover, by (3.3.8), (3.3.9), (3.5.6) and (3.5.2), we have

$$\begin{aligned} \sup_{\varphi\in\mathbb{T}^d} \|\Psi^{\pm 1}(\varphi) - \mathrm{Id}\|_{\mathcal{L}(H^s,H^{s-\delta})} &\le \gamma^{-1}C_s\varepsilon\,, \\ \sup_{\varphi\in\mathbb{T}^d} \|\Psi^{\pm 1}(\varphi)\|_{\mathcal{L}(H^s,H^s)} &\le 1 + \gamma^{-1}C_s\varepsilon\,, \end{aligned} \tag{3.5.7}$$

for some $C_s > 0$. This concludes the proof.

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3.6 Appendix : Technical lemmata

3.6.1 Proof of Lemma 3.2.6

Proof of Lemma 3.2.6. The bounds (3.2.37), (3.2.38) and (3.2.42) can be deduced by using the properties of the semi-norm in (3.2.21)-(3.2.24) and the definition in (3.2.31). We give the proof the bound (3.2.41) of item (*iii*). The bound (3.2.39) is similar. We have that

$$B(\omega) := (\omega\cdot\partial_\varphi)^{-1}A(\omega) = \sum_{0\neq l\in\mathbb{Z}^d} \frac{1}{\mathrm{i}\omega\cdot l} e^{il\cdot\varphi}A(\omega;l). \tag{3.6.1}$$

Thus

$$\begin{aligned}\Big(\mathcal{N}_{m,r-\alpha,p}((\omega\cdot\partial_\varphi)^{-1}A)\Big)^2 &\overset{(3.2.31)}{\leq} \sum_{0\neq l\in\mathbb{Z}^d}\frac{1}{|\omega\cdot l|^2}\langle l\rangle^{2(r-\alpha)}\mathcal{N}^2_{m,p}(A(l))\\ &\overset{(3.2.40)}{\leq} \frac{1}{\gamma^2}\sum_{l\in\mathbb{Z}^d\setminus\{0\}}\langle l\rangle^{2(r-\alpha)}|l|^{2\alpha}\mathcal{N}^2_{m,p}(A(l))\\ &\leq \frac{C}{\gamma^2}\sum_{0\neq l\in\mathbb{Z}^d}\langle l\rangle^{2r}\mathcal{N}^2_{m,p}(A(l)) = \frac{C}{\gamma^2}\Big(\mathcal{N}_{m,r,p}(A)\Big)^2 .\end{aligned} \tag{3.6.2}$$

To estimate $\mathcal{N}^{lip,\mathcal{O}}_{m,r-(2\alpha+1),p}(B)$ (see (3.2.16)) we reason as follow. We first note that

$$\begin{aligned}B(\omega_1)-B(\omega_2) &= \sum_{0\neq l\in\mathbb{Z}^d}\frac{1}{\mathrm{i}\omega_1\cdot l}e^{il\cdot\varphi}\Big(A(\omega_1;l)-A(\omega_2;l)\Big)\\ &\quad+\sum_{0\neq l\in\mathbb{Z}^d}\frac{(\omega_1-\omega_2)\cdot l}{\mathrm{i}(\omega_1\cdot l)(\omega_2\cdot l)}e^{il\cdot\varphi}A(\omega_2;l).\end{aligned}$$

Moreover, by using (3.2.40) and that $\mathcal{O}$ is compact, we have

$$\left|\frac{(\omega_1-\omega_2)\cdot l}{\mathrm{i}(\omega_1\cdot l)(\omega_2\cdot l)}\right| \leq C\frac{1}{\gamma^2}|l|^{2\alpha+1}|\omega_1-\omega_2|\,.$$

Therefore reasoning as in (3.6.2) we get

$$\frac{\mathcal{N}_{m,r-(2\alpha+1),p}(B(\omega_1)-B(\omega_2))}{|\omega_1-\omega_2|} \lesssim \frac{1}{\gamma}\mathcal{N}^{lip,\mathcal{O}}_{m,r,p}(A)+\frac{1}{\gamma^2}\mathcal{N}^{sup,\mathcal{O}}_{m,r,p}(A)\,. \tag{3.6.3}$$

Combining (3.6.2), (3.6.3) and recalling (3.2.17), (3.6.1) we obtained

$$\begin{aligned}\mathcal{N}^{\gamma,\mathcal{O}}_{m,r-(2\alpha+1),p}(B) &\lesssim \frac{1}{\gamma}\mathcal{N}^{sup,\mathcal{O}}_{m,r,p}(A)+\gamma\left(\mathcal{N}^{lip,\mathcal{O}}_{m,r,p}(A)+\frac{1}{\gamma^2}\mathcal{N}^{sup,\mathcal{O}}_{m,r,p}(A)\right)\\ &\lesssim \frac{1}{\gamma}\left(\mathcal{N}^{sup,\mathcal{O}}_{m,r,p}(A)+\gamma\mathcal{N}^{lip,\mathcal{O}}_{m,r,p}(A)\right)\end{aligned}$$

which is bound (3.2.41). □

3.6.2 Properties of the s-decay norm

In this appendix s_0 is some fixed number satisfying $s_0>(d+n)/2$.

Lemma 3.6.1. *Let $\alpha>0$. Then (recall* (3.2.61), (3.2.55)*)*

$$[\![\mathcal{D}^{\pm\alpha}A\mathcal{D}^{\mp\alpha}]\!]^{\gamma,\mathcal{O}}_s \leq_s [\![A]\!]^{\gamma,\mathcal{O}}_{s+\alpha}\,, \tag{3.6.4}$$

$$[\![\mathcal{D}^{\pm\alpha}(\Pi_N A)\mathcal{D}^{\mp\alpha}]\!]_s^{\gamma,\mathcal{O}} \leq_s N^{\alpha}[\![A]\!]_s^{\gamma,\mathcal{O}} \,. \tag{3.6.5}$$

Proof. The bounds (3.6.4), (3.6.5) follow by reasoning as in the proof of Lemma $A.1$ in [FG19] and using the (3.2.59). □

Lemma 3.6.2. *Let A be a matrix as in* (3.2.52) *with finite $[\![\cdot]\!]_s$-norm (see* (3.2.55)*). Then (recall* (3.2.54)*) one has*

$$\|A(\varphi)\|_{\mathcal{L}(H^s,H^s)} \leq_s |A(\varphi)|_s \leq_s [\![A]\!]_{s+s_0}\,, \quad \forall\, \varphi \in \mathbb{T}^d\,.$$

Proof. See Lemma 2.4 in [BBM14]. □

Lemma 3.6.3. *Assume that*

$$C(s)[\![A]\!]_{s_0}^{\gamma,\mathcal{O}} \leq 1/2 \tag{3.6.6}$$

for some large $C(s) > 0$ depending on $s \geq s_0$. Then the map $\Phi := \mathrm{Id} + \Psi$ defined as

$$\Phi := e^{\mathrm{i}A} := \sum_{p\geq 0} \frac{1}{p!}(\mathrm{i}A)^p\,, \tag{3.6.7}$$

satisfies

$$[\![\Psi]\!]_s^{\gamma,\mathcal{O}} \leq_s [\![A]\!]_s^{\gamma,\mathcal{O}}\,. \tag{3.6.8}$$

Proof. For any $n \geq 1$, using (3.2.58), we have, for some $C(s) > 0$,

$$\begin{aligned} &[\![A^n]\!]_{s_0} \leq [C(s_0)]^{n-1}[\![A]\!]_{s_0}^n\,,\\ &[\![A^n]\!]_s \leq n[C(s)[\![A]\!]_{s_0}]^{n-1}C(s)[\![A]\!]_s\,, \quad \forall\, s \geq s_0\,. \end{aligned}$$

The same holds also for the norm $[\![\cdot]\!]_s^{\gamma,\mathcal{O}}$. Hence

$$[\![\Psi]\!]_s^{\gamma,\mathcal{O}} \leq [\![A]\!]_s^{\gamma,\mathcal{O}} \sum_{p\geq 1} \frac{C(s)^p}{p!}([\![A]\!]_{s_0}^{\gamma,\mathcal{O}})^{p-1}\,,$$

for some (large) $C(s) > 0$. By the smallness condition (3.6.6) one deduces the bounds (3.6.8). □

Lemma 3.6.4. *Let $\alpha, \beta \in \mathbb{R}$. Then*

$$[\![AM]\!]_{\alpha+\beta,s}^{\gamma,\mathcal{O}} \leq_s [\![A]\!]_{\alpha,s+|\beta|}^{\gamma,\mathcal{O}}[\![M]\!]_{\beta,s_0+|\alpha|}^{\gamma,\mathcal{O}} + [\![A]\!]_{\alpha,s_0+|\beta|}^{\gamma,\mathcal{O}}[\![M]\!]_{\beta,s+|\alpha|}^{\gamma,\mathcal{O}}\,, \tag{3.6.9}$$

$$[\![(\mathrm{Id} - \Pi_N)M]\!]_{\beta,s}^{\gamma,\mathcal{O}} \leq_s N^{-\mathfrak{s}}[\![M]\!]_{\beta,s+\mathfrak{s}}^{\gamma,\mathcal{O}}\,, \qquad \mathfrak{s} \geq 0\,. \tag{3.6.10}$$

Moreover, if $\alpha \le \beta < 0$ *then*

$$[\![AM]\!]^{\gamma,\mathcal{O}}_{\beta,s} \le_s [\![A]\!]^{\gamma,\mathcal{O}}_{\alpha,s}[\![M]\!]^{\gamma,\mathcal{O}}_{\beta,s_0} + [\![A]\!]^{\gamma,\mathcal{O}}_{\alpha,s_0}[\![M]\!]^{\gamma,\mathcal{O}}_{\beta,s} \,. \tag{3.6.11}$$

Proof. To prove (3.6.9) one reasons as in Lemma $A.3$ in [FG19]. The (3.6.11) and (3.6.10) follow by Lemma 3.6.1. □

Lemma 3.6.5. *One has*

$$\|\mathcal{D}^\beta A h\|_{\ell_s} \le_s [\![A]\!]_{\beta,s}\|h\|_{\ell_{s_0}} + [\![A]\!]_{\beta,s_0}\|h\|_{\ell_s} \,, \tag{3.6.12}$$

for any $h \in \ell_s$ *(see* (3.2.15)*) and* $\beta \in \mathbb{R}$.

Proof. One reasons as in Lemma $A.4$ in [FG19]. □

3.6.3 Flows of pseudo differential operators

Lemma 3.6.6. *Fix* $m \le 0$, $0 \le \delta \le 1$, $r > d/2$ *and* $\rho \ge 0$ *and consider* $S_1 \in \mathcal{A}^{\gamma,\mathcal{O}}_{m,r}$ *and* $S_2 \in \mathcal{A}^{\gamma,\mathcal{O}}_{\delta,r}$ *(see Definition 3.2.3). Assume also that*

$$[S_2, K_0] = 0 \,, \quad \langle S_2 h, v\rangle = \langle h, S_2 v\rangle \tag{3.6.13}$$

where $\langle\cdot,\cdot\rangle$ *is the standard* L^2 *scalar product. Let us define*

$$\Phi_1^\tau := \Phi_1^\tau(\varphi) := e^{\tau \mathrm{i} S_1} \,, \qquad \Phi_2^\tau := \Phi_2^\tau(\varphi) := e^{\tau \mathrm{i} S_2} \,. \tag{3.6.14}$$

For any $s \ge 0$ *there are* $\varepsilon_0, C, p > 0$ *such that, for any* $0 < \varepsilon \le \varepsilon_0$, *if*

$$\mathcal{N}^{\gamma,\mathcal{O}}_{m,r,p}(S_1) + \mathcal{N}^{\gamma,\mathcal{O}}_{\delta,r,p}(S_2) \le \varepsilon, \tag{3.6.15}$$

then the following holds true :

(i) *the map* Φ_1^τ *satisfies*

$$\sup_{\varphi\in\mathbb{T}^d} \|\Phi_1^\tau(\varphi) - \mathrm{Id}\|^{\gamma,\mathcal{O}}_{\mathcal{L}(H^s;H^{s-m})} \le C\mathcal{N}^{\gamma,\mathcal{O}}_{m,r,p}(S_1) \,, \tag{3.6.16}$$

$$\sup_{\varphi\in\mathbb{T}^d} \|(\partial_\varphi^k \Phi_1^\tau)(\varphi)\|^{\gamma,\mathcal{O}}_{\mathcal{L}(H^s,H^s)} \le C\mathcal{N}^{\gamma,\mathcal{O}}_{\delta,r,p}(S_1) \,, \quad 0 \le k \le r \,, \tag{3.6.17}$$

for any $\tau \in [0,1]$ *;*

(ii) the map Φ_2^τ satisfies

$$\sup_{\varphi\in\mathbb{T}^d}\|\Phi_2^\tau(\varphi)\|_{\mathcal{L}(H^s,H^s)}\leq(1+C\mathcal{N}^{\gamma,\mathcal{O}}_{\delta,r,p}(S_2))\,, \tag{3.6.18}$$

$$\sup_{\varphi\in\mathbb{T}^d}\|(\Phi_2^\tau(\varphi)-\mathrm{Id})\|_{\mathcal{L}(H^s,H^{s-\delta})}\leq C\mathcal{N}^{\gamma,\mathcal{O}}_{\delta,r,p}(S_2)\,, \tag{3.6.19}$$

$$\sup_{\varphi\in\mathbb{T}^d}\|(\partial_\varphi^k\Phi_2^\tau)(\varphi)\|_{\mathcal{L}(H^s,H^{s-k\delta})}\leq C\mathcal{N}^{\gamma,\mathcal{O}}_{\delta,r,p}(S_2)\,,\quad 1\leq k\leq r\,, \tag{3.6.20}$$

for any $\tau\in[0,1]$ and any $\omega\in\mathcal{O}$. Moreover the following bounds on the Lipschitz norm hold true :

$$\sup_{\varphi\in\mathbb{T}^d}\|\Phi_2^\tau(\varphi)\|^{\gamma,\mathcal{O}}_{\mathcal{L}(H^s,H^{s-1})}\leq(1+C\mathcal{N}^{\gamma,\mathcal{O}}_{\delta,r,p}(S_2))\,, \tag{3.6.21}$$

$$\sup_{\varphi\in\mathbb{T}^d}\|(\Phi_2^\tau(\varphi)-\mathrm{Id})\|^{\gamma,\mathcal{O}}_{\mathcal{L}(H^s,H^{s-\delta-1})}\leq C\mathcal{N}^{\gamma,\mathcal{O}}_{\delta,r,p}(S_2)\,, \tag{3.6.22}$$

$$\sup_{\varphi\in\mathbb{T}^d}\|(\partial_\varphi^k\Phi_2^\tau)(\varphi)\|^{\gamma,\mathcal{O}}_{\mathcal{L}(H^s,H^{s-k\delta-1})}\leq C\mathcal{N}^{\gamma,\mathcal{O}}_{\delta,r,p}(S_2)\,,\quad 1\leq k\leq r\,, \tag{3.6.23}$$

for any $\tau\in[0,1]$.

Proof. We shall prove the result for the map Φ_2^τ. The estimates on Φ_1^τ can be obtained in the same way. Notice that the operator Φ_2^τ solves the problem

$$\begin{cases}\partial_\tau\Phi_2^\tau(\varphi)=\mathrm{i}S_2(\varphi)\Phi_2^\tau(\varphi)\\ \Phi_2^0(\varphi)=\mathrm{Id}\,.\end{cases} \tag{3.6.24}$$

The existence of the flow Φ_2^τ in $\mathcal{L}(H^s,H^s)$ can be obtained following the line of chapter 5 in [Tay91]. Then using (3.6.24) and the assumption (3.6.13) one can check that

$$\partial_\tau\|\Phi_2^\tau h\|^2_{H^s}=0\qquad\Rightarrow\qquad\|\Phi_2^\tau h\|_{H^s}\leq\|h\|_{H^s}\,,$$

for any $\tau\in[0,1]$, $h\in H^s$ and $\varphi\in\mathbb{T}^d$. This is the (3.6.18). Let us now define

$$\Gamma^\tau(\varphi):=\Phi_2^\tau(\varphi)-\mathrm{Id}\,.$$

It solve the problem

$$\partial_\tau\Gamma^\tau(\varphi)=\mathrm{i}S_2(\varphi)\Gamma^\tau(\varphi)+\mathrm{i}S_2(\varphi)\,,\qquad\Gamma^0(\varphi)=0\,.$$

By Duhamel formula have

$$\Gamma^\tau(\varphi)h=\int_0^\tau\Phi_2^\tau\Phi_2^{-\sigma}(\varphi)\mathrm{i}S_2(\varphi)hd\sigma\,.$$

Therefore the bound (3.6.19) follows by (3.6.18) and the estimates on S_2. Similarly the operator $(\partial_\varphi \Phi_2^\tau)(\varphi)$ satisfies

$$\begin{cases} \partial_\tau (\partial_\varphi \Phi_2^\tau)(\varphi) = \mathrm{i} S_2(\varphi)(\partial_\varphi \Phi_2^\tau)(\varphi) + \mathrm{i}(\partial_\varphi S_2)(\varphi)\Phi_2^\tau(\varphi)\,, \\ (\partial_\varphi \Phi_2^0)(\varphi) = 0\,. \end{cases} \tag{3.6.25}$$

We have that

$$\sup_{\varphi \in \mathbb{T}^d} \|(\partial_\varphi S_2)(\varphi)\Phi_2^\tau(\varphi) h\|_{H^{s-\delta}} \lesssim \|h\|_{H^s} \mathcal{N}^{\gamma,\mathcal{O}}_{\delta,r,p}(S_2)$$

by (3.6.18) and the fact that $r > d/2$. Hence, using Duhamel formula and the (3.6.18), we deduce the (3.6.20) for $k = 1$. The (3.6.20) for $k > 1$ can be obtained in the same way by differentiating (3.6.25). The Lipschitz bounds (3.6.21)-(3.6.23) follows reasoning as in the estimates of $\partial_\varphi \Phi_2^\tau(\varphi)$. The bounds (3.6.16), (3.6.17) can be deduced reasoning as done above and using the fact that the generator $\mathrm{i}S_1(\varphi)$ is a *bounded* pseudodifferential operator. □

Lemma 3.6.7. *Let $r_1 \geq 0$ and $r > r_1 + d/2$, $\delta > 0$, $\rho_1 > 0$, $\rho := \rho_1 + \delta r_1 + 1$ and consider $R \in \mathcal{R}^{\gamma,\mathcal{O}}_{\rho,r}$ (see Definition 3.2.4). Consider also the map $\Phi_2(\varphi) := \Phi_2^\tau(\varphi)_{|\tau=1}$, where $\Phi_2^\tau(\varphi)$ is given in Lemma 3.6.6. Then $G_2(\varphi) := \Phi_2(\varphi) R(\varphi) \Phi_2^{-1}(\varphi)$ belongs to $\mathcal{R}^{\gamma,\mathcal{O}}_{\rho_1,r_1}$. Moreover for any $s \geq 0$ there exist p and C such that*

$$|G_2|^{\gamma,\mathcal{O}}_{\rho_1,r_1,s} \leq |R|^{\gamma,\mathcal{O}}_{\rho,r,s}(1 + C\mathcal{N}^{\gamma,\mathcal{O}}_{\delta,r,p}(S_2))\,. \tag{3.6.26}$$

Proof. We need to prove that the map $\varphi \mapsto \Gamma(\varphi)$ is in $H^{r_1}(\mathbb{T}^d; \mathcal{L}(H^s; H^{s+\rho_1}))$. We note that

$$\begin{aligned} |G_2|_{\rho_1,r_1,s} &\lesssim \sum_{k=0}^{r_1} \sup_{\varphi\in\mathbb{T}^d} \|(\partial_\varphi^k G_2)(\varphi)\|_{\mathcal{L}(H^s;H^{s+\rho_1})} \\ &\lesssim \sum_{k=0}^{r_1} \sum_{\substack{k_1+k_2+k_3=k \\ k_i\geq 0}} \sup_{\varphi\in\mathbb{T}^d} \|(\partial_\varphi^{k_1}\Phi_2)(\varphi)(\partial_\varphi^{k_2} R)(\varphi)(\partial_\varphi^{k_3}\Phi_2^{-1})(\varphi)\|_{\mathcal{L}(H^s;H^{s+\rho_1})}\,. \end{aligned} \tag{3.6.27}$$

We estimate separately each summand in (3.6.27). First of all notice that, by the definition of the norm in (3.2.33) and the fact that $r > r_1 + d/2$, one has

$$\sup_{\varphi\in\mathbb{T}^d} \|\partial_\varphi^{k_2}(R(\varphi))\|_{\mathcal{L}(H^s;H^{s+\rho})} \lesssim |R|_{\rho,r,s}\,. \tag{3.6.28}$$

Hence the summand in (3.6.27) with $k_1 = k_3 = 0$ is trivially bounded by the right hand side in (3.6.26). If at least one between k_1, k_2 is different from zero we have, for any $h \in \mathbb{H}^s$,

$$\begin{aligned} &\|(\partial_\varphi^{k_1}\Phi_2)(\varphi)(\partial_\varphi^{k_2}R)(\varphi)(\partial_\varphi^{k_3}\Phi_2^{-1})(\varphi)h\|_{H^{s+\rho_1}} \\ &\overset{(3.6.20)}{\lesssim} \mathcal{N}^{\gamma,\mathcal{O}}_{\delta,r,p}(S_2)\|(\partial_\varphi^{k_2}R)(\varphi)(\partial_\varphi^{k_3}\Phi_2^{-1})(\varphi)h\|_{H^{s+\rho_1+k_1\delta}} \\ &\overset{(3.6.28)}{\lesssim} \mathcal{N}^{\gamma,\mathcal{O}}_{\delta,r,p}(S_2)|R|_{\rho,r,s}\|(\partial_\varphi^{k_3}\Phi_2^{-1})(\varphi)h\|_{H^{s+\rho_1+k_1\delta-\rho}} \\ &\overset{(3.6.15)}{\lesssim} \mathcal{N}^{\gamma,\mathcal{O}}_{\delta,r,p}(S_2)|R|_{\rho,r,s}\|h\|_{H^{s+\rho_1+(k_1+k_3)\delta-\rho}} \,. \end{aligned}$$

Notice that $+\rho_1 + (k_1+k_3)\delta - \rho \le 0$ since $k_1 + k_3 \le r_1$ and that the estimate above is uniform in $\varphi \in \mathbb{T}^d$. Hence, together with the (3.6.27), it implies the (3.6.26) for the norm $|\cdot|_{\rho_1,r_1,s}$. The Lipschitz bounds are obtained similarly taking into account the extra loss of derivatives appearing in the estimates (3.6.21)-(3.6.23). □

Similarly we prove in the bounded case :

Lemma 3.6.8. *Let $r_1 \ge 0$ and $r > r_1 + d/2$, $\rho > 0$ and consider $R \in \mathcal{R}^{\gamma,\mathcal{O}}_{\rho,r}$. Consider also the map $\Phi_1(\varphi) := \Phi_1^\tau(\varphi)_{|\tau=1}$, where $\Phi_1^\tau(\varphi)$ is given in Lemma 3.6.6. Then $G_1(\varphi) := \Phi_1(\varphi)R(\varphi)\Phi_1^{-1}(\varphi)$ belongs to $\mathcal{R}^{\gamma,\mathcal{O}}_{\rho,r}$. Moreover for any $s \ge 0$ there exist p and C such that*

$$|G_1|^{\gamma,\mathcal{O}}_{\rho,r_1,s} \le C|R|^{\gamma,\mathcal{O}}_{\rho,r,s}(1 + \mathcal{N}^{\gamma,\mathcal{O}}_{m,r,p}(S)) \,. \tag{3.6.29}$$

Data Availability. The data that supports the findings of this study are available within the article.

4

Birkhoff normal form for abcd Boussinesq system on the circle

4.1 Introduction

We consider the *abcd*-Boussinesq system

$$\begin{cases} (1-b\partial_{xx})\partial_t\eta + \partial_x(a\partial_{xx}u + u + u\eta) = 0 & (t,x) \in \mathbb{R}\times\mathbb{T} \\ (1-d\partial_{xx})\partial_t u + \partial_x(c\partial_{xx}\eta + \eta + \frac{1}{2}u^2) = 0 \end{cases} \tag{4.1.1}$$

Where η, u are real functions with zero momentum

$$\int_{\mathbb{T}} \eta(t,x)\ \mathrm{d}x = \int_{\mathbb{T}} u(t,x)\ \mathrm{d}x = 0.$$

Here the independent variable x corresponds to distance along the channel and t is proportional to elapsed time. The quantity $\eta = \eta(x,t)$ corresponds to the depth of the water at the point x and time t. The variable $u(x,t)$ is proportional to the horizontal velocity at the height θh, where θ is a fixed constant in the interval $[0,1]$ and h is the undisturbed water depth.

We are interested in the long time behavior of small amplitude solution of (4.1.1). In this paper, we would like to prove a Birkhoff normal form result for (4.1.1). The idea comes from Theorem 2.13 of [BG06] (see also [Bam03], [BDGS07],[Gre07], [Bam08], [Del12]), where the authors prove a Birkhoff normal form result for Hamiltonian with nonlinearity satisfying a tame modulus condition and nonresonant frequencies. In [BG06], the authors applied that result to many partial differential equations : NLW equations, NLS equations in one dimension and higher dimension, coupled NLS in one dimension. In this paper, we state a similar result for Boussinesq systerm.

As in general, the proof of a Birkhoff normal form result is obtained by constructing a sequence of canonical transformations to eliminate the non-normalized part of the nonlinearity step by step. This procedure relates to solving homological equations in each step. The idea in [BG06] is to use a so-called tame inequality

$$\|uv\|_s \leq C(\|u\|_s\|v\|_1 + \|v\|_s\|u\|_1) \tag{4.1.2}$$

and that if $u \in H^s$ depends only on high modes $u = \sum_{|k|\geq N} u_k e^{i2\pi kx}$ then

$$\|u\|_1 \leq \frac{\|u\|_s}{N^{s-1}}. \tag{4.1.3}$$

This term is small when N large enough. Then in the Birkhoff procedure, the nonlinearity, which satisfies the tame properties and has many high modes in expansion, is small and controllable. The study of frequency plays an important role in order to eliminate the remainder part of non-integrable resonant terms. However, contrary [BG06], where external potentials were used to verify the nonresonant condition, we obtain the nonresonant condition by studying frequencies as functions of parameters a, b, c, d.

Our main contribution in this work is to verify that the nonlinearity in Boussinesq system satisfies a tame property and that the frequencies satisfy a nonresonant condition(section 4) then we prove our Birkhoff normal form result(section 5). In section 2, we state our main theorem and explain the scheme of our proof. In section 3, we recall notations of tame property and nonresonant condition, which was introduced in [BG06]. We give the proof of local well-posedness in Appendix B.

The system (4.1.1) was originally derived by Bona, Chen and Saut [BCS02],[BCS04] in the vein of the Boussinesq original derivation [Bou72]. The equation is derived to study the two dimensional, incompressible and irrotational water wave in the shallow water regime. The *abcd* Boussinesq equation and its extensions have been studied extensively in the literature (see [SL08], [BCL05], [BLS08], [LPS12], [SX12], [SX17], [LY97]). Here, the parameters a, b, c, d satisfy the consistency conditions (see [BCS02], [BCS04])

$$a + b + c + d = \frac{1}{3}. \tag{4.1.4}$$

To consider all possible value of the parameters (a, b, c, d) present in the equation is too complicated, here we concentrate our study to the "Hamiltonian generic" case namely the case where

$$b = d > 0, \qquad a, c < 0.$$

In a next paper, we would like to study the Boussinesq equation in the KdV-KdV regime

($b = d = 0$). It is worth saying that to prove a Birkhoff normal form result for this case is much more complicated since in this case the nonlinearity is unbounded and its tame property is failed. However, a recent paper [BG20] gives tools to deal with this problem.

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4.2 Statement of the main theorem

We expand the solution in Fourier variables

$$u(x) = \sum_{k\in\mathbb{Z}^*} u_k e^{2i\pi kx}, \qquad \eta(x) = \sum_{k\in\mathbb{Z}^*} \eta_k e^{2i\pi kx} \tag{4.2.1}$$

note that $\overline{u}_k = u_{-k}$, $\overline{\eta}_k = \eta_{-k}$ since u and η are real, then (4.1.1) reads

$$\begin{cases} \partial_t \eta_k = -\frac{i2\pi k}{1+4\pi^2 bk^2}((1-4\pi^2 ak^2)u_k + \sum_{j+l=k} u_j\eta_l) \\ \partial_t u_k = -\frac{i2\pi k}{1+4\pi^2 bk^2}((1-4\pi^2 ck^2)\eta_k + \frac{1}{2}\sum_{j+l=k} u_j u_l) \end{cases} \quad k\in\mathbb{Z}^*. \tag{4.2.2}$$

Simplify, we denote

$$D_k = \frac{2\pi k}{1+4b\pi^2k^2}, \quad \omega_k = \sqrt{(1-4a\pi^2k^2)(1-4c\pi^2k^2)}$$
$$\Omega_k = D_k\omega_k.$$

Before we state our result, we need to put the system (4.2.2) into a more convenient form by using a linear symplectic change of variables

$$\psi_k = \frac{1}{\sqrt{2}}(\alpha_k u_k + \alpha_k^{-1}\eta_k), \quad \phi_k = \frac{1}{\sqrt{2}}(\alpha_k u_k - \alpha_k^{-1}\eta_k)$$

where $\alpha_k = \left(\frac{1-4a\pi^2k^2}{1-4c\pi^2k^2}\right)^{\frac{1}{4}} = \alpha_{-k}$. Then the system can be written as

$$\begin{cases} \partial_t\psi_k &= -iD_k\nabla_{\psi_{-k}}H \\ \partial_t\phi_k &= iD_k\nabla_{\phi_{-k}}H \end{cases} \quad k\in\mathbb{Z}^*, \tag{4.2.3}$$

where H is the Hamiltonian

$$H = H_0 + P$$

with

$$H_0 = \sum_{k\in\mathbb{N}*} \omega_k(\psi_k\psi_{-k} + \phi_k\phi_{-k})$$
$$P = \frac{1}{4\sqrt{2}} \sum_{j+l+h=0} \alpha_h\alpha_j^{-1}\alpha_l^{-1}(\psi_j + \phi_j)(\psi_l + \phi_l)(\phi_h - \psi_h)$$

Through all the paper, we identify the couple function (ψ, ϕ) with its series of Fourier coefficients $(\psi, \phi) = (\psi_k, \phi_k)_{k\in\mathbb{Z}^*}$. We also define the Sobolev space $(s \geq 0)$

$$H^s := \{z = (\psi, \phi) = (\psi_k, \phi_k)_{k\in\mathbb{Z}^*} \in \ell^2(\mathbb{T}) \times \ell^2(\mathbb{T}) \mid \|z\|_s^2 = \sum_{k\in\mathbb{Z}^*} |k|^{2s}(|\phi_k|^2 + |\psi_k|^2) < \infty\}. \tag{4.2.4}$$

Denote by $N_k = |(\psi_k, \phi_k)|^2 = |\phi_k|^2 + |\psi_k|^2$ the actions of the flow $(\psi, \phi)(t)$. We also denote $I_k = |\phi_k|^2$, $J_k = |\psi_k|^2$ then $N_k = I_k + J_k = \alpha^2|u_k|^2 + \alpha^{-2}|\eta_k|^2$. Denote by $B_s(R)$ the open ball centered at the origin of radius R in $\mathcal{P}_s$.

Poisson bracket In the phase space H^s, we define the Poisson bracket

$$\begin{aligned}\{P, Q\} &= \sum_{k\in\mathbb{Z}^*} D_k\partial_{\psi_k}P\partial_{\psi_{-k}}Q - \sum_{k\in\mathbb{Z}^*} D_k\partial_{\phi_k}P\partial_{\phi_{-k}}Q \\ &= \sum_{k\in\mathbb{N}^*} D_k(\partial_{\psi_k}P\partial_{\psi_{-k}}Q - \partial_{\psi_{-k}}P\partial_{\psi_k}Q) - \sum_{k\in\mathbb{N}^*} D_k(\partial_{\phi_k}P\partial_{\phi_{-k}}Q - \partial_{\phi_{-k}}P\partial_{\phi_k}Q).\end{aligned}$$

so that the equation (4.2.3) reads

$$\begin{cases} i\partial_t\psi_k &= \{\psi_k, H\} \\ i\partial_t\phi_k &= \{\phi_k, H\} \end{cases} \qquad k \in \mathbb{Z}^*. \tag{4.2.5}$$

Frequencies Let us write $\Omega_k = \frac{2\pi k}{1+4b\pi^2k^2}\sqrt{pk^4 + ek^2 + 1}$ where $p = 16\pi^4 ac$, $e = -4\pi^2(a + c)$. Here we assume that $b = d \geq 0$, and we consider two different cases : the "generic" case $b = d > 0$, $a, c < 0$ and the Kdv-Kdv case $b = d = 0$, $a, c > 0$.

In the generic case, since $a + c + 2b = \frac{1}{3}$, we have $b = \frac{1}{6} - (a + c)/2 > \frac{1}{6}$ and

$$0 < p = 16\pi^4 ac \leq 16\pi^4(b - \frac{1}{6})^2 \qquad e = -4\pi^2(a + c) = 4\pi^2(2b - \frac{1}{3}) \geq 0.$$

We denote $\mathcal{I}_b := (0, p_0 = 16\pi^4(b - \frac{1}{6})^2)$. In the Kdv-Kdv case, in order for frequencies to be real, we assume $a, c > \frac{1}{4\pi^2}$. Since $a + c = \frac{1}{3}$, we have

$$\frac{4}{3}\pi^2 - 1 < p \leq \frac{4}{9}\pi^4.$$

Denote $\mathcal{I}_0 = (\frac{4}{3}\pi^2 - 1, \frac{4}{9}\pi^4]$.

For any $r \geq 3$, we define

$$\alpha_* := (\frac{7}{2}r^3 + r^2 + 7r)^2;\ N_*(r,\alpha,\mu) := [\mu^{-\frac{1}{2r\alpha}}];\ \ s_*(r,\alpha) = 2\alpha r^2 + 2.$$

Then our main result is stated as follow

Theorem 4.2.1. *Let $r \geq 1$, $0 < \mu \ll 1$. Then for any $s \geq s_*, \alpha > \alpha_*$ there exists a subset $\mathcal{I}^\mu \subset \mathcal{I}_b$ and a constant $C = C(r,s,b)$ such that*

$$|\mathcal{I}^\mu - \mathcal{I}_b| \leq C|\mathcal{I}_b|\mu^{\frac{\alpha+1}{r+2}} \tag{4.2.6}$$

and for any $p \in \mathcal{I}^\mu$, for $|t| \leq \mu^{-r+3/2}$, there exists a transformation $\mathcal{T} : B_s(\mu/3) \to B_s(\mu)$ satisfying

$$H \circ \mathcal{T} = H_0 + Z + \mathcal{R}. \tag{4.2.7}$$

here Z is a polynomial of degree at most $r+2$ that commutes with the actions N_k, i.e.,

$$\{Z, N_k\} = 0 \qquad\qquad \forall k \in \mathbb{Z}^* \tag{4.2.8}$$

and $\mathcal{R} \in C^\infty(B_s(\mu))$ fulfills the estimate

$$\sup_{\|(\psi,\phi)\|_s \leq \mu/3} \|X_{\mathcal{R}}\|_s \leq C\mu^{r+\frac{3}{2}}. \tag{4.2.9}$$

The canonical transformations and its inverse are close to identity

$$\sup_{\|(\phi,\psi)\|_s \leq \mu/3} \|(\phi,\psi) - \mathcal{T}(\phi,\psi)\|_s \leq C_s\mu^2 \qquad \sup_{\|(\phi,\psi)\|_s \leq \mu/3} \|(\phi,\psi) - \mathcal{T}^{-1}(\phi,\psi)\|_s \leq C_s\mu^2. \tag{4.2.10}$$

Corollary 4.2.2. *Fix $r \geq 1$, assume Ω is nonresonant, then for any s sufficiently large, there exist μ_0 and c such that if the initial data is in $B_s(\mu)$, $\mu \leq \mu_0$ then*

— $\|(u,\eta)(t)\|_s \leq 2\mu$ *for* $t \leq c\mu^{-r+3/2}$

— $\alpha_k^2(|u_k|^2(t) - |u_k|^2(0)) + \alpha_k^{-2}(|\eta_k(t)|^2 - |\eta_k(0)|^2) \leq \frac{\mu^3}{k^{2s}}$ *for* $t \leq c\mu^{-r+3/2};\ k \in \mathbb{Z}^*$.

The difference between the "generic" system and the Kdv-Kdv system comes from D_k. In the generic case $b > \frac{1}{6}$, we have $|D_k| \leq 1$, which makes it easy to prove tame property, while in the Kdv-Kdv case $D_k = 2\pi k$ is unbounded so that the nonlinear term P does not satisfies tame property. In the fullness of time, we would like to study this case, which is much more complicated.

4.2.1 Scheme of the proof

In order to prove the Theorem 4.2.1, we search iterative changes of variables $\mathcal{T}_n$, $1 \leq n \leq r$, such that

$$H \circ \mathcal{T}_n = H_0 + Z_4(I, J) + Z_6(I, J, Y) + \cdots + Z_n(I, J, Y) + R_{n+1}. \tag{4.2.11}$$

Where R_{n+1} is a perturbation term which is of the size $\mu^{n+\frac{3}{2}}$ and

$$I = (I_k)_{k\in\mathbb{Z}^*}, \quad J = (J_k)_{k\in\mathbb{Z}^*}, \quad Y = (\prod_{\sum k_j=0} Y_{k_j}),\ Y_k = \psi_k \phi_k.$$

Here we call Y_k the pseudo actions, which satisfy $\{Y_k, H_0\} = 0,\ k \in \mathbb{Z}^*$. Moreover

$$\{I_k + J_k, \prod_{\sum k_j=0} Y_{k_j}\} = 0 \qquad \forall k, k_j \in \mathbb{Z}^*. \tag{4.2.12}$$

The condition $\sum k_j = 0$ appear naturally since H comes from an integration of a real function on the circle. As a consequence of the appearance of Y, the exchange of energy mainly occurs between I_k and J_k for a long time $|t| \leq \mu^{-r+\frac{3}{2}}$, $\forall k \in \mathbb{Z}^*$.

4.3 Normal form and tame modulus

In this section, we recall some notations and lemmas used in [BG06].

4.3.1 Nonresonant condition

Definition 4.3.1. Fix two positive parameters κ and α, and a positive integer N. A function Z is said to be in $(\kappa, \alpha, N)-$normal form with respect to Ω if $Z_{kl} \neq 0$ implies

$$|\Omega(k-l)| < \frac{\kappa}{N^\alpha} \quad \textbf{and} \quad \sum_{|j|\geq N+1} |k_j| + |l_j| \leq 2.$$

Definition 4.3.2 (Nonresonant condition)**.** Let r be a positive integer, we say that the frequency Ω is nonresonant if there exist $\kappa > 0$, and $\alpha \in \mathbb{R}$ such that for any N large enough one has

$$|\sum_{j\in\mathbb{N}^*} \Omega_j k_j| \geq \frac{\kappa}{N^\alpha} \tag{4.3.1}$$

for any $k \in \mathbb{Z}^\infty$, fulfilling $0 \neq |k| := \sum_j |k_j| \leq r+2$, $\sum_{|j|>N} |k_j| \leq 2$.

Remark 4.3.3. If the frequency Ω is nonresonant then any polynomial which is in $(\kappa, \alpha, N)-$normal form depends only on I, J and Y.

Example 4.3.4. If the frequency Ω is nonresonant then a polynomial of order 6, which is in $(\kappa, \alpha, N)-$normal form with respect to Ω would be

$$Z_6 = \sum_{k,l \in \mathbb{Z}^*} a_{k,l} Y_k Y_l Y_{-k-l} + \sum_{k,l,h \in \mathbb{Z}^*} b_{k,l,h} I_k I_l I_h + c_{k,l,h} I_k I_l J_h + d_{k,l,h} I_k J_l J_h + e_{k,l,h} J_k J_l J_h$$

where $a_{k,l}, b_{k,l,h}, c_{k,l,h}, d_{k,l,h}, e_{k,l,h}$ are real constants.

4.3.2 Tame modulus

Let us consider a homogeneous polynomial of degree $r+1$ $f : \mathcal{P}_s \to \mathbb{R}$, which can be written as

$$f(z) := \sum_{|j|=r+1} f_j z^j, \qquad j = (\ldots, j_{-l1}, j_{-l2}, \ldots, j_{-11}, j_{-12}, j_{11}, j_{12}, \ldots, j_{l1}, j_{l2}, \ldots)$$
$$z^j := (\psi, \phi)^j = \ldots \psi_{-l}^{j_{-l1}} \phi_{-l}^{j_{-l2}} \ldots \psi_l^{j_{l1}} \phi_l^{j_{l2}} \ldots, \qquad |j| := \sum_l (|j_{l1}| + |j_{l2}|),$$

We can associate f to a symmetric $r+1-$linear form $\tilde{f}$ as

$$f(z) = \tilde{f}(z^{(1)}, \ldots, z^{(r+1)}) = \sum_{|j|=r+1} f_{j_1, \ldots, j_{r+1}} z_{j_1}^{(1)} \ldots z_{j_{r+1}}^{(r+1)}, \qquad j_l = (j_{l1}, j_{l2}).$$

We say that f is bounded if there exists a constant C such that

$$|f(z)| \le C \|z\|_s^{r+1}, \qquad \forall z \in \mathcal{P}_s,$$

or equivalently

$$|\tilde{f}(z^{(1)}, \ldots, z^{(r+1)})| \le C \|z^{(1)}\|_s \ldots \|z^{(r+1)}\|_s.$$

Now consider $X_f := (\frac{2\pi k}{1+4\pi^2 bk^2} \partial_{\psi_k} f, -\frac{2\pi k}{1+4\pi^2 bk^2} \partial_{\phi_k} f)$ the vector field of f. We write it as

$$X_f(z) = \sum_l X_l(z) e_l$$

where e_l is l-th standard basis of $\mathcal{P}_s$ and $X_l(z)$ is a real valued homogeneous polynomial of degree r. We also consider the $r-$linear form $\tilde{X}_f$ so that $X_f(z) = \tilde{X}_f(z, z, \ldots, z)$.

Definition 4.3.5. Let f is a homogeneous polynomial of degree r, we define its modulus

$\lfloor f \rceil$ by

$$\lfloor f \rceil := \sum_{|j|=r} |f_j| z^j.$$

We also define the modulus of its vector field X_f as

$$\lfloor X_f \rceil := \sum_{l \in \mathbb{Z}\setminus\{0\}} \lfloor X_l \rceil e_l.$$

Definition 4.3.6. Let $s \geq 1$, a homogeneous polynomial vector field X of degree r is said to be an $s-$tame map if there exists a constant C_s such that

$$\|\tilde{X}(z^{(1)}, \dots, z^{(r)})\|_s \leq C_s \frac{1}{r} \sum_{l=1}^{r} \|z^{(1)}\|_1 \dots \|z^{(l-1)}\|_1 \|z^{(l)}\|_s \|z^{(l+1)}\|_1 \dots \|z^{(r)}\|_1. \tag{4.3.2}$$

If a map is s-tame for any $s \geq 1$ then it is said to be tame. Moreover if X is the vector field of a homogeneous polynomial f and $\lfloor X_f \rceil$ is an $s-$tame map also, then we define

$$|f|_s := sup \frac{\|\tilde{X}_{\lfloor f \rceil}(z^{(1)}, \dots, z^{(r)})\|_s}{\frac{1}{r}\sum_{l=1}^{r} \|z^{(1)}\|_1 \dots \|z^{(l-1)}\|_1 \|z^{(l)}\|_s \|z^{(l+1)}\|_1 \dots \|z^{(r)}\|_1}$$

the s tame norm of f. We denote by T^s_M the set of function has finite s tame norm.

Remark 4.3.7. If $f \in T^s_M$ is a homogeneous polynomial of degree $r+1$ then one has

$$\|X_f(z)\|_s \leq \|X_{\lfloor f \rceil}\|_s \leq |f|_s \|z\|_1^{r-1} \|z\|_s. \tag{4.3.3}$$

The important property of s tame norm is that the Poisson bracket of two homogeneous polynomials in T^s_M is also in T^s_M (see lemma 4.12 in [BG06]).

Lemma 4.3.8. *Assume that $f, g \in T^s_M$ are two homogeneous polynomials of degrees $n+1$ and $m+1$ respectively, then $\{f, g\}$ is a homogeneous polynomial of order $n+m$ in T^s_M with*

$$|\{f, g\}|_s \leq 2(n+m)|f|_s |g|_s. \tag{4.3.4}$$

For a non homogeneous polynomial, we consider its Taylor expansion

$$f = \sum f_r$$

where f_r is homogeneous function of degree r. Then we define

$$T_{s,R} := \{f = \sum f_r \mid f_r \in T^s_M;\ \langle |f| \rangle_{s,R} := \sum_{r \geq 2} |f_r|_s R^{r-1} < \infty\}. \tag{4.3.5}$$

Following lemma 4.13 in [BG06] we have

Lemma 4.3.9. *Let $f, g \in T_{s,R}$, then for any $d < R$ we have $\{f; g\} \in T_{s,R-d}$ and*

$$\langle|\{f; g\}|\rangle_{s,R-d} \leq \frac{2}{d}\langle|f|\rangle_{s,R}\langle|g|\rangle_{s,R}.$$

Proof. Write $f = \sum_j f_j$ and $g = \sum_k g_k$ with f_j homogeneous of degree j and similarly for g. One has

$$\{f, g\} = \sum_{j,k}\{f_j, g_k\}.$$

We have estimate for each term of the series

$$\begin{aligned}
\langle|\{f_j; g_k\}|\rangle_{s,R-d} &= |\{f_j, g_k\}|_s(R-d)^{j+k-3} \\
&\leq 2|f_j|_s|g_k|_s(j+k-2)(R-d)^{j+k-3} \\
&\leq 2|f_j|_s|g_k|_s\frac{1}{d}R^{j+k-2} \\
&= \frac{2}{d}\langle|f_j|\rangle_{s,R}\langle|g_k|\rangle_{s,R}
\end{aligned}$$

here we use the estimate

$$k(R-d)^{k-1} < \frac{R^k}{d}, \qquad \forall 0 < d < R.$$

□

Then as in lemma 4.14 in [BG06], we deduce

Lemma 4.3.10. *Let $g, \chi \in T_{s,R}$ be two analytic functions. Denote $g_l = \{g_{l-1}, \chi\}$ for all $l > 0$ with $g_0 = g$. Then for any $0 < d < R$, one has $g_n \in T_{s,R-d}$ and*

$$\langle|g_n|\rangle_{s,R-d} \leq \langle|g|\rangle_{s,R}(\frac{2e}{d}\langle|\chi|\rangle_{s,R})^n. \tag{4.3.6}$$

Proof. Denote $\delta = \frac{d}{n}$. Apply iteratively Lemma 4.3.9, one has

$$\langle|g_l|\rangle_{s,R-\delta l} \leq \frac{2}{\delta}\langle|g_{l-1}|\rangle_{s,R-\delta l-1}\langle|\chi|\rangle_{s,R} \leq \cdots \leq \frac{2^n}{n!}(\frac{n}{d}\langle|\chi|\rangle_{s,R})^n\langle|g|\rangle_{s,R}. \tag{4.3.7}$$

Using the inequality $n^n < N!e^n$, one has the thesis.

□

Lemma 4.3.11. *Let χ, g be two analytic functions with Hamiltonian vector fields analytic* *in* $B_s(R)$. *Fix* $0 < d < R$ *assume* $\|X_\chi\|_{s,R} < \frac{d}{3}$, *then for* $|t| \leq 1$, *one has*

$$\sup_{\|z\|_s \leq R-d} \|\varphi_\chi^t(z) - z\|_s \leq \|X_\chi\|_{s,R} \tag{4.3.8}$$

$$\|X_{g\circ\varphi_\chi^t}\|_{s,R-d} \leq (1 + \frac{3}{d}\|X_\chi\|_{s,R})\|X_g\|_{s,R}. \tag{4.3.9}$$

For the proof see [Bam99] proof of lemma 8.2

Remark 4.3.12. For any positive integer number N, we write $z = \overline{z} + \widehat{z}$ with $\overline{z} = (\phi_j, \psi_j)_{|j|\leq N}$ and $\widehat{z} = (\phi_j, \psi_j)_{|j|\geq N}$. Then if homogeneous polynomial $f \in T_M^s$ of order $r+1$ has a zero of order three in $\widehat{z}$, one has

$$\begin{aligned} \|X_f\|_{s,R} &\leq |f|_s(\frac{1}{r}\sum_{l=1}^{r} \|z^{(1)}\|_1 \ldots \|z^{(l-1)}\|_1\|z^{(l)}\|_s\|z^{(l+1)}\|_1 \ldots \|z^{(r)}\|_1) \\ &\leq |f|_s\|\widehat{z}\|_1\|z\|_1^{r-3}\|z\|_s \leq \frac{|f|_s R^r}{N^{s-1}} = \frac{\langle|f|\rangle_{s,R}}{N^{s-1}}. \end{aligned} \tag{4.3.10}$$

Here we use estimates

$$\begin{aligned} &\|\widehat{z}\|_1 \leq \frac{\|z\|_s}{N^{s-1}} \\ &\|\overline{z}\|_s \leq \|z\|_s, \qquad \|\widehat{z}\|_s \leq \|z\|_s \\ &\|z\|_1 \leq \|z\|_s. \end{aligned}$$

For a non homogeneous polynomial $f \in T_{s,R}$ of order less or equal $r+2$

$$f = \sum_{j\leq r+2} f_j$$

then

$$\|X_f\|_{s,R} \leq \sum_{j\leq r+2} \|X_{f_j}\|_{s,R} \leq \frac{\sum_{j\leq r+2} |f_j|_s R^{j-1}}{N^{s-1}} = r\frac{\langle|f|\rangle_{s,R}}{N^{s-1}}. \tag{4.3.11}$$

4.3.3 Homological equation

In each step of Birkhoff normal form procedure, we need to solve a homological equation

$$-i\{H_0, \chi\} + Z = f \tag{4.3.12}$$

with Z is in $(\kappa, \alpha, N)-$normal form. Then after each step, we have the new nonlinearity term of size $\{Z_4, \chi\} + \{f, \chi\}$. Then we need to estimate the nonlinearity χ and the normal form Z.

Lemma 4.3.13. *Let f be a polynomial in T_M^s which is at most quadratic in the variables $\hat{z}$. Assume nonresonant condition, then there exist $\chi, Z \in T_{s,R}$ with Z in $(\kappa, \alpha, N)-$normal form solving*

$$\{H_0, \chi\} + Z = f. \tag{4.3.13}$$

Moreover Z and χ satisfy the estimates

$$\langle|\chi|\rangle_{s,R} \leq \frac{N^\alpha}{\kappa}\langle|f|\rangle_{s,R}, \qquad \langle|Z|\rangle_{s,R} \leq \langle|f|\rangle_{s,R}. \tag{4.3.14}$$

For the proof see [BG06] proof of lemma 4.7.

4.3.4 Remind the idea of [BG06]

Here we recall the idea of the proof of the Theorem 2.13 in [BG06]. The proof consists in using iterative Lie transformations. Consider an Hamiltonian function χ with the corresponding Hamiltonian equation

$$(\dot{\psi}, \dot{\phi}) = X_\chi((\psi, \phi)).$$

Denote by φ_χ^t the corresponding flow and call $\varphi_\chi = \varphi_\chi^1$ the Lie transform generated by χ. Then for any polynomial g, we have an important property

$$\frac{d}{dt}(g \circ \varphi_\chi^t) = \{g, \chi\} \circ \varphi_\chi^t.$$

To prove the theorem we find Hamiltonian functions χ_n and Hamiltonians H_n such that

$$\begin{cases} H_n = H_0 + Z_n + f_n + \mathcal{R}_n \\ H_{n+1} = H_n \circ \varphi_{\chi_n} \quad \tau = \varphi_{\chi_0} \circ \cdots \circ \varphi_{\chi_r} \\ H_{n+1} = H_n - \{H_n, \chi_n\} + \sum_{n\geq 2} \frac{(-1)^n}{n!} ad_{\chi_n}(H_n) \\ Z_{n+1} = Z_n + f_n + -\{H_0, \chi_n\} \\ f_{n+1} = -\{f_n, \chi_n\} + \sum_{l\geq 2}^r \frac{-1^l}{l!} ad_{\chi_n}^l (H_0 + Z_n + f_n) \\ \mathcal{R}_{n+1} = \mathcal{R}_n \circ \varphi_{\chi_n} + \frac{(-1)^{r+1}}{(r+1)!} \int_0^1 ad_{\chi_n}^{r+1}(H_0 + Z_n + f_n) \circ \varphi_{\chi_n}^\tau d\tau. \end{cases} \tag{4.3.15}$$

Here f_n is a polynomial of order r, $\mathcal{R}_n$ is a small term. This leads us to solve the homological equation

$$\{H_0, \chi\} + Z = f \tag{4.3.16}$$

with Z is in $(\kappa, \alpha, N)-$normal form with respect to Ω. Then by [BG06] if $f_0 = P$ has the tame modulus property and frequencies Ω_k satisfy a nonresonant condition, there exists χ solving (4.3.16) and

$$\langle|X_\chi|\rangle_{s,R} \leq C\frac{N^\alpha}{\kappa}\langle|X_f|\rangle_{s,R}, \qquad \langle|X_Z|\rangle_{s,R} \leq C\langle|X_f|\rangle_{s,R}. \tag{4.3.17}$$

In our case, we formally expand f in Taylor series

$$f(\phi,\psi) = \sum_{j,l} f_{j,l}\Pi_{k,h}\psi_k^{j_k}\phi_h^{l_h}$$

and similarly for χ, Z. The equation (4.3.16) becomes

$$(\Omega(j-l))\chi_{j,l} + Z_{j,l} = \sum_{k\in\mathbb{N}^*}(\Omega_k(j_k - j_{-k} - l_k + l_{-k}))\chi_{j,l} + Z_{j,l} = f_{j,l}. \tag{4.3.18}$$

Here we use

$$\{H_0, \psi_k^{j_k}\phi_h^{l_h}\} = (\Omega_k j_k - \Omega_h l_h)\psi_k^{j_k}\phi_h^{l_h}, \qquad \Omega_k = -\Omega_{-k}.$$

Then when Ω satisfies the nonresonant condition, we define the solution for the homological equation as following

$$\chi_{jl} := \frac{f_{jl}}{(\Omega(j-l))} \quad \textbf{with} \quad |\Omega(j-l)| \geq \frac{\kappa}{N^\alpha}$$
$$Z_{jl} := f_{jl} \quad \textbf{with} \quad |\Omega(j-l)| < \frac{\kappa}{N^\alpha}.$$

4.4 Boussinesq equation's properties

4.4.1 Nonlinearity

Now consider

$$P = \sum_{k_1+k_2+k_3=0} \alpha_{k_1}^{-1}\alpha_{k_2}^{-1}\alpha_{k_3}(\psi_{k_1}+\phi_{k_1})(\psi_{k_2}+\phi_{k_2})(\psi_{k_3}-\phi_{k_3}).$$

We prove that P has tame modulus.

Lemma 4.4.1. *The nonlinearity P has tame modulus.*

Proof. One has that its modulus is

$$\lfloor P\rceil = \sum_{k_1+k_2+k_3=0} |\alpha_{k_1}^{-1}\alpha_{k_2}^{-1}\alpha_{k_3}|(\psi_{k_1}+\phi_{k_1})(\psi_{k_2}+\phi_{k_2})(\psi_{k_3}+\phi_{k_3})$$

and

$$\lfloor \tilde{X}_P \rceil = (|D_k|\partial_{\phi_{-k}}\lfloor P\rceil, |D_k|\partial_{\phi_{-k}}\lfloor P\rceil)_{k\in\mathbb{Z}^*}$$

with

$$\begin{aligned}
|D_k|\partial_{\phi_k}\lfloor P\rceil = |D_k|\partial_{\psi_k}\lfloor P\rceil &= \frac{1}{2\sqrt{2}}|D_k|\left(\sum_{j+l+k=0}\frac{1}{2}|\alpha_k\alpha_j^{-1}\alpha_l^{-1}|(\psi_j+\phi_j)(\psi_l+\phi_l)\right)\\
&+\frac{1}{2\sqrt{2}}|D_k|\left(\sum_{j+l+k=0}|\alpha_k^{-1}\alpha_j^{-1}\alpha_l|(\psi_j+\phi_j)(\psi_l+\phi_l)\right)\\
&=\frac{1}{2\sqrt{2}}|D_k|\sum_{j+l+k=0}(\frac{1}{2}|\alpha_k\alpha_j^{-1}\alpha_l^{-1}|+|\alpha_k^{-1}\alpha_j^{-1}\alpha_l|)(\psi_j+\phi_j)(\psi_l+\phi_l).
\end{aligned}$$

In the case $b \neq 0$, we have that $|D_k| = \frac{|2\pi k|}{1+4\pi^2 bk^2} \leq \frac{1}{2\pi|b|}$. Since $|\alpha_k| \leq 1$, we verify $s-$tame map property of $\lfloor X_P\rceil$ as

$$\begin{aligned}
\|\lfloor X_P\rceil\|_s^2 &= \frac{1}{4}\sum_{k\in\mathbb{Z}^*}\langle k\rangle^{2s}|D_k|^2\Big(\sum_{j+l+k=0}(\frac{1}{2}|\alpha_k\alpha_j^{-1}\alpha_l^{-1}|+|\alpha_k^{-1}\alpha_j^{-1}\alpha_l|)(\psi_j+\phi_j)(\psi_l+\phi_l)\Big)^2\\
&\leq \frac{9}{64\pi^2 b^2}\sum_{k\in\mathbb{Z}^*}\langle k\rangle^{2s}\Big(\sum_{j+l+k=0}(\psi_j+\phi_j)(\psi_l+\phi_l)\Big)^2\\
&\leq \frac{9}{16\pi^2 b^2}\left(\|z^{(1)}\|_s^2\|z^{(2)}\|_1^2+\|z^{(1)}\|_1^2\|z^{(2)}\|_s^2\right)
\end{aligned}$$

i.e.

$$\|\lfloor X_P\rceil\|_s \leq \frac{3}{4\pi|b|}\left(\|z^{(1)}\|_s\|z^{(2)}\|_1+\|z^{(1)}\|_1\|z^{(2)}\|_s\right) \tag{4.4.1}$$

So that P has tame modulus. □

4.4.2 Frequencies

We now consider the frequencies

$$\Omega_k = 2\pi\frac{k}{2+4\pi^2bk^2}\sqrt{(1-4a\pi^2k^2)(1-4c\pi^2k^2)} = 2\pi\frac{k}{1+4\pi^2bk^2}\sqrt{pk^4+ek^2+1}$$

where

$$e=-4\pi^2(a+c)>0,\quad p=16\pi^4ac\in(0,p_0=16\pi^4(\frac{1}{6}-b)^2)=\mathcal{I}_b.$$

The parameter s and p are well-defined, positive and independent. In this subsection, we study frequencies Ω_k as functions of p. More precisely, the goal of this part is to prove the following theorem, which has been demonstrated in slightly different contexts in [Bam03],[BG06] and [EGK16].

Theorem 4.4.2. *There exists a set $\mathcal{J} \in \mathcal{I}$ of full measure such that for any $p \in \mathcal{J}$, fixed $r \geq 1$, there exist $\kappa = \kappa(r,p) > 0$, $\alpha = \alpha(r,p) > 0$ such that for any N large enough then*

$$|\sum_{j\geq 1} \Omega_j k_j| \geq \frac{\kappa}{N^\alpha}, \tag{4.4.2}$$

for any $k \in \mathbb{Z}^\infty$, fulfilling $0 \neq |k| = \sum_j |k_j| \leq r+2$, $\sum_{|j|>N} |k_j| \leq 2$.

In order to prove Theorem 4.4.2, we need to construct some lemmas.

Lemma 4.4.3. *For any $r \leq N$, consider r indexes $j_1 < j_2 < j_3 < ... j_r \leq N$; consider the determinant*

$$D =: \begin{vmatrix} \Omega_{j_1} & \Omega_{j_2} & \dots & \Omega_{j_r} \\ \frac{d\Omega_{j_1}}{dp} & \frac{d\Omega_{j_2}}{dp} & \dots & \frac{d\Omega_{j_r}}{dp} \\ . & . & \dots & . \\ . & . & \dots & . \\ \frac{d^{r-1}\Omega_{j_1}}{dp^{r-1}} & \frac{d^{r-1}\Omega_{j_2}}{dp^{r-1}} & \dots & \frac{d^{r-1}\Omega_{j_r}}{dp^{r-1}} \end{vmatrix}$$

Then

$$\begin{aligned} D &= \pm \left[\prod_{j=1}^{r-1} \frac{(2j-3)!}{2^{j-2}(j-2)!2^j}\right] \left(\prod_{\ell=1}^{r} \Omega_{j_\ell}\right) \left(\prod_{1\leq \ell<k\leq r} (x_{j_\ell} - x_{j_k})\right) \\ &\geq C(r,b)\frac{1}{N^{\frac{1}{2}(5r^2-3r)}}. \end{aligned}$$

Here $x_j = \frac{j^4}{pj^4+ej^2+1}$.

Proof. We compute the iterative derivatives of Ω_k with respect to p in the spirit of section 3 in [EGK16]

$$\begin{aligned} \frac{d^j\Omega_k}{dp^j} &= 2\pi \frac{k}{1+4\pi^2 bk^2} \frac{(2j-1)!}{2^{j-1}(j-1)!2^j} \frac{(-1)^j k^{4j}}{(pk^4+ek^2+1)^{j-\frac{1}{2}}} \\ &= \frac{(-1)^j(2j-1)!}{2^{j-1}(j-1)!2^j} \Omega_k x_k^j. \end{aligned}$$

Substituting this into the determinant then

$$D = \pm \left[\prod_{j=1}^{r} \frac{(2j-3)!}{2^{j-2}(j-2)!2^j}\right]\left(\prod_{\ell=1}^{r}\Omega_{j_\ell}\right) \times \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ x_{j_1} & x_{j_2} & x_{j_3} & \dots & x_{j_r} \\ x_{j_1}^2 & x_{j_2}^2 & x_{j_3}^2 & \dots & x_{j_r}^2 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ x_{j_1}^{r-1} & x_{j_2}^{r-1} & x_{j_3}^{r-1} & \dots & x_{j_r}^{r-1} \end{vmatrix}$$

$$= \pm \left[\prod_{j=1}^{r} \frac{(2j-3)!}{2^{j-2}(j-2)!2^j}\right]\left(\prod_{\ell=1}^{r}\Omega_{j_\ell}\right)\left(\prod_{1\le \ell<k\le r}(x_{j_\ell}-x_{j_k})\right).$$

Since $b \ge \frac{1}{6}$, we have estimate

$$|\prod_{\ell=1}^{r}\Omega_{j_\ell}| = \prod_{\ell=1}^{r} 2\pi\frac{|j_\ell|}{1+4\pi^2 b j_\ell^2}\sqrt{p j_\ell^4 + e j_\ell^2 + 1} \ge \frac{1}{(4\pi b)^r}\prod_{\ell=1}^{r}\frac{1}{|j_\ell|}$$

and

$$\prod_{1\le\ell<k\le r}|(x_{j_\ell}-x_{j_k})| = \prod_{1\le\ell<k\le r}\frac{|j_\ell^2+j_k^2+e j_\ell^2 j_k^2||j_\ell^2-j_k^2|}{(p j_\ell^4+e j_\ell^2+1)(p j_k^4+e j_k^2+1)}$$
$$\ge \prod_{1\le\ell<k\le r}\frac{2(|j_\ell|+|j_k|)}{(p+e+1)^2 j_\ell^3 j_k^3}$$
$$\ge \prod_{1\le\ell<k\le r}\frac{2(|j_\ell|+|j_k|)}{(4\pi^2(b-\frac{1}{6})+1)^4 j_\ell^3 j_k^3}.$$

Then

$$D \ge |\prod_{j=1}^{r}\frac{(2j-3)!}{2^{j-2}(j-2)!2^j}\frac{1}{(4\pi b)^r}\prod_{\ell=1}^{r}\frac{1}{|j_\ell|}\prod_{1\le\ell<k\le r}\frac{2(|j_\ell|+|j_k|)}{(4\pi^2(b-\frac{1}{6})+1)^4 j_\ell^3 j_k^3} \ge C(r,b)\frac{1}{N^{\frac{1}{2}(5r^2-3r)}}$$

where

$$C(r,b,p) = |\prod_{j=1}^{r}\frac{(2j-3)!}{2^{j-2}(j-2)!2^j}|\frac{1}{(4\pi b)^r}\frac{2^{r(r-1)/2}}{(4\pi^2(b-\frac{1}{6})+1)^{2r(r-1)}}.$$

□

Now we recall a result from [BGG85] appendix B

Lemma 4.4.4. *Let $u^{(1)}, \cdots, u^{(r)}$ be r independent vectors in $\mathbb{R}^r$ of norm at most one,* *and let $w \in \mathbb{R}^r$ be any non-zero vector. Then there exists $i \in [1, \cdots, r]$ such that*

$$|\langle u^{(i)}, w\rangle| \geq \frac{|w||det(u^{(1)}, \cdots, u^{(r)})|}{r^{\frac{3}{2}}}.$$

By [E02] (see also [EGK16 ; XYQ97]), we have the lemma

Lemma 4.4.5. *Suppose that $g(p)$ be a C^r-smooth function on an interval $\mathcal{J} \subset \mathbb{R}$ such that $|g^{'}(p)|_{C^{r-1}} \leq \beta$. Let $J_h := \{p \in \mathcal{J} : \ |g(p)| < h\}$, $h > 0$. If $\max_{1\leq k\leq r} \min_p |g^{(k)}(p)| \geq d$ then $|J_h| \leq C_r(\frac{\beta}{h} + 1)(\frac{h}{d})^{1/r}$.*

Now let us consider the function $g(p) = |k|^{-1}(\sum_{|a|=1}^N k_a\Omega_a + c)$, with $|k| \leq r < N$, c is a fixed constant, then $|g^{'}(p)|_{C^n} \leq C(n, b)$. Define

$$R_{kc}(\kappa, \alpha) := \{p \in \mathcal{I}_b \mid |g(p)| \leq \frac{\kappa}{|k|N^\alpha}\}$$

By lemma 4.4.4 one has $\max_{1\leq k\leq r} \min_p |\partial^k g(p)| \geq C(r, b)N^{-\frac{1}{2}(5r^2-3r)-2}$ so that

$$|R_{kc}(\kappa, \alpha)| \leq \frac{2}{C(r, b)}\kappa^{1/r}N^\tau$$

with $\tau = \frac{1}{2}(5r - 3r) + 4 - \frac{\alpha}{r}$.

Lemma 4.4.6. *Fix $\alpha > \frac{5}{2}r^3 + r^2 + 7r$, and let $I_\kappa = \mathcal{I} \setminus \bigcup_{kc} R_{c,k}(\kappa, \alpha)$ then for any $p \in I_\kappa$ and for any $c \in \beta\mathbb{Z}$, $k \in \mathbb{Z}^N$ with $0 \neq |k| \leq r$, $\beta > 0$, one has*

$$|\sum_{|a|=1}^N k_a\Omega_a + c| \geq \frac{\kappa}{N^\alpha}$$

Moreover

$$|\mathcal{I} - I_\kappa| \leq C\kappa^{\frac{1}{r}}$$

where $C = \frac{2\sqrt{p+e+1}}{C(r,b)\pi b}$.

Proof. Since

$$|\Omega_k| = |\frac{2\pi k}{1 + 4\pi^2 bk^2}\sqrt{pk^4 + ek^2 + 1}| \leq \frac{\sqrt{p + e + 1}}{2\pi b}|k|$$

We have that $|\sum_{|a|=1}^{N} k_a\Omega_a + c| \geq 1$ for all $|c| \geq \frac{\sqrt{p+e+1}}{2\pi b} rN$. hence

$$
\begin{aligned}
|\bigcup_{kc} R_{c,k}(\kappa,\alpha)| &\leq \sum_{|k|\leq r, |c|\leq CrN} |R_{c,k}(\kappa,\alpha)| \\
&\leq 2\frac{\sqrt{p+e+1}}{2\pi b} rN \frac{2}{C(r,b)} r^{2r} N^{1+\tau} \kappa^{1/r} \\
&\leq \frac{2\sqrt{p+e+1}}{C(r,b)\pi b} \kappa^{1/r}.
\end{aligned}
$$

□

Lemma 4.4.7. *For any $0 < \kappa \ll 1$, there exist $\alpha' > 0$ and a set $\mathcal{J}_\kappa$ satisfying*

$$|\mathcal{I} - \mathcal{J}_\kappa| \leq C\kappa^{\frac{\alpha+1}{r+2}}$$

Such that $\forall p \in \mathcal{J}_\kappa$ one has

$$|\Omega^N \cdot k + e_1\Omega_j + e_2\Omega_l| \geq \frac{\kappa}{N^{\alpha'}} \tag{4.4.3}$$

for any $k \in \mathbb{Z}^N$, $|e_i| \leq 1$, $|j| \geq |l| > N$, and $|k| + |e_1| + |e_2| \neq 0$, $|k| \leq r+2$.

Proof. One has an estimate for Ω_j (and Ω_l also)

$$\Omega_j = 2\pi\frac{\sqrt{ac}}{b} j + a_j \qquad \textbf{with } |a_j| \leq \frac{C}{j}.$$

Hence

$$e_1\Omega_j + e_2\Omega_l = 2\pi\frac{\sqrt{ac}}{b}(e_1 j + e_2 l) + e_1 a_j + e_2 a_l.$$

If $j, l > CN^\alpha/\kappa$ then $e_1 a_j + e_2 a_l$ is just an irrelevant terms, so that the estimate (4.4.3) follows from Lemma 4.4.6 with $\beta = 2\pi\frac{\sqrt{ac}}{b}$. For the case $j, l \leq CN^\alpha/\kappa$, we reapplies the lemma with N, r replaced by $N' = CN^\alpha/\kappa$ and $r' = r+2$. Then the nonresonance condition (4.4.3) holds provided $\alpha' = \alpha^2 \approx r^6$ and the complement set is measured by a constant times $\kappa^{\frac{\alpha+1}{r+2}}$.

□

Proof for Theorem 4.4.2. Define $\mathcal{J} := \bigcup_{\kappa>0} \mathcal{J}_\kappa$ then it satisfies the Theorem. □

4.5 Proof of the Theorem 4.2.1

Since P has tame modulus, there exists a positive number $\mu_\#$ such that

$$\langle |P| \rangle_{s,\mu} \leq A\mu^2, \qquad \forall 0 < \mu \leq \mu_\# \tag{4.5.1}$$

here $A = \frac{3}{2\pi|b|}$. Fix $r_* > 1$, $N > 1$, for any $r \leq r_*$, we define μ_r and μ_* by

$$\mu_* := \frac{\kappa}{48 e r_* N^\alpha A} \tag{4.5.2}$$

$$\mu_r := \mu(1 - \frac{r}{2r_*}) \qquad \forall \mu_r \leq \mu_\#, \quad \frac{u_r}{\mu_*} \leq \frac{1}{2}. \tag{4.5.3}$$

Then by induction, we can construct sequences of Hamiltonian χ_r, H_r, Z_r and continuous functions f_r for $0 < r \leq r^*$ as in (4.3.15). Precisely

Proposition 4.5.1. *Fix $r_* > 1$, $N > 1$, for any $r \leq r_*$, there exists a canonical transformation $\varphi_r = \varphi_{\chi_r}$ which puts H in the form*

$$H_r := H \circ \varphi_r = H_0 + Z_r + f_r + \mathcal{R}_r^N + \mathcal{R}_r^T \tag{4.5.4}$$

here Z_r is a polynomial of degree at most $r+2$ having a zero of order 3 at the origin and is in $(\kappa, \alpha, N)-$normal form ; f_r is a polynomial having a zero of order $r+3$ at the origin. Moreover

$$\langle |Z_r| \rangle_{s,\mu_r} \leq \begin{cases} 0 & r = 0 \\ A\mu^2 \sum_{l=0}^{r-1} (\frac{\mu}{\mu_*})^l & r \geq 1 \end{cases} \tag{4.5.5}$$

$$\langle |f_r| \rangle_{s,\mu_r} \leq A\mu^2 (\frac{\mu}{\mu_*})^r \tag{4.5.6}$$

the remainder terms $\mathcal{R}_r^N, \mathcal{R}_r^T \in C^\infty(B_s(\mu_r))$ satisfy

$$\|X_{\mathcal{R}_r^T}\|_{s,\mu_r} \leq 2^r \mu^2 (\frac{\mu}{\mu_*})^r \tag{4.5.7}$$

$$\|X_{\mathcal{R}_r^N}\|_{s,\mu_r} \leq \frac{\mu^2}{N^{s-1}} \tag{4.5.8}$$

for any $s \geq 1$. The transformation φ_r satisfies

$$\sup_{(\psi,\phi) \in B_s(\mu_r)} \|(I - \varphi_r)(\psi, \phi)\|_s \leq A\mu^2 (\frac{\mu}{\mu_*})^r. \tag{4.5.9}$$

Similarly, the inverse transformation fulfills the same estimate.

Proof. We proceed by induction. It is trivial for the case $r = 0$. For $r > 0$ we write

$$f_r = f_r^0 + f_r^N \tag{4.5.10}$$

where f_r^0 contains at most terms quadratic in $\widehat{z}$, and f_r^N is the remainder of the expansion. Since both f_r^0 and f_r^N are truncations of f_r, then

$$\langle|f_r^N|\rangle_{s,\mu_r} \leq \langle|f_r|\rangle_{s,\mu_r}, \qquad \langle|f_r^0|\rangle_{s,\mu_r} \leq \langle|f_r|\rangle_{s,\mu_r}.$$

By lemma 4.3.13, one has

$$\{H_0, \chi_r\} + Z_{r+1} - Z_r = f_r^0 \tag{4.5.11}$$

and

$$\langle|\chi_r|\rangle_{s,\mu_r} \leq \frac{N^\alpha}{\kappa}\langle|f_r^0|\rangle_{s,\mu_r}, \qquad \langle|Z_{r+1} - Z_r|\rangle_{s,\mu_r} \leq \langle|f_r^0|\rangle_{s,\mu_r}. \tag{4.5.12}$$

Then by lemma 4.3.10, one has

$$\begin{aligned}
\langle|f_{r+1}^0|\rangle_{s,\mu_{r+1}} &= \langle|\{Z_r - H_0, \chi_r\} + \{f_r^0, \chi_r\} + \sum_{n\geq 2}^{l_r}\frac{(-1)^n}{n!}ad_{\chi_r}(Z_r + f_r^0)|\rangle_{s,\mu_{r+1}} \\
&\leq \frac{4r_*}{\mu}\langle|Z_r - H_0|\rangle_{s,\mu_r}\langle|\chi_r|\rangle_{s,\mu_r} + \frac{4r_*}{\mu}\langle|f_r^0|\rangle_{s,\mu_r}\langle|\chi_r|\rangle_{s,\mu_r} + \sum_{n\geq 2}^{l_r}\langle|Z_r + f_r^0|\rangle_{s,\mu_r}(\frac{4r_*e}{R}\langle|\chi_r|\rangle_{s,\mu_r}) \\
&\leq \frac{4r_*}{\mu}A\mu_r^2\frac{N^\alpha}{\kappa}\langle|f_r^0|\rangle_{s,\mu_r} + \frac{2\frac{4r_*e}{R}\langle|\chi_r|\rangle_{s,\mu_r}}{1 - \frac{4r_*e}{R}\langle|\chi_r|\rangle_{s,\mu_r}}\langle|f_r^0|\rangle_{s,\mu_r} \\
&\leq \frac{\langle|f_r^0|\rangle_{s,\mu_r}\mu}{\mu_*}.
\end{aligned}$$

Here we use $\frac{4r_*e}{\mu}\langle|\chi_r|\rangle_{s,\mu_r} \leq \frac{1}{8}\frac{\mu}{\mu_*} \leq \frac{1}{8}$, $\langle|Z_r - H_0|\rangle_{s,\mu_r} \leq A\mu_r^2$, which we get by induction and $\frac{4r_*}{\mu_*}A\mu_r^2\frac{N^\alpha}{\kappa} \leq \frac{\mu}{4\mu_*}$, which is due to the definition of μ_*. Then

$$\langle|f_{r+1}^0|\rangle_{s,\mu_{r+1}} \leq \frac{\mu}{\mu_*}\langle|f_r^0|\rangle_{s,\mu_r} \leq (\frac{\mu}{\mu_*})^{r+1}A\mu^2,$$

and thus

$$\langle|\chi_{r+1}|\rangle_{s,\mu_{r+1}} \leq \frac{N^\alpha}{\kappa}\langle|f_{r+1}^0|\rangle_{s,\mu_{r+1}} \leq \frac{N^\alpha}{\kappa}(\frac{\mu}{\mu_*})^{r+1}A\mu^2. \tag{4.5.13}$$

Then

$$\sup_{(\psi,\phi)\in B_s(\mu_r)} \|(I-\varphi_r)(\psi,\phi)\|_s \leq \langle|\chi_r|\rangle_{s,\mu_r} \leq \frac{N^\alpha}{\kappa}(\frac{\mu}{\mu_*})^{r+1}A\mu^2, \tag{4.5.14}$$

$$\sup_{(\psi,\phi)\in B_s(\mu_r)} \|(I-\varphi_r^{-1})(\psi,\phi)\|_s \leq \langle|\chi_r|\rangle_{s,\mu_r} \leq \frac{N^\alpha}{\kappa}(\frac{\mu}{\mu_*})^{r+1}A\mu^2. \tag{4.5.15}$$

For the remainder terms, one has

$$\mathcal{R}^T_{r+1} = \mathcal{R}^T_r \circ \phi_{\chi_r} + \int_0^1 \frac{(-1)^{l_r+1}}{(l_r+1)!} ad^{l_r+1}_{\chi_r}(Z_r + f^0_r)\circ \varphi^\tau_{\chi_r} d\tau, \tag{4.5.16}$$

$$\mathcal{R}^N_{r+1} = f^N_r \circ \varphi_{\chi_r}. \tag{4.5.17}$$

Here R^N_r contains at least terms cubic in $\hat{z}$. The two remainder terms $\mathcal{R}^T_{r+1}, \mathcal{R}^N_{r+1}$ are in $C^\infty(B_s(\mu_{r+1}))$. For the first term, one has

$$\begin{aligned}
\|X_{\mathcal{R}^T_{r+1}}\|_s &\leq \|X_{\mathcal{R}^T_r\circ\varphi_{\chi_r}}\|_s + \frac{1}{(l_r+1)!}\|\int_0^1 X_{ad^{l_r+1}_{\chi_r}(Z_r+f^0_r)\circ\varphi^\tau_{\chi_r}} d\tau\|_s \\
&\leq (1+\frac{3}{d}\|X_{\chi_r}\|_s)(\|X_{\mathcal{R}^T_r}\|_s + \frac{1}{(l_r+1)!}\|X_{\frac{i^{l_r+1}}{(l_r+1)!}ad^{l_r+1}_{\chi_r}(Z_r+f^0_r)}\|_s) \\
&\leq 2^{r+1}\mu^2(\frac{\mu}{\mu_*})^{r_*+1}.
\end{aligned}$$

For the second term

$$\begin{aligned}
\|X_{R^N_r}\|_s \leq \frac{1}{N^{s-1}}\langle|R^N_r|\rangle_{s,\mu_{r+1}} &= \frac{1}{N^{s-1}}\langle|f^N_r\circ\varphi_{\chi_r}|\rangle_{s,\mu_{r+1}} \\
&\leq \frac{1}{N^{s-1}}(1+\frac{3}{d}\langle|\chi_r|\rangle_{s,\mu_r})\langle|f^N_r|\rangle_{s,\mu_r} \\
&\leq \frac{\mu^2}{N^{s-1}}.
\end{aligned}$$

□

Proof of theorem 4.2.1. We take $\mathcal{T} = \varphi_{\chi_1}\circ\cdots\circ\varphi_{\chi_r}$, then $Z = H_0 + Z_r$ is in (κ,α,N) normal form. We need to choose N in order to obtain $\mathcal{R} = \mathcal{R}^T_r + \mathcal{R}^N_r$ is small. We have

$$\begin{aligned}
\|X_{\mathcal{R}}\|_{s,\frac{\mu}{2}} &\leq \|X_{\mathcal{R}^T_r}\|_{s,\frac{\mu}{2}} + \|X_{\mathcal{R}^N_r}\|_{s,\frac{\mu}{2}} \\
&\leq 2^r\mu^2(\frac{\mu}{\mu_*})^r + \frac{\mu^2}{N^{s-1}} \\
&\leq C\mu^2(\mu N^\alpha)^r + \frac{\mu^2}{N^{s-1}}.
\end{aligned}$$

Choose $N = N_* = [\mu^{-\frac{1}{2r\alpha}}]$ and $s > 2\alpha r^2 + 1$ then the last term is smaller than μ^{r+1}. The transformation $\mathcal{T}$ and its inverse fulfill the estimate (4.2.10). $\square$

4.6 Appendix A

Proof of lemma 4.3.4. Denote $X := X_f$, $Y := X_g$. One has

$$X = \sum_{k,l_1,\cdots,l_n} D_k X_k^{l_1,\cdots,l_n} z_{l_1}\cdots z_{l_n} e_k \tag{4.6.1}$$

$$Y = \sum_{k,j_1,\cdots,j_m} D_k Y_k^{j_1,\cdots,j_m} z_{j_1}\cdots z_{j_n} e_k \tag{4.6.2}$$

here we write z_l to represent ϕ_l or ψ_l. Remind that $D_k = \frac{2\pi k}{1+4\pi^2 bk^2}$. Then

$$\begin{aligned}\lfloor X_{\{f,g\}}\rceil(z) = &\sum_{\substack{k,l_1,\cdots,l_n\\ j_1,\cdots,j_m}} |nD_k D_{l_n} X_k^{l_1,\cdots,l_n} Y_{l_n}^{j_1,\cdots,j_m} \pm mD_k D_{l_n} X_{l_n}^{l_1,\cdots,l_{n-1},j_m} Y_k^{j_1,\cdots,j_{m-1},l_n}|\\ &\times e_k z_{l_1}\cdots z_{l_{n-1}} z_{j_1}\cdots z_{j_n}.\end{aligned}$$

One has

$$\begin{aligned}\|\lfloor X_{\{f,g\}}\rceil(z)\|_s^2 \leq\ & 2\sum_{\substack{k,l_1,\cdots,l_n\\ j_1,\cdots,j_m}} (|nD_k D_{l_n} X_k^{l_1,\cdots,l_n} Y_{l_n}^{j_1,\cdots,j_m}|^2 + |mD_k D_{l_n} X_{l_n}^{l_1,\cdots,l_{n-1},j_m} Y_k^{j_1,\cdots,j_{m-1},l_n}|^2)\\ & \langle k\rangle^{2s} \times (z_{l_1}\cdots z_{l_{n-1}} z_{j_1}\cdots z_{j_n})^2\\ & \leq 2n^2 \sum_{k,l_1,\cdots,l_n} (|D_k X_k^{l_1,\cdots,l_n}|^2 \langle k\rangle^{2s} (z_{l_1}\cdots z_{l_{n-1}})^2 \sum_{l_n,j_1,\cdots,j_m} |D_{l_n} Y_{l_n}^{j_1,\cdots,j_m}|^2 (z_{j_1}\cdots z_{j_n})^2\\ & + 2m^2 \sum_{j_m,l_1,\cdots,l_n} (|D_{l_n} X_{l_n}^{l_1,\cdots,l_{n-1},j_m} (z_{l_1}\cdots z_{l_{n-1}})^2\\ & \times \sum_{k,l_n,j_1,\cdots,j_{m-1}} |D_k Y_k^{j_1,\cdots,j_{m-1},l_n}|^2 (z_{j_1}\cdots z_{j_n})^2|^2 \langle k\rangle^{2s}\\ & \leq 2n^2 |f|_s^2 |g|_0^2 (\frac{1}{n}\sum \|z_{l_1}\|_1^2\cdots\|z_{l_i}\|_1^2 \|z_{l_i}\|_s^2 \|z_{l_{i+1}}\|_1^2\cdots\|z_{l_n}\|_1^2)\|z_{j_1}\|_1^2\cdots\|z_{j_m}\|_1^2\\ & + 2m^2 |f|_0^2 |g|_s^2 (\frac{1}{m}\sum \|z_{j_1}\|_1^2\cdots\|z_{j_i}\|_1^2 \|z_{j_i}\|_s^2 \|z_{j_{i+1}}\|_1^2\cdots\|z_{l_n}\|_1^2)\|z_{j_1}\|_1^2\cdots\|z_{j_m}\|_1^2\\ \leq & 2(n+m)^2 |f|_s^2 |g|_s^2 (\frac{1}{n+m-1}\sum \|z_{l_1}\|_1^2\cdots\|z_{l_i}\|_1^2 \|z_{l_i}\|_s^2 \|z_{l_{i+1}}\|_1^2\cdots\|z_{l_{n+m-1}}\|_1^2).\end{aligned}$$

$\square$

4.7 Appendix B

In this section, we prove the local well-posedness for the Boussinesq system (see chapter 16 [**Tay3**]).

4.7.1 The case $b \neq 0$

Existence

By the change of variables, the well-posedness of (4.2.3) implies the well-posedness of (4.1.1). The idea here is to obtain a solution to (4.1.1) as a limit of solutions $(\psi_\varepsilon, \phi_\varepsilon)$ to

$$\begin{cases} \partial_t \psi_{\varepsilon,k} &= -iD_k \nabla_{\psi_{\varepsilon,-k}} H(\psi_\varepsilon, \phi_\varepsilon) \\ \partial_t \phi_{\varepsilon,k} &= iD_k \nabla_{\phi_{\varepsilon,-k}} H(\psi_\varepsilon, \phi_\varepsilon). \end{cases} \tag{4.7.1}$$

Here $(\psi_\varepsilon, \phi_\varepsilon)$ is defined by

$$\psi_\varepsilon(t) = \sum_{k \in \mathbb{Z}^*} \varphi(\varepsilon k) \psi_k(t) e^{i2\pi kx} \tag{4.7.2}$$

$$\phi_\varepsilon(t) = \sum_{k \in \mathbb{Z}^*} \varphi(\varepsilon k) \phi_k(t) e^{i2\pi kx}. \tag{4.7.3}$$

Where $\varphi \in C_0^\infty(\mathbb{R})$ is an even, real valued satisfying $\varphi(x) = 1$ for any $x \in [-1, 1]$. Since φ has support bounded in $\mathbb{R}$, the sum (4.7.2) and (4.7.3) are finite. As a consequence, the system (4.7.7) is of finite ODEs. By Cauchy-Lipschitz theorem, we know that there is a unique solution to this system, for t close to 0. First, we will prove that the solution $(\psi_\varepsilon, \phi_\varepsilon)(t)$ exists in an interval $[-T, T]$ independent of $\varepsilon \in (0, 1]$.

Lemma 4.7.1. *Let $s \geq 1$, $0 < \|(\psi_\varepsilon, \phi_\varepsilon)(0)\|_s = \mu$, then $\|(\psi_\varepsilon, \phi_\varepsilon)(t)\|_s \leq 2\mu$ $\forall t \in [-\frac{b\pi}{3\mu}, \frac{b\pi}{3\mu}]$ independent of $\varepsilon \in (0, 1]$.*

Proof. Since P has tame modulus one has an estimate

$$\begin{aligned} \partial_t \|(\psi_\varepsilon, \phi_\varepsilon)(t)\|_s^2 &= |\{\|(\psi_\varepsilon, \phi_\varepsilon)(t)\|_s^2, P\}| \\ &\leq \|(\psi_\varepsilon, \phi_\varepsilon)(t)\|_s \|\lfloor X_P \rceil (\psi_\varepsilon, \phi_\varepsilon)\|_s \\ &\leq \frac{3}{2\pi b} \|(\psi_\varepsilon, \phi_\varepsilon)(t)\|_s^2 \|(\psi_\varepsilon, \phi_\varepsilon)(t)\|_1. \end{aligned}$$

By Gronwall's inequality, one has that $\|(\psi_\varepsilon, \phi_\varepsilon)(t)\|_s \leq 2\mu$ for all $t \in [0, \frac{b\pi}{3\mu}]$, where we assume $\|(\psi(x, 0), \phi(x, 0))\|_s = \mu \ll 1$. In the same way, we also have $\|(\psi_\varepsilon, \phi_\varepsilon)(-t)\|_s \leq$

2μ for all $t \in [0, \frac{b\pi}{3\mu}]$. Denote $I = [-\frac{b\pi}{3\mu}, \frac{b\pi}{3\mu}]$, then $(\psi_\varepsilon, \phi_\varepsilon)(t)$ is uniformly bounded in $C(I, H^s) \cap C^1(I, H^{s-1})$. □

Lemma 4.7.2. *Let $s \geq 1$, the sequence $\{(\psi_\varepsilon, \phi_\varepsilon)(t)\}_{\varepsilon\in(0,1]}$ is Cauchy in H^s for all $t \in I$.*

Proof. We consider $(v, w)(t) = (\psi_{\varepsilon_1}, \phi_{\varepsilon_1})(t) - (\psi_{\varepsilon_2}, \phi_{\varepsilon_2})(t)$. Then for any $N > 0$, there exists $\varepsilon_N \in I$ such that for any $\varepsilon_1, \varepsilon_2 \leq \varepsilon_N$, one has

$$\|(\psi_{\varepsilon_1}, \phi_{\varepsilon_1})(0) - (\psi_{\varepsilon_2}, \phi_{\varepsilon_2})(0)\|_s \leq \frac{1}{N^s}\mu.$$

The key point here is $(\psi_{\varepsilon_1}, \phi_{\varepsilon_1})_k(0) = (\psi_{\varepsilon_2}, \phi_{\varepsilon_2})_k(0)$ for all $|k| \leq N$. Moreover $(v, w)(t)$ satisfies equation

$$\begin{cases} \partial_t v_k &= -iD_k(\nabla_{\psi_{\varepsilon_1,-k}} H(\psi_{\varepsilon_1}, \phi_{\varepsilon_1}) - \nabla_{\psi_{\varepsilon_2,-k}} H(\psi_{\varepsilon_2}, \phi_{\varepsilon_2})) \\ \partial_t w_k &= iD_k(\nabla_{\phi_{\varepsilon_1,-k}} H(\psi_{\varepsilon_1}, \phi_{\varepsilon_1}) - \nabla_{\phi_{\varepsilon_2,-k}} H(\psi_{\varepsilon_2}, \phi_{\varepsilon_2})). \end{cases} \tag{4.7.4}$$

Then

$$\begin{aligned} \partial_t \|(v, w)(t)\|_s^2 = &\sum_{k\in N^*} \langle k\rangle^{2s}(|\partial_t v_k \overline{v}_k| + |\partial_t w_k \overline{w}_k|) \\ &\leq \|(v, w)(t)\|_s(\|\lfloor X_P\rceil((v, w), (\psi_{\varepsilon_1}, \phi_{\varepsilon_1}))\|_s + \|\lfloor X_P\rceil((v, w), (\psi_{\varepsilon_2}, \phi_{\varepsilon_2}))\|_s) \\ &\leq \frac{3}{2\pi b}\|(v, w)(t)\|_s^2(\|(\psi_{\varepsilon_1}, \phi_{\varepsilon_1})(t)\|_1 + \|(\psi_{\varepsilon_2}, \phi_{\varepsilon_2})(t)\|_1) \\ &+ \frac{3}{2\pi b}\|(v, w)(t)\|_s\|(v, w)(t)\|_1(\|(\psi_{\varepsilon_1}, \phi_{\varepsilon_1})(t)\|_s + \|(\psi_{\varepsilon_2}, \phi_{\varepsilon_2})(t)\|_s) \end{aligned}$$

By Gronwall's inequality, one has that $\|(\psi_\varepsilon, \phi_\varepsilon)(t)\|_s \leq 4\frac{\mu}{N^s}$ for all $t \in I$. □

Lemma 4.7.3. *Let $s \geq 1$ and $b \neq 0$, then provided the initial datum $(\psi_\varepsilon(0), \phi_\varepsilon(0)) \in H^s, \varepsilon \in (0, 1]$, with $\|(\psi_\varepsilon(0), \phi_\varepsilon(0))\|_s = \mu$ small enough, the system (4.2.3) admits a solution $(\psi, \phi)(t)$ for all $t \in I = [\frac{-b\pi}{3\mu}, \frac{b\pi}{3\mu}]$ with*

$$(\psi, \phi) \quad in \quad L^\infty(I, H^s) \cap Lip(I, H^{s-1}) \qquad \forall s \geq 1. \tag{4.7.5}$$

Proof. By lemma 4.7.2, the bounded function $(\psi_\varepsilon, \phi_\varepsilon)$ converges to (ψ, ϕ) in $L^\infty(I, H^s)$ when $\varepsilon \to 0$. Similarly

$$\lim_{\varepsilon\to 0} X_{H(\psi_\varepsilon,\phi_\varepsilon)} \to X_{H(\psi,\phi)} \quad \textbf{in} \quad L^\infty(I, H^{s-1}),$$

while as a result $\partial_t(\psi_\varepsilon, \phi_\varepsilon) \to \partial_t(\psi, \phi)$ in $L^\infty(I, H^{s-1})$. Thus $(\psi, \phi) \in L^\infty(I, H^s) \cap Lip(I, H^{s-1})$ is a solution of (4.2.3). □

Lemma 4.7.4. *The solution $(\psi, \phi)(t)$ given in Lemma 4.7.3 is in $C(I, H^s)$, $\forall s \geq 1$.*

Proof. It is enough to prove $\|(\psi,\phi)(t)\|_s$ is continuous of t. Similar as lemma 4.7.1, one has

$$\partial_t\|(\psi,\phi)(t)\|_s^2 \leq \frac{3}{2\pi b}\|(\psi,\phi)(t)\|_s^2\|(\psi,\phi)(t)\|_1$$

so that $(\psi,\phi)\in C(I,H^s)$ and $\|(\psi,\phi)(t)\|_s \leq 2\mu$ for all $t\in I$. □

Uniqueness

Assume that $(\psi^{(1)},\phi^{(1)})$, $(\psi^{(2)},\phi^{(2)})$ are two solutions of (4.2.3) satisfying $(\psi^{(1)},\phi^{(1)})(x,0) = (\psi^{(2)},\phi^{(2)})(x,0)$. Denote $(\nu,\upsilon) = (\psi^{(1)},\phi^{(1)}) - (\psi^{(2)},\phi^{(2)})$, then by computation one has

$$\begin{cases} (\nu,\upsilon)(x,0) & = 0 \\ \partial_t\|(\nu,\upsilon)\|_s^2 & \leq C\|(\nu,\upsilon)\|_s(|(\psi^{(1)},\phi^{(1)})\|_s^2 + \|(\psi^{(2)},\phi^{(2)})\|_s^2) \end{cases} \tag{4.7.6}$$

for some constant C. By Gronwall's inequality it is proved that $(\nu,\upsilon)\equiv 0$. So that the solution (ψ,ϕ) is unique.

Finally we have proved

Proposition 4.7.5. *Let $s\geq 1$, $b>0$ the system* (4.1.1) *is locally well-posed for sufficiently small initial datum in H^s.*

4.7.2 The case b = 0

In this case, since $D_k = 2\pi k$ is not bounded, we do not have tame property, so that the proof for existence and uniqueness of (ψ,ϕ) solution has to be modified. As in the case $b\neq 0$, we define

$$u_\varepsilon(t) = \sum_{k\in\mathbb{Z}}\varphi(\varepsilon k)u_{\varepsilon,k}(t)e^{i2\pi kx} \qquad u_{\varepsilon,k}(0) = u_k(0)$$
$$\eta_\varepsilon(t) = \sum_{k\in\mathbb{Z}}\varphi(\varepsilon k)\eta_{\varepsilon,k}(t)e^{i2\pi kx} \qquad \eta_{\varepsilon,k}(0) = \eta_k(0).$$

which are solutions to

$$\begin{cases} \partial_t\eta_{\varepsilon,k} & = -i2\pi k((1-4\pi^2ak^2)u_{\varepsilon,k} + \sum_{j+l=k}u_{\varepsilon,j}\eta_{\varepsilon,l}) \\ \partial_t u_{\varepsilon,k} & = -i2\pi k((1-4\pi^2ck^2)\eta_{\varepsilon,k} + \frac{1}{2}\sum_{j+l=k}u_{\varepsilon,j}u_{\varepsilon,l}). \end{cases} \tag{4.7.7}$$

Then one has

$$\partial_t \sum_{k\in\mathbb{N}^*} |k|^{2s}(\alpha_k^2|\eta_{\varepsilon,k}|^2) + \alpha_k^{-2}|u_{\varepsilon,k}|^2) = 2\pi\frac{1}{2}\sum_{j+l+k=0}(\alpha_l^2 l^{2s+1} + \alpha_k^2 k^{2s+1})u_{\varepsilon,j}\eta_{\varepsilon,l}\eta_{\varepsilon,k}$$
$$+ 2\pi\frac{1}{6}\sum_{j+l+k=0}(\alpha_l^{-2}l^{2s+1} + \alpha_k^{-2}k^{2s+1} + \alpha_j^{-2}j^{2s+1})u_{\varepsilon,j}u_{\varepsilon,l}u_{\varepsilon,k}.$$

Here $\alpha_k^2 = (\frac{1-4\pi^2ak^2}{1-4\pi^2ck^2})^{1/2}$. One has

$$\alpha_k^2 = \sqrt{\frac{a}{c}} + O(\frac{1}{k}); \qquad \alpha_k^{-2} = \sqrt{\frac{c}{a}} + O(\frac{1}{k}).$$

So that

$$|\alpha_l^2 l^{2s+1} + \alpha_k^2 k^{2s+1}| \leq \sqrt{\frac{a}{c}}|l^{2s+1} + k^{2s+1}| + |\alpha_l^2 - \sqrt{\frac{a}{c}}||l^{2s+1}| + |\alpha_k^2 - \sqrt{\frac{a}{c}}||k^{2s+1}| \leq C|j|(|l|^{2s} + |k|^{2s})$$

Since $j+l+k=0$, we could estimate the term $|j|(|l|^{2s}+|k|^{2s})$ by $\mu_1^s\mu_2^s\mu_3$, where μ_1, μ_2, μ_3 are the first, the second and the third large term among $|j|, |l|, |k|$. Similarly, one has

$$|\alpha_l^{-2}l^{2s+1} + \alpha_k^{-2}k^{2s+1} + \alpha_j^{-2}j^{2s+1}| \leq C\mu_1^s\mu_2^s\mu_3.$$

Then applying the Young inequality, we have

$$|\partial_t \sum_{k\in\mathbb{N}^*} |k|^{2s}(\alpha_k^2|\eta_{\varepsilon,k}|^2) + \alpha_k^{-2}|u_{\varepsilon,k}|^2)| \leq C\|(u_\varepsilon, \eta_\varepsilon)\|_s^3$$

for some constant C. By Gronwall's inequality, one has that $\|(\psi_\varepsilon, \phi_\varepsilon)(t)\|_s \leq 2\mu$ for all $t \in [0, \frac{2}{C\mu}]$. Denote $I = [-\frac{2}{C\mu}, \frac{2}{C\mu}]$, then $(\psi_\varepsilon, \phi_\varepsilon)(t)$ is uniformly bounded in $C(I, H^s) \cap C^1(I, H^{s-3})$.

Lemma 4.7.6. *Let $s \geq 1$, the sequence $\{(\psi_\varepsilon, \phi_\varepsilon)(t)\}_{\varepsilon\in(0,1]}$ is Cauchy in H^s for all $t \in I$.*

Proof. We consider $(v, w)(t) = (\psi_{\varepsilon_1}, \phi_{\varepsilon_1})(t) - (\psi_{\varepsilon_2}, \phi_{\varepsilon_2})(t)$. Then for any $N > 0$, there exists $\varepsilon_N \in I$ such that for any $\varepsilon_1, \varepsilon_2 \leq \varepsilon_N$, one has

$$\|(\psi_{\varepsilon_1}, \phi_{\varepsilon_1})(0) - (\psi_{\varepsilon_2}, \phi_{\varepsilon_2})(0)\|_s \leq \frac{1}{N^s}\mu.$$

Moreover $(v, w)(t)$ satisfies

$$\begin{aligned}
\partial_t \|(v,w)(t)\|_s^2 &= \sum_{k \in N^*} \langle k \rangle^{2s} (|\partial_t v_k \overline{v}_k| + |\partial_t w_k \overline{w}_k|) \\
&\leq 2\pi \frac{1}{2} \sum_{j+l+k=0} |\alpha_l^2 l^{2s+1} + \alpha_k^2 k^{2s+1}| |u_{\tilde{\varepsilon},j}| (|\eta_{\tilde{\varepsilon},l} w_k| + |w_l| |\eta_{\tilde{\varepsilon},k}|) \\
&+ 2\pi \frac{1}{6} \sum_{j+l+k=0} |\alpha_l^{-2} l^{2s+1} + \alpha_k^{-2} k^{2s+1} + \alpha_j^{-2} j^{2s+1}| \\
&\times (|u_{\tilde{\varepsilon},j}| u_{\tilde{\varepsilon},l} v_k| + |u_{\tilde{\varepsilon},j}| u_{\tilde{\varepsilon},k} v_l| + |u_{\tilde{\varepsilon},k}| u_{\tilde{\varepsilon},l} v_j|) \\
&\leq C \|(v,w)(t)\|_s (\|(\psi_{\varepsilon_1}, \phi_{\varepsilon_1})(t)\|_s^2 + \|(\psi_{\varepsilon_2}, \phi_{\varepsilon_2})(t)\|_s^2).
\end{aligned}$$

Here $u_{\tilde{\varepsilon},l} = u_{\varepsilon_1,l} + u_{\varepsilon_2,l}$ and $\eta_{\tilde{\varepsilon},l} = \eta_{\varepsilon_1,l} + \eta_{\varepsilon_2,l}$, $\forall l \in \mathbb{Z}^*$. By Gronwall's inequality, one has that $\|(v,w)(t)\|_s \leq 4\frac{\mu}{N^s}$ for all $t \in I$. □

As in the case $b \neq 0$, one has $\lim_{\varepsilon \to 0}(u_\varepsilon, \eta_\varepsilon) \to (u, \eta) \in C(I, H^s)$ solution of Boussinesq systems. Similarly, we have the uniqueness for (u, η).

Proposition 4.7.7. *Let* $s \geq 1$, $b = 0$ *the system* (4.1.1) *is locally well-posed for sufficiently small initial datum in* H^s.

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DOCTORAT BRETAGNE LOIRE / MATHSTIC

UNIVERSITÉ DE NANTES

Titre : titre (en français) Trois résultats sous forme normale pour les équations de Schrödinger et le système abcd de Boussinesq

Mot clés : Théorie KAM, forme normale de Birkhoff, réductibilité, l'équation de Hamilton ,l'équation de Schrödinger,le système abcd de type Boussinesq

Résumé : On montre des résultats de forme normale pour des EDPs Hamiltoniennes : l'équation de Schrödinger non linéaire quintique sur le cercle, l'équation de Schrödinger sur une variété Zoll et le système abcd de type Boussinesq sur le cercle. Ces résultats sont démontrés à l'aide de procédure KAM et de procédure de forme normale de Birkhoff. On déduit des résultats de forme normale le comportement en temps long des solutions au voisinage de zéro.

Title: titre (en anglais) Three normal form results for Schrödinger equations and abcd Boussinesq system

Keywords: KAM theory, Birkhoff normal form, Reducibility, Hamiltonian equations, Schrödinger equation, abcd Boussinesq system

Abstract: We prove normal form results for Hamiltonian PDEs: the quintic nonlinear Schrödinger equation on the circle, the Schrödinger equation on a Zoll manifold and the abcd Boussinesq system on the circle. These results are proved via KAM procedure and Birkhoff normal form procedure. As corollaries of normal form results, one deduces the long time behavior of solutions near to zero.