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Knot invariants in embedded contact homology

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Contents

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« Dum loquimur, fugerit invida aetas:
carpe diem, quam minimum credula postero. »

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Introduction

Version française

Un entrelacs avec n composantes dans une variété Y de dimension 3 est l'image par un plongement lisse $\sqcup_{i=1}^{n} S^1 \hookrightarrow Y$ de copies du cercle. Deux entrelacs L_1 et L_2 sont dit équivalents si il existe une isotopie ambiante de Y qui envoie L_1 sur L_2. Une multitude d'invariants d'entrelacs ont été définis et ils peuvent être de beaucoup de types différents. Le plus simple est peut-être le nombre n de composantes connexes d'un entrelacs : un entrelacs avec une seule composante connexe est appelé nœud. Un exemple d'invariant plus sophistiqué, mais encore classique, est le polynôme d'Alexander, qui associe à un entrelacs L de S^3 un polynôme de Laurent.

Durant les quinze dernières années, de nouvelles méthodes, impliquant l'utilisation de structures géométriques additionnelles sur les variétés, ont mené à la découverte de nouveaux types d'invariants pour les entrelacs, ainsi que pour les variétés de dimension trois et quatre. Une structure symplectique sur une variété M orientée de dimension paire $2n$ consiste en une 2-forme ω telle que $(\omega)^{2n}$ est une forme de volume positive sur M. Étant donnée une variété Y fermée et orientée, Ozsváth et Szabó dans [42] ont construit une variété symplectique auxiliaire $(M(Y), \omega)$ équipée d'une structure presque complexe J ; ensuite ils ont défini quatre $\mathbb{Z}$-modules $CF^*(M(Y), J, \omega)$, avec $* \in \{\infty, +, -, \widehat{\ }\}$, munis de différentielles obtenues en comptant certaines courbes holomorphes dans $M(Y)$. Les groupes d'homologie associés ne dépendent pas des choix faits pour la variété auxiliaire ni des structures géométriques et sont des invariants topologiques de Y :

$$HF^{\infty}(Y), \quad HF^{+}(Y), \quad HF^{-}(Y), \quad \widehat{HF}(Y).$$

Ces groupes sont les *homologies de Heegaard Floer* respectivement en version *infinie, plus, moins et chapeau.*

De plus Ozsváth et Szabó dans [44] et Rasmussen dans [50] prouvent qu'un nœud homologiquement trivial K dans Y induit une filtration sur les complexes de chaînes de Heegaard Floer. La première page de chaque suite spectrale associée (dans les versions relatives) s'avère être un invariant topologique de K : ce sont les groupes bigradués

$$HFK^{\infty}(K,Y), \quad HFK^{+}(K,Y), \quad HFK^{-}(K,Y), \quad \widehat{HFK}(K,Y),$$

nommés *homologies de Heegaard Floer pour les nœuds* (dans les versions respectives). Ces homologies sont des invariants puissants pour le couple (K,Y). Par exemple dans [44] et [50], il a été prouvé que $\widehat{HFK}(K,S^3)$ *catégorifie le polynôme d'Alexander de* K. C'est-à-dire la chose suivante.
Si $C := \{(C_{*,i}, \partial_i)\}_{i\in\mathbb{Z}}$ est une collection de complexes de chaînes, sa caractéristique d'Euler graduée est $\chi(C) := \sum_i \chi(C_{*,i})t^i$, où $\chi(C_{*,i})$ est la caractéristique d'Euler standard de $C_{*,i}$ et t est une variable formelle. Par les propriétés de la caractéristique d'Euler, ce polynôme est invariant si on le calcule à partir de l'homologie de C. Comme nous l'avons mentionné, $\widehat{HFK}(K,Y)$ est un groupe bigradué : le fait qu'il catégorifie le polynôme d'Alexander signifie que

$$\chi(\widehat{HFK}(K,S^3)) \doteq \Delta_K,$$

où $\doteq$ veut dire que les deux côtes sont égaux à multiplication par un monôme monique près. Cela a été la première catégorification du polynôme d'Alexander ; une seconde (en homologie de Seiberg-Witten-Floer) a été découverte plus tard par Kronheimer et Mrowka ([35]).

Dans [46] Ozsváth et Szabó développent une construction similaire pour les entrelacs L dans S^3 et obtiennent les invariants

$$HFL^{-}(L,S^3) \quad \text{et} \quad \widehat{HFL}(L,S^3),$$

les *homologies de Heegaard Floer pour les entrelacs* de L. Ces homologies sont dotées d'un $\mathbb{Z}^n$-degré additionnel, où n est le nombre des composantes connexes de L. De plus ils prouvent que $HFL^{-}(L,S^3)$ *catégorifie le polynôme d'Alexander à multivariables*, qui est une généralisation du polynôme d'Alexander classique. En particulier ils montrent que :

$$\chi\left(HFL^{-}(L,S^3)\right) \doteq \begin{cases} \Delta_L(t_1,\ldots,t_n) & \text{si } n>1 \\ \\ \Delta_L(t)/(1-t) & \text{si } n=1. \end{cases} \tag{0.1}$$

En général la théorie d'Heegaard Floer a fourni une grande quantité d'informations sur la topologie des nœuds et des variétés de dimension trois. De plus elle donne de puissants instruments pour la compréhension de certaines structures géométriques sur les variétés. Un exemple est la présence d'un *invariant de structures de contact* dans Heegaard Floer ([45]). Une forme de contact sur une variété Y orientée est une 1-forme lisse α telle que $\alpha\wedge d\alpha$ est une forme de volume positive. Une structure de contact sur Y est une distribution de plans ξ

pour laquelle il existe une forme de contact α avec $\xi = \ker \alpha$. Deux structures de contact ξ_1 et ξ_2 sont dites équivalentes s'il existe une isotopie de Y dont la différentielle envoie ξ_1 sur ξ_2.

Un ingrédient clé pour la définition de l'invariant de structures de contact dans Heegaard Floer est la correspondance de Giroux entre structures de contact et les *décompositions en livre ouvert*. De manière informelle, une décomposition en livre ouvert d'une variété Y de dimension trois consiste en un triplet (L, S, ϕ), où L est un entrelacs dans Y, S est une surface compacte orientée avec bord et $\phi : S \to S$ est un difféomorphisme préservant l'orientation, tel que si $\mathcal{N}(L)$ est un petit voisinage tubulaire de L, alors $Y \setminus \mathcal{N}(L)$ est homéomorphe à $\frac{S\times[0,1]}{(x,1)\sim(\phi(x),0)}$. Alors L, S et ϕ sont respectivement la reliure, la page et la monodromie du livre ouvert. Si l'entrelacs L est la reliure d'un livre ouvert, on dit que L est un *entrelacs fibré*.

Étant donnée une décomposition en livre ouvert (L, S, ϕ) de Y, Thurston et Wilkenkemper ([57]), ont décrit une méthode pour construire une structure de contact "adaptée" à (L, S, ϕ). Plus tard, dans [19], Giroux a découvert une façon d'associer à une structure de contact ξ sur Y une décomposition en livre ouvert de Y adaptée (à isotopie près) à ξ. Par conséquent, il a montré qu'il existe une correspondance biunivoque entre classes d'isotopie de structures de contact et (classes d'équivalence de) décompositions en livre ouvert. Ce résultat est un apport fondamental à l'étude des interactions entre la géométrie de contact et la topologie en basse dimension.

Les livres ouverts ont été utilisés dans une série d'articles par Colin, Ghiggini et Honda pour prouver l'équivalence entre l'homologie de Heegaard Floer et *l'homologie de contact plongée* pour les trois-variétés. Celle-ci est une autre théorie homologique, définie par Hutchings, qui associe à une variété de contact (Y, α) deux modules gradués $ECH(Y, \alpha)$ et $\widehat{ECH}(Y, \alpha)$. Une forme de contact α détermine de façon unique un champ de vecteurs non singulier R_α, appelé *champ de Reeb*. Les générateurs des complexes de chaînes des homologies ECH sont alors certains produits formels d'*orbites de Reeb*, c'est-à-dire des orbites fermées de R_α.

Théorème 0.1 (Colin, Ghiggini, Honda)**.**

$$HF^+(-Y) \cong ECH(Y, \alpha)$$
$$\widehat{HF}(-Y) \cong \widehat{ECH}(Y, \alpha),$$

où $-Y$ est la variété Y avec l'orientation opposée.

Après ce théorème, il est naturel de chercher un analogue en homologie de contact plongée de l'homologie de Heegaard Floer pour les nœuds. Dans [13] les auteurs définissent une version suturée de l'homologie de contact plongée, en analogie avec l'*homologie de Heegaard Floer suturée* développée par Juhász dans [32]. Celle-ci peut être pensée comme une version relative de ECH pour les variétés à bord. En particulier, étant donné un nœud K dans une variété Y de dimension trois, ils ont défini une *version chapeau de l'homologie de contact plongée pour les nœuds*

$$\widehat{ECK}(K, Y, \alpha).$$

De manière informelle, celle-ci est l'homologie ECH dans la version chapeau de la variété de contact $(Y \setminus \mathcal{N}(K), \alpha)$, où $\mathcal{N}(K)$ est un petit voisinage tubulaire de K dans Y et α est une forme de contact convenable. Dans [13] la conjecture suivante est énoncée :

Conjecture 0.2.

$$\widehat{HFK}(-K, -Y) \cong \widehat{ECK}(K, Y, \alpha).$$

L'objectif principal de ce travail est de donner des indices sur la véracité de cette conjecture. Tout d'abord on définit une *version complète de homologie de contact plongée*

$$ECK(K, Y, \alpha)$$

pour les nœuds K dans des variétés Y de dimension trois munies d'une forme de contact α convenable. Puis on généralise les définitions au cas des entrelacs L avec plusieurs composantes connexes et on obtient des homologies

$$ECK(L, Y, \alpha) \quad \text{et} \quad \widehat{ECK}(L, Y, \alpha).$$

Conjecture 0.3. *Pour chaque nœud K ou entrelacs L dans Y, il existe une forme de contact α telle que :*

$$\begin{array}{rcl} ECK(K, Y, \alpha) & \cong & HFK^{-}(K, Y) \\ & \textit{et} & \\ \widehat{ECK}(L, Y, \alpha) & \cong & \widehat{HFK}(-L, -Y), \\ ECK(L, Y, \alpha) & \cong & HFK^{-}(L, Y). \end{array}$$

Ensuite on calcule les caractéristiques d'Euler graduées des homologies ECK pour les nœuds et entrelacs dans les sphères d'homologie et on montre le résultat suivant :

Théorème 0.4. *Soit L un entrelacs avec n composantes dans une sphère d'homologie Y. Alors il existe une forme de contact α pour laquelle*

$$\chi(ECK(L, Y, \alpha)) \doteq \mathrm{ALEX}(Y \setminus L).$$

Ici $\mathrm{ALEX}(Y \setminus L)$ est le *quotient d'Alexander* du complémentaire de L dans Y. Ce théorème est prouvé en utilisant une reformulation dynamique de ALEX due à Fried ([16]). Des relations classiques entre $\mathrm{ALEX}(S^3 \setminus L)$ et Δ_L impliquent le résultat suivant.

Théorème 0.5. *Soit L un entrelacs dans S^3 avec n composantes. Alors il existe une forme de contact α telle que :*

$$\chi\left(ECK(L,S^3,\alpha)\right) \doteq \begin{cases} \Delta_L(t_1,\dots,t_n) & \text{if } n>1 \\ \\ \Delta_L(t)/(1-t) & \text{if } n=1 \end{cases}$$

et

$$\chi\left(\widehat{ECK}(L,S^3,\alpha)\right) \doteq \begin{cases} \Delta_L(t_1,\dots,t_n)\cdot\prod_{i=1}^n(1-t_i) & \text{if } n>1 \\ \\ \Delta_L(t) & \text{if } n=1. \end{cases}$$

Cela implique que l'homologie ECK *est une catégorification du polynôme d'Alexander à multivariables.*
Une comparaison entre ces formules et l'equation 0.1 (et les expressions analogues pour les versions chapeau de HFK et HFL prouvées en [46]), implique en plus le corollaire suivant.

Corollaire 0.6. *Dans S^3 les conjectures 0.2 et 0.3 sont vraies au niveau des caractéristiques d'Euler.*

Dans le dernier chapitre de cette thèse on commence à examiner la Conjecture 0.2 au niveau de l'homologie. Comme remarqué ci-dessus $\widehat{HFK}_{*,*}(K,Y)$ et $\widehat{ECK}_{*,*}(K,Y,\alpha)$ sont des modules bigradués. Le degré filtré vient d'une filtration induite par K sur des complexes des chaînes convenables $\widehat{CF}_*(Y)$ et $\widehat{ECC}_*(Y,\alpha)$ pour $\widehat{HF}_*(Y)$ et, respectivement, $\widehat{ECH}_*(Y,\alpha)$.

Dans leur série d'articles, Colin, Ghiggini et Honda définissent des morphismes de complexes des chaînes

$$\begin{aligned} \Phi : \widehat{CF}_*(-Y) &\to \widehat{ECC}_*(Y,\alpha) \\ &\text{et} \\ \Psi : \widehat{ECC}_*(Y,\alpha) &\to \widehat{CF}_*(-Y) \end{aligned}$$

qui induisent des isomorphismes en homologie, inverse l'un de l'autre. Ici les complexes $\widehat{CF}_*(-Y)$ et $\widehat{ECC}_*(Y,\alpha)$ sont définis de façon appropriée à partir d'une décomposition en livre ouvert (K,S,ϕ) de Y avec reliure connexe. Soit H (resp. G) une homotopie de chaînes entre $\Psi\circ\Phi$ (resp. $\Phi\circ\Psi$) et le morphisme identité sur $\widehat{CF}(-Y)$ (resp. $\widehat{ECC}(Y,\alpha)$). Tous ces morphismes sont définis en comptant certaines courbes holomorphes dans des variétés symplectiques avec bord. Par des résultats classiques sur les suites spectrales, si l'on prouve que tous les morphismes ci-dessus préservent les filtrations données par K sur $\widehat{CF}_*(-Y)$ et $\widehat{ECC}_*(Y,\alpha)$, alors la conjecture 0.2 est vraie pour les nœuds fibrés.

En section 7.3 on prouvera le résultat suivant :

Théorème 0.7. *Soit (K, S, ϕ) une décomposition en livre ouvert de Y. Alors il existe α pour laquelle Φ préserve la filtration du nœud donnée par K sur les complexes.*

Une des difficultés principales dans la preuve du théorème 0.7 vient du fait que les courbes holomorphes comptées par Φ n'intersectent jamais K : par conséquent on ne peut pas appliquer certains arguments standards utilisés normalement dans des contextes similaires, comme par exemple dans la preuve que la différentielle de ECH respecte la filtration du nœud ([13]).

Notre stratégie consistera tout d'abord à modifier la monodromie et la forme de contact près du bord de $Y \setminus \mathcal{N}(K)$, afin de pouvoir définir les filtrations du nœud en HF et en ECH d'une façon similaire. Ensuite nous appliquerons des arguments de Wendl à propos de feuilletages holomorphes ([59]) pour vérifier que Φ' ne compte pas certaines courbes holomorphes qui ne respectent clairement pas la filtration. Finalement nous prouverons que toutes les courbes holomorphes comptées par Φ' préservent la filtration.

Ainsi le théorème 0.7 peut être considéré comme la première étape d'une preuve de la conjecture 0.2.

English version

A link with n components in a 3-manifold Y is the image of an embedding $\sqcup_{i=1}^{n} S^1 \hookrightarrow Y$. Two links L_1 and L_2 are said equivalent if there exists an ambient isotopy of Y carrying L_1 to L_2. A multitude of link invariants of different types has been defined and they can be of very different types. The simplest to define is perhaps the natural number n of the connected components of a link: a 1-component link is called a knot. An example of more sophisticated, but still "classical" invariant is *the Alexander polynomial* Δ_K, which associates to any link L in S^3 a Laurent polynomial.

In the last fifteen years new methods involving additional geometric structures on manifolds led to the discovery of new invariants of links, as well as of three and four manifolds.

A symplectic structure on an oriented even dimensional manifold M consists in a closed two form ω such that $\omega \wedge \ldots \wedge \omega$ is a positive volume form on M. Given a closed oriented three-manifold Y, in [42] Ozsváth and Szabó built an auxiliary even dimensional manifold $M(Y)$ equipped with a symplectic form ω and an almost complex structure J; then they defined four $\mathbb{Z}$-modules $CF^*(M(Y), J, \omega)$, with $* \in \{\infty, +, -, \widehat{\ }\}$, endowed with differentials, obtained by counting certain holomorphic curves in $M(Y)$. The associated homology groups do not depend on the choices made for the auxiliary manifold and the geometric structures and are topological invariants of Y, indicated

$$HF^\infty(Y), \quad HF^+(Y), \quad HF^-(Y), \quad \widehat{HF}(Y).$$

These groups are the *Heegaard Floer homologies of* Y in the infinity, plus, minus and hat version respectively.

Moreover Ozsváth and Szabó in [44] and Rasmussen in [50] proved that any homologically trivial knot K in Y induces a "knot filtration" on the Heegaard Floer chain complexes. The first pages of the associated spectral sequences (in each versions) result then to be topological invariants of K: these are bigraded homology groups

$$HFK^\infty(K,Y), \quad HFK^+(K,Y), \quad HFK^-(K,Y), \quad \widehat{HFK}(K,Y)$$

called *Heegaard Floer knot homologies* (in the respective versions). These homologies are powerful invariants for the couple (K, Y). For instance in [44] and [50], it has been proved that $\widehat{HFK}(K, S^3)$ *categorifies the Alexander polynomial of* K. This means the following. Given a collection of finite dimensional chain complexes $C = \{(C_{*,i}, \partial_i)\}_{i\in\mathbb{Z}}$, its graded Euler characteristic is $\chi(C) = \sum_i \chi(C_{*,i})t^i$, where $\chi(C_{*,i})$ is the standard Euler characteristic of $C_{*,i}$ and t is a formal variable. By the properties of the Euler characteristic, this polynomial does not change by taking the homology of C. As said, $\widehat{HFK}(K, Y)$

is a bigraded collection of moduli: the fact that it categorifies the Alexander polynomial of K means that:

$$\chi(\widehat{HFK}(K,S^3)) \doteq \Delta(K),$$

where $\doteq$ means that the two sides are equal up to change sign and multiply by a monic monomial. This was the first categorification of the Alexander polynomial; a second one (in Seiberg-Witten-Floer homology) has been discovered later by Kronheimer and Mrowka ([35]).

In [46] Ozsváth and Szabó developed a similar construction for any link L in S^3 and got invariants

$$HFL^-(L,S^3) \qquad \text{and} \qquad \widehat{HFL}(L,S^3)$$

for L, which they called *Heegaard Floer link homologies*. Now these homologies come with an additional $\mathbb{Z}^n$ degree, where n is the number of the connected components of L. Ozsváth and Szabó proved moreover that $HFL^-(L,S^3)$ *categorifies the multivariable Alexander polynomial of* L, which is a generalization of the classic Alexander polynomial. They found in particular that:

$$\chi\left(HFL^-(L,S^3)\right) \doteq \begin{cases} \Delta_L(t_1,\ldots,t_n) & \text{if } n>1 \\ \\ \Delta_L(t)/(1-t) & \text{if } n=1. \end{cases} \tag{0.2}$$

In general, Heegaard Floer homology theory can give a huge amount of information about the topology of links and three-manifolds. Moreover it turned out to provide powerful tools in the understanding of certain geometric structures on the underling manifolds. One example is the presence of an *invariant of contact structures* in Heegaard Floer ([45]). A contact form in an oriented three manifold Y is a smooth one form α such that $\alpha \wedge d\alpha$ is a positive volume form. A contact structure on Y is a plane field ξ for which there exists a contact form α such that $\xi = \ker\alpha$. Two contact structures ξ_1 and ξ_2 are equivalent if there exists an isotopy of Y whose differential carries ξ_1 to ξ_2.

A key ingredient in the definition of the invariant of contact structures in Heegaard Floer is the Giroux one-to-one correspondence between contact structures and *open book decompositions*. Roughly speaking, an open book decomposition of a three manifold Y consists in a triple (L,S,ϕ), where L is a link in Y, S is a compact oriented surface with boundary and $\phi\colon S \to S$ is an orientation preserving diffeomorphism such that, if $\mathcal{N}(L)$ is a small tubular neighborhood of L, then $Y \setminus \mathcal{N}(L)$ is homeomorphic to $\frac{S\times[0,1]}{(x,1)\sim(\phi(x),0)}$. In this case L, S and ϕ are called the binding, the page and, respectively, the monodromy of the open book. If the knot K is the binding of some open book, then it is called a *fibered knot*. Given an open book decomposition (L,S,ϕ) for Y, Thurston and Wilkenkemper ([57]) described how to get a contact structure "adapted" to (L, S, ϕ). In [19], Giroux discovered a way to associate to a contact structure ξ on Y an open book decomposition of Y adapted (up to isotopy), in the sense of Thurston and Wilkenkemper, to ξ. By consequence he showed that there exists a one-to-one correspondence between isotopy classes of contact structures and (equivalence classes of) open book decompositions. This result gave a fundamental contribution to the study of the deep interactions intercurring between contact geometry and low dimensional topology.

Open book decompositions were used in a series of papers by Colin, Ghiggini and Honda to prove the equivalence between Heegaard Floer homology and *embedded contact homology* for three manifolds. The last one is another Floer homology theory, first defined by Hutchings, which associates to a contact manifold (Y, α) two graded modules

$$ECH(Y, \alpha) \quad \text{and} \quad \widehat{ECH}(Y, \alpha).$$

A contact form α determines univocally a non singular vector field R_α, called *Reeb vector field*. The generators of the ECH chain groups are then certain formal products of *Reeb orbits*, i.e. closed orbits of R_α.

Theorem 0.1 (Colin, Ghiggini, Honda, [8]-[12])**.**

$$HF^+(-Y) \cong ECH(Y, \alpha)$$
$$\widehat{HF}(-Y) \cong \widehat{ECH}(Y, \alpha),$$

where $-Y$ is the manifold Y with the inverted orientation.

In light of Theorem 0.1, it is a natural problem to find an embedded contact counterpart of Heegard Floer knot homology. In analogy with the *sutured Heegaard Floer theory* developed by Juhász ([32]), in [13] the authors define a sutured version of embedded contact homology. This can be thought of as a version of embedded contact homology for manifolds with boundary. In particular, given a knot K in a contact three manifold (Y, ξ), using sutures they define a *hat version of embedded contact knot homology*

$$\widehat{ECK}(K, Y, \alpha).$$

Roughly speaking, this is the hat version of ECH homology for the contact manifold with boundary $(Y \setminus \mathcal{N}(K), \alpha)$, where $\mathcal{N}(K)$ is a suitable thin tubular neighborhood of K in Y and α is a contact form for ξ satisfying specific compatibility conditions with K. In [13] the following conjecture is stated:

Conjecture 0.2.

$$\widehat{ECK}(K, Y, \alpha) \quad \cong \quad \widehat{HFK}(-K, -Y).$$

The aim of this thesis is to provide evidences of the veracity of this conjecture. The first thing we do is to define a *full version of embedded contact knot homology*

$$ECK(K, Y, \alpha)$$

for knots K in any contact three manifold (Y, ξ) endowed with a (suitable) contact form α for ξ. Moreover we generalize the definitions to the case of links L with more then one components to obtain homologies

$$ECK(L, Y, \alpha) \qquad \text{and} \qquad \widehat{ECK}(L, Y, \alpha).$$

We state then the following:

Conjecture 0.3. *For any knot K and link L in Y, there exist contact forms for which:*

$$\begin{array}{rcl} ECK(K, Y, \alpha) & \cong & HFK^-(K, Y) \\ & \textit{and} & \\ \widehat{ECK}(L, Y, \alpha) & \cong & \widehat{HFK}(-L, -Y), \\ ECK(L, Y, \alpha) & \cong & HFK^-(L, Y). \end{array}$$

Next we compute the graded Euler characteristics of the ECK homologies for knots and links in homology three-spheres and we prove the following:

Theorem 0.4. *Let L be an n-component link in a homology three-sphere Y. Then there exists a contact form α such that*

$$\chi(ECK(L, Y, \alpha)) \doteq \mathrm{ALEX}(Y \setminus L).$$

Here $\mathrm{ALEX}(Y \setminus L)$ is the *Alexander quotient* of the complement of L in Y. The theorem is proved using Fried's dynamic reformulation of ALEX ([16]). Classical relations between $\mathrm{ALEX}(S^3 \setminus L)$ and Δ_L imply the following result:

Theorem 0.5. *Let L be any n-component link in S^3. Then there exists a contact form α for which:*

$$\chi\left(ECK(L, S^3, \alpha)\right) \doteq \begin{cases} \Delta_L(t_1, \dots, t_n) & \textit{if } n > 1 \\ \Delta_L(t)/(1-t) & \textit{if } n = 1 \end{cases}$$

and

$$\chi\left(\widehat{ECK}(L, S^3, \alpha)\right) \doteq \begin{cases} \Delta_L(t_1, \dots, t_n) \cdot \prod_{i=1}^n (1-t_i) & \textit{if } n > 1 \\ \Delta_L(t) & \textit{if } n = 1. \end{cases}$$

This implies that the homology ECK *is a categorification of the multivariable Alexander polynomial.*
Moreover comparing last theorem with Equation 0.2 (and the analogue expressions for the hat versions of HFK and HFL proved in [46]), it follows that

Corollary 0.6. *In S^3, Conjectures 0.2 and 0.3 hold at level of Euler characteristics.*

In the last chapter of this thesis we begin to investigate Conjecture 0.2 at the homology level. As mentioned, both $\widehat{HFK}_{*,*}(K,Y)$ and $\widehat{ECK}_{*,*}(K,Y,\alpha)$ are bigraded modules. The further $\mathbb{Z}$-degree comes from a filtration induced by K on suitable chain complexes $\widehat{CF}_*(Y)$ and $\widehat{ECC}_*(Y,\alpha)$ for $\widehat{HF}_*(Y)$ and, respectively, $\widehat{ECH}_*(Y,\alpha)$.

In their series of papers, Colin-Ghiggini-Honda define chain maps

$$\begin{array}{rcl} \Phi : \widehat{CF}_*(-Y) & \longrightarrow & \widehat{ECC}_*(Y,\alpha) \\ & \text{and} & \\ \Psi : \widehat{ECC}_*(Y,\alpha) & \longrightarrow & \widehat{CF}_*(-Y) \end{array}$$

that induce isomorphisms in homology, which are the inverse of one another. Let H (resp. G) be chain homotopies between $\Psi \circ \Phi$ (resp. $\Phi \circ \Psi$) and the identity map of $\widehat{CF}(-Y)$ (resp. $\widehat{ECC}(Y,\alpha)$). All these maps are defined by counting certain holomorphic curves in symplectic four manifolds with boundary. By standard results about spectral sequences, if one can prove that all the maps above are filtered with respect of the knot filtrations on $\widehat{CF}_*(-Y)$ and $\widehat{ECC}_*(Y,\alpha)$, then conjecture 0.2 is true at least for the fibered knots.

In section 7.3 we prove the following:

Theorem 0.7. *Let (K,S,ϕ) be an open book decomposition of a three manifold Y. Then, there exists a contact form α for which Φ preserves the knot filtrations given by K on the complexes.*

One of the main difficulties in proving Theorem 0.7 comes from the fact that the holomorphic curves counted by Φ never cross K, so that we can not directly apply some standard argument common in this kind of situation and used, for example, in [13] to prove that the ECH-differential respects the knot filtration.

Our strategy will consist first in modifying the monodromy and the contact form near the boundary of $Y \setminus \mathcal{N}(K)$ in order to define the knot filtrations in HF and ECH in a similar way. Then we will apply Wendl's arguments about holomorphic foliations ([59]) to check that some specific holomorphic curve that evidently do not respect the knot filtrations are not in fact counted by Φ. Finally we will prove that all the remaining holomorphic curves counted by Φ respect the filtrations

For what said before, Theorem 0.7 can be viewed as the first step of a proof of conjecture 0.2.

I

1

Contact geometry and holomorphic curves

In this chapter we give basic definitions and results about contact geometry, open book decompositions of three manifolds and holomorphic curves. Moreover in Section 1.3 we recall some notions about Morse Bott theory in contact geometry.

1.1 Contact geometry

Let us begin by introducing some basic objects in three dimensional contact geometry. We refer the reader to [20] and the other cited references for the details.

A (co-oriented) *contact form* on a three dimensional oriented manifold Y is a $\alpha \in \Omega^1(Y)$ such that $\alpha \wedge d\alpha$ is a positive volume form. A *contact structure* is a smooth plane field ξ on Y such that there exists a contact form α for which $\xi = \ker \alpha$. The *Reeb vector field* of α is the (unique) vector field R_α determined by the equations $d\alpha(R_\alpha, \cdot) = 0$ and $\alpha(R_\alpha) = 1$. A *simple Reeb orbit* is a closed oriented orbit of $R = R_\alpha$, i.e. it is the image δ of an embedding $S^1 \hookrightarrow Y$ such that R_P is positively tangent to δ in any $P \in \delta$. A *Reeb orbit* is an m-fold cover of a simple Reeb orbit, with $m \geq 1$.
The form α determines an *action* $\mathcal{A}$ on the set of its Reeb orbits defined by $\mathcal{A}(\gamma) = \int_\gamma \alpha$. By definition $\mathcal{A}(\gamma) > 0$ for any non empty orbit γ.

A basic result in contact geometry asserts that the flow of the Reeb vector field (abbreviated Reeb flow) $\phi = \phi_R$ preserves ξ, that is $(\phi_t)_*(\xi_P) = \xi_{\phi_t(P)}$ for any $t \in \mathbb{R}$ (see [20, Chapter 1]). Given a Reeb orbit δ, there exists $T \in \mathbb{R}^+$ such that $(\phi_T)_*(\xi_P) = \xi_P$ for any $P \in \delta$; if T is the smallest possible, the isomorphism $\mathfrak{L}_\delta := (\phi_T)_* : \xi_P \to \xi_P$ is called the (*symplectic*) *linearized first return map* of R in P.

The orbit δ is called *non-degenerate* if 1 is not an eigenvalue $\mathfrak{L}_\delta$. There are two types of non-degenerate Reeb orbits: *elliptic* and *hyperbolic*. δ is elliptic if the eigenvalues of $\mathfrak{L}_\delta$ are on the unit circle and is hyperbolic if they are real. In the last case we can make a further distinction: δ is called *positive* (*negative*) hyperbolic if the eigenvalues are both positive (resp. negative).

Definition 1.1. *The* Lefschetz sign *of a non-degenerate Reeb orbit δ is*

$$\epsilon(\delta) := \operatorname{sign}(\det(\mathbb{1} - \mathfrak{L}_\delta)) \in \{+1, -1\}.$$

Observation 1.2. *It is easy to check that $\epsilon(\delta) = +1$ if δ is elliptic or negative hyperbolic and $\epsilon(\delta) = -1$ if δ is positive hyperbolic.*

To any non-degenerate orbit δ and a trivialization τ of $\xi|_\delta$ we can associate also the *Conley-Zehnder index* $\mu_\tau(\delta) \in \mathbb{Z}$ of δ with respect to τ. Even if we do not give a precise definition (that can be found for example in [14] or [23]) we will provide an explicit description of this index (see [28, section 3.2]).

Given $P \in \delta$, using the basis of $\xi|_\delta$ determined by τ we can regard the differentials $\phi_{t*} : \xi_P \to \xi_{\phi_t(P)}$ of the Reeb flow as a path in $t \in [0, T]$ of 2×2 symplectic matrices. In particular $\phi_{0*} : \xi_P \to \xi_P$ is the identity matrix and, if T is as above, $\phi_{T*} : \xi_P \to \xi_P$ is a matrix representation for $\mathfrak{L}_\delta$.

If δ is elliptic, following this path for $t \in [0, T]$, ϕ_{T*} will represent a rotation by some angle $2\pi\theta$ with $\theta \in \mathbb{R} \setminus \mathbb{Z}$ (since δ is non degenerate). Then $\mu_\tau(\delta) = 2\lfloor\theta\rfloor + 1$, where $\lfloor\theta\rfloor$ is the highest integer smaller then θ.

Otherwise, if δ is hyperbolic, then the symplectic matrix of ϕ_{T*} rotates the eigenvectors of $\mathfrak{L}_\delta$ by an angle $k\pi$ with $k \in 2\mathbb{Z}$ if δ is positive hyperbolic and $k \in 2\mathbb{Z} + 1$ if δ is negative hyperbolic. Then $\mu_\tau(\delta) = k$.

Observation 1.3. *Even if $\mu_\tau(\delta)$ depends on τ, its parity depends only on δ. Indeed, if δ is elliptic, then $\mu_\tau(\delta) \equiv 1 \mod 2$. Moreover suppose that δ is hyperbolic and $\mu_\tau(\delta) = k$; if τ' differs from τ by a twist of an angle $2n\pi$ with $n \in \mathbb{Z}$, the rotation by $k\pi$ on the eigenvectors will be composed with a rotation by $2n\pi$. Then $\mu_{\tau'}(\delta) = k + 2n \equiv k \mod 2$.*

Corollary 1.4. *If δ is non-degenerate then for any τ*

$$(-1)^{\mu_\tau(\delta)} = -\epsilon(\delta).$$

Definition 1.5. *Given $X \subseteq Y$, we will indicate by $\mathcal{P}(X)$ the set of simple Reeb orbits of α contained in X. An* orbit set *(or* multiorbit*) in X is a formal finite product $\gamma = \prod_i \gamma_i^{k_i}$, where $\gamma_i \in \mathcal{P}(X)$ and $k_i \in \mathbb{N}$ is the* multiplicity *of γ_i in γ, with $k_i \in \{0, 1\}$ whenever γ_i is hyperbolic. The set of multiorbits in X will be denoted by $\mathcal{O}(X)$.*

Note that the empty set is considered as an orbit, called *empty orbit* and it is indicated by $\emptyset$.

An orbit set $\gamma = \prod_i \gamma_i^{k_i}$ belongs to the homology class $[\gamma] = \sum_i k_i[\gamma_i] \in H_1(Y)$ (unless stated otherwise, all homology groups will be taken with integer coefficients). Moreover the action of γ is defined by $\mathcal{A}(\gamma) = \sum_i k_i \int_{\gamma_i} \alpha$.

1.2 Holomorphic curves

In this section we recall some definitions and properties about holomorphic curves in dimension 4. We refer the reader to [38] and [39] for the general theory and [28] and [9]-[12] for an approach more specialized to our context.

Let X be an oriented even dimensional manifold. An *almost complex structure* on X is an isomorphism $J : TX \to TX$ such that $J(T_P X) = T_P X$ and $J^2 = -id$. If (X_1, J_1) and (X_2, J_2) are two even dimensional manifolds endowed with an almost complex structure, a map $u : (X_1, J_1) \to (X_2, J_2)$ is *pseudo-holomorphic* if it satisfies the *Cauchy-Riemann equation*

$$du \circ J_1 = J_2 \circ du.$$

Definition 1.6. *A* pseudo-holomorphic curve *in a four-dimensional manifold (X, J) is a pseudo-holomorphic map $u : (F, j) \to (X, J)$, where (F, j) is a Riemann surface.*

Note that here we do not require that F is connected.

In this thesis we will be particularly interested in pseudo-holomorphic curves (that sometimes we will call simply holomorphic curves) in “symplectizations” of contact three manifolds. Let (Y, α) be a contact three-manifold and consider the four-manifold $\mathbb{R} \times Y$. Call s the $\mathbb{R}$-coordinate and let $R = R_\alpha$ be the Reeb vector field of α. The almost complex structure J on $\mathbb{R} \times Y$ is *adapted to* α if

1. J is s-invariant;
2. $J(\xi) = \xi$ and $J(\partial_s) = R$ at any point of $\mathbb{R} \times Y$;
3. $J|_\xi$ is compatible with $d\alpha$, i.e. $d\alpha(\cdot, J\cdot)$ is a Riemannian metric.

For us, a holomorphic curve u in the symplectization of (Y, α) is a holomorphic curve $u : (F, j) \to (\mathbb{R} \times Y, J)$, where:

i. J is adapted to α;

ii. $(\dot{F}, j)$ is a Riemann surface obtained from a closed surface F by removing a finite number of points (called *punctures*);

iii. for any puncture x there exists a neighborhood $U(x) \subset F$ such that $U(x) \setminus \{x\}$ is mapped by u asymptotically to a cover of a cylinder $\mathbb{R} \times \delta$ over an orbit δ of R in a way that $\lim_{y \to x} \pi_{\mathbb{R}}(u(y)) = \pm\infty$, where $\pi_{\mathbb{R}}$ is the projection on the $\mathbb{R}$-factor of $\mathbb{R} \times Y$.

We say that x is a *positive puncture* of u if in the last condition above the limit is $+\infty$: in this case the orbit δ is a *positive end* of u. If otherwise the limit is $-\infty$ then x is a *negative puncture* and δ is a *negative end* of u.

If δ is the Reeb orbit associated to the puncture x, then u near x determines a cover of δ: the number of sheets of this cover is the *local x-multiplicity of δ in u*. The sum of the x-multiplicities over all the punctures x associated to δ is the *(total) multiplicity* of δ in u.

If γ (γ') is the orbit set determined by the set of all the positive (negative) ends of u counted with multiplicity, then we say that u is a *holomorphic curve from γ to γ'*.

Example 1.7. *A* cylinder over an orbit set *γ of Y is the holomorphic curve $\mathbb{R} \times \gamma \subset \mathbb{R} \times Y$.*

Observation 1.8. *Note that if there exists a holomorphic curve u from γ to γ', then $[\gamma] = [\gamma'] \in H_1(Y, \mathbb{Z})$.*

We state now some result about holomorphic curves that will be useful later.

Lemma 1.9 (see for example [58])**.** *If u is a holomorphic curve in the symplectization of (Y, α) from γ to γ', then $\mathcal{A}(\gamma) \geq \mathcal{A}(\gamma')$.*

This lemma follows by applying the Stokes' theorem and using the fact that $d\alpha$ is always non negative on a holomorphic curve.

Theorem 1.10 ([39], Lemma 2.4.1)**.** *Let $u : (F, j) \to (\mathbb{R} \times Y, J)$ be a non-constant holomorphic curve in (X, J), then the critical points of $\pi_{\mathbb{R}} \circ u$ are isolated. In particular, if π_Y denotes the projection $\mathbb{R} \times Y \to Y$, $\pi_Y \circ u$ is transverse to R_α away from a set of isolated points.*

From now on if u is a map with image in $\mathbb{R} \times Y$, we will set $u_{\mathbb{R}} := \pi_{\mathbb{R}} \circ u$ and $u_Y := \pi_Y \circ u$.

Holomorphic curves also enjoy the following property, which will be important for us: see for example [21].

Theorem 1.11 (Positivity of intersection; Gromov, McDuff, Micallef-White)**.** *Let u and v be two distinct holomorphic curves in a four manifold (W, J). Then $\#(\mathrm{Im}(u) \cap \mathrm{Im}(v)) < \infty$. Moreover, if P is an intersection point between $\mathrm{Im}(u)$ and $\mathrm{Im}(v)$, then its contribution m_P to the algebraic intersection number $\langle \mathrm{Im}(u), \mathrm{Im}(v) \rangle$ is strictly positive, and $m_P = 1$ if and only if u and v are embeddings near P that intersect transversely in P.*

When the almost complex structure does not play an important role or is understood it will be omitted from the notations.

1.3 Morse-Bott theory

The Morse-Bott theory in contact geometry has been first developed by Bourgeois in [3]. We present in this section some basic notions and applications, mostly as presented in [9].

Definition 1.12. *A* Morse-Bott torus *(briefly M-B torus) in a* 3*-dimensional contact manifold* (Y, α) *is an embedded torus* T *in* Y *foliated by a family* γ_t, $t \in S^1$, *of Reeb orbits, all in the same class in* $H_1(T)$*, that are non-degenerate in the Morse-Bott sense. Here this means the following. Given any* $P \in T$ *and a positive basis* (v_1, v_2) *of* ξ_P *where* $v_2 \in T_P(T)$ *(so that* v_1 *is transverse to* $T_P(T)$*), then the differential of the first return map of the Reeb flow on* ξ_P *is of the form*

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

for some $a \neq 0$*. If* $a > 0$ *(resp.* $a < 0$*) then* T *is a* positive *(resp.* negative*) M-B torus.*

We say that α is a *Morse-Bott contact form* if all the Reeb orbits of α are either isolated and non-degenerate or come in S^1-families foliating M-B tori.

As explained in [3] and [9, section 4] it is possible to modify the Reeb vector field in a small neighborhood of a M-B torus T preserving only two orbits, say e and h, of the S^1-family of Reeb orbits associated to T.

Moreover, for any fixed $L > 0$, the perturbation can be done in a way that e and h are the only orbits in a neighborhood of T with action less then L.

If T is a positive (resp. negative) M-B torus and τ is the trivialization of ξ along the orbits given pointwise by the basis (v_1, v_2) above, then one can make the M-B perturbation in a way that h is positive hyperbolic with $\mu_\tau(h) = 0$ and e is elliptic with $\mu_\tau(e) = 1$ (resp. $\mu_\tau(e) = -1$).

The orbits e and h can be seen as the only two critical points of a Morse function $f_T : S^1 \to \mathbb{R}$ defined on the S^1-family of Reeb orbits foliating T and with maximum corresponding to the orbit with higher C-Z index. Often M-B tori will be implicitly given with such a function.

Observation 1.13. *It is important to remark that, before the perturbation,* T *is foliated by Reeb orbits of* α *and so these are non-isolated. Moreover the form of the differential of the first return map of the flow of* ξ *implies that these orbits are also degenerate.*

After the perturbation, T *contains only two isolated and non degenerate orbits, but other orbits are created in a neighborhood of* T *and these orbits can be non-isolated and degenerate. See Figure 1.1 later for an example of M-B perturbations.*

Proposition 1.14 ([3], Section 3)**.** *For any M-B torus T and any $L \in \mathbb{R}$ there exists a M-B perturbation of T such that, with the exception of e and h, all the periodic orbits in a neighborhood of T have action greater then L.*

A torus T foliated by Reeb orbits all in the same class of $H_1(T)$ (like for example a Morse-Bott torus) can be used to obtain constraints about the behaviour of a holomorphic curve near T.

Following [9, Section 5], if γ is any of the Reeb orbits in T, we can define the *slope of* T as the equivalence class $s(T)$ of $[\gamma] \in H_1(T, \mathbb{R}) - \{0\}$ up to multiplication by positive real numbers.

Let $T \times [-\epsilon, \epsilon]$ be a neighborhood of $T = T \times \{0\}$ in Y with coordinates (ϑ, t, y) such that $(\partial_\vartheta, \partial_t)$ is a positive basis for $T(T)$ and ∂_y is directed as a positive normal vector to T.

Suppose that $u : (F, j) \to (\mathbb{R} \times Y, J)$ is a holomorphic curve in the symplectization of (Y, α); by Theorem 1.10, there exist at most finitely many points in $T \times [-\epsilon, \epsilon]$ where $u_Y(F)$ is not transverse to R_α. Then, if $T_y := T \times \{y\}$ and $u(F)$ intersects $\mathbb{R} \times T_y$, we can associate a slope $s_{T_y}(u)$ to $u_Y(F) \cap T_y$, for any $y \in [-\epsilon, \epsilon]$: this is defined exactly like $s(T)$, where $u_Y(F) \cap T_y$ is considered with the orientation induced by $\partial\left(u_Y(F) \cap (T \times [-\epsilon, y])\right)$.

Observation 1.15. *Note that if u has no ends in $T \times [y, y']$, then*

$$\partial(u_Y(F) \cap T \times [y, y']) = u_Y(F) \cap T_{y'} - u_Y(F) \cap T_y$$

and $s_{T_y}(u) = s_{T'_y}(u)$.

The following Lemma is a consequence of the positivity of intersection in dimension four (see [9, Lemma 5.2.3]).

Lemma 1.16 (Blocking Lemma)**.** *Let T be linearly foliated by Reeb trajectories with slope $s = s(T)$ and u a holomorphic curve be as above.*

1. *If u is homotopic, by a compactly supported homotopy, to a map whose image is disjoint from $\mathbb{R} \times T$, then $u_Y(F) \cap T = \emptyset$.*
2. *Let $T \times [-\epsilon, \epsilon]$ be a neighborhood of $T = T \times \{0\}$. Suppose that, for some $y \in [-\epsilon, \epsilon] \setminus \{0\}$, u has no ends in $T \times (0, y]$ if $y \in (0, \epsilon]$ or in $T \times [y, 0)$ if $y \in [-\epsilon, 0)$. If $s_{T_y}(u) = \pm s(T)$ then u has an end which is a Reeb orbit in T.*

Let now x be a puncture of F whose associated end is an orbit γ in T; if there exists a neighborhood $U(x)$ of x in F such that $u_Y(U(x) \setminus \{x\}) \cap T = \emptyset$ then γ is a *one sided end* of u in x. This is equivalent to requiring that $u_Y(U(x))$ is contained either in $T \times (-\epsilon, 0)$ or in $T \times (0, \epsilon)$.

The following is proved in [9] (Lemma 5.3.2).

Lemma 1.17 (Trapping Lemma). *If T is a positive (resp. negative) M-B torus and $\gamma \subset T$ is a one sided end of u associated to the puncture x, then x is positive (resp. negative).*

Even if we do not give here the proofs of the last two lemmas, we will extensively use similar arguments later (in particular in Chapter 7).

1.4 Open books

Definition 1.18. *Given a surface S and a diffeomorphism $\phi\colon S \to S$, the* mapping torus *of (S, ϕ) is the three dimensional manifold*

$$N(S, \phi) := \frac{S \times [0, 2]}{(x, 2) \sim (\phi(x), 0)}.$$

In this paper we use the following definition of open book decomposition of a 3-manifold Y. This is not the original definition but a more specific version based on [9].

Definition 1.19. *An* open book decomposition for Y *is a triple (L, S, ϕ) such that*

- *$L = K_1 \sqcup \ldots \sqcup K_n$ is an n-component link in Y;*
- *S is a smooth, compact, connected, oriented surface with an n-components boundary;*
- *$\phi : S \to S$ is an orientation preserving diffeomorphism such that on a small neighborhood $\{1, \ldots, n\} \times [0, 1] \times S^1$ of $\partial S = \{1, \ldots, n\} \times \{1\} \times S^1$, with coordinates (y, ϑ) near each component, it acts by*

$$(y, \vartheta) \xmapsto{\phi} (y, \vartheta - y + 1) \tag{1.1}$$

  (and in particular $\phi|_{\partial S} = id_{\partial S}$);
- *for each K_i there exists a tubular neighborhood $\mathcal{N}(K_i) \subset Y$ of K_i such that Y is diffeomorphic to $N(S, \phi) \sqcup_{i=1}^{n} \mathcal{N}(K_i)$ where the union symbol means that for any i, $\{i\} \times \{1\} \times S^1 \times \frac{[0,2]}{0\sim 2}$ is glued to $\mathcal{N}(K_i)$ in a way that, for any $\vartheta \in S^1$, $\{i\} \times \{1\} \times \{\vartheta\} \times \frac{[0,2]}{(0\sim 2)}$ is identified with a meridian of K_i in $\partial\mathcal{N}(K_i)$.*

The link L is called the binding, *the surfaces $S \times \{t\}$ are the* pages *and the diffeomorphism ϕ is the* monodromy *of the open book.*

When we are interested mostly in the mapping torus part of an open book decomposition we will use a notation of the form (S, ϕ), omitting the reference to its binding. Sometimes we will call (S, ϕ) an *abstract open book*.

Following [9], we will often consider each $\mathcal{N}(K_i)$ as a union of a copy of $\frac{[0,2]}{(0\sim 2)} \times [1, 2] \times S^1$, endowed with the extension of the coordinates (t, y, ϑ), glued along $\{y = 2\}$ to a smaller neighborhood $V(K_i)$ of K_i. The gluing is done in a way that the sets $\{\vartheta = \mathrm{const.}\}$ are identified with meridians for K and the sets $\{t = \mathrm{const.}\}$ are identified to longitudes.

By the Giroux's work in [19] there is a one to one correspondence between contact structures (up to isotopy) and open book decompositions (up to *Giroux stabilizations*) of Y. In order to simplify the notations, we consider here open books with connected binding.

Given (K, S, ϕ) we can follow the Thurston-Wilkenkemper construction ([57]) to associate to it an *adapted contact form* α on Y as explained in [9, section 2]. In N the resulting Reeb vector field $R = R_\alpha$ enjoys the following properties:

– R is transverse to the pages $S \times \{t\}$ $\forall t \in [0, 2]$;
– the first return map of R is isotopic to ϕ;
– each torus $T_y = S^1 \times \frac{[0,2]}{(0\sim 2)} \times \{y\}$, for $y \in [0, 1]$, is linearly foliated by Reeb orbits and the first return map of R on T_y is

$$(y, \vartheta) \mapsto (y, \vartheta - y + 1).$$

The last implies that when the set of orbits foliating T_y comes in an S^1-family, T is Morse-Bott.

To explain the behaviour of R on $\mathcal{N}(K)$, let us extend the coordinates (ϑ, t, y) to $V \setminus K \cong \frac{[0,2]}{(0\sim 2)} \times [2, 3) \times S^1$, where $K = \{y = 3\}$. For $y \in [0, 3)$ set $T_y = \frac{[0,2]}{(0\sim 2)} \times \{y\} \times S^1$. Given a curve $\gamma(x) = (\gamma_t(x), y, \gamma_\vartheta(x))$ in T_y we can define the *slope* of γ in x_0 by

$$s_{T_y}(\gamma, x_0) = \frac{\gamma_t'(x_0)}{\gamma_\vartheta'(x_0)} \in \mathbb{R} \cup \{\pm\infty\}.$$

In particular if a meridian has constant slope, this must be $+\infty$ and ∂S has slope 0. Note that the slope of T_y as given by

$$s(T_y) = \frac{\gamma_t(x)}{\gamma_\vartheta(x)} \in \mathbb{R} \cup \{\pm\infty\},$$

where now γ is a parametrization of a Reeb trajectory in T_y and $x \in \mathrm{Im}(\gamma)$. Note in particular that if $s(T_y)$ is irrational then T_y does not contain Reeb orbits, and if T_y is foliated by meridians (like T_1) then $s(T_y) = +\infty$.

On $\frac{[0,2]}{(0\sim 2)} \times [1, 2] \times S^1$ the contact form will depend on a small real constant $\delta > 0$: call α_δ the contact form on all Y. Let $f_\delta : [1, 3) \to \mathbb{R}$ be a smooth function such that:

– f_δ has minimum in $y = 1.5$ of value $-\delta$;
– $f_\delta(1) = f_\delta(2) = 0$;
– $f_\delta(y) = -y + 1$ near $\{y = 1\}$;
– $f_\delta'(y) < 0$ for $y \in [1, 1.5)$ and $f_\delta'(y) > 0$ for $y \in (1.5, 3)$.

Then the Reeb vector field R of α_δ in $\mathcal{N}(K) \setminus int(V)$ is such that:
– R is transverse to the annuli $\{t\} \times [1,2] \times S^1$ $\forall t \in \frac{[0,2]}{0\sim 2}$;
– the tori T_y, $y \in [1,2]$ are foliated by Reeb orbits with constant slope and first return map given by $(y, \vartheta) \mapsto (y, \vartheta + f_\delta(y))$.

Finally in V each torus T_y is linearly foliated by Reeb orbits whose slope vary in $(C, +\infty]$ for y going from 3 (not included) to 2 and, where C is a positive real number. Moreover K is also a Reeb orbit.

Note that for every δ, T_1 is a negative M-B torus foliated by orbits with constant slope $+\infty$. As explained in 1.3 we can perturb the associated S^1-family of orbits into a pair of simple Reeb orbits (e, h), where e is an elliptic orbit with C-Z index -1 and h is positive hyperbolic with index 0 (the indexes are computed with respect to the trivialization given by the torus).

Similarly the positive M-B torus T_2 is also foliated by orbits with constant slope $+\infty$ and a M-B perturbation gives a pair of simple Reeb orbits (e_+, h_+) in T_2, where e_+ is elliptic of index 1 and h_+ is hyperbolic of index 0 (in the papers [9]-[12] the orbits e_+ and h_+ are called e' and h' respectively).

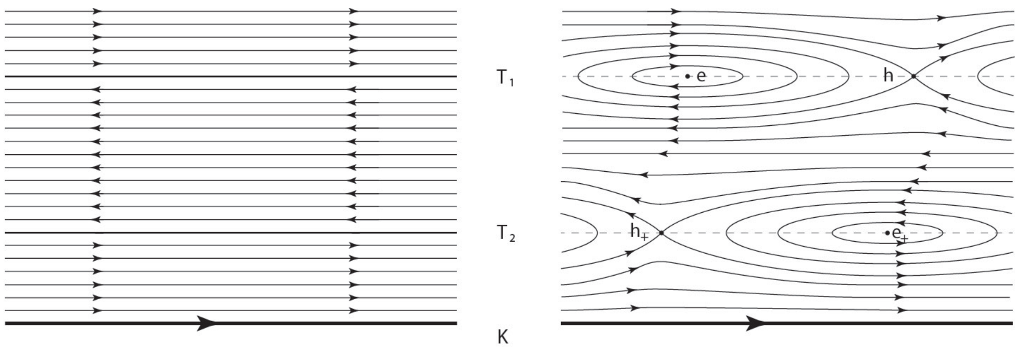

Figure 1.1: Reeb dynamic before and after a M-B perturbation of the tori T_1 and T_2. Both pictures take place in a page of the open book. Each flow line represents an invariant subset of S under the Reeb flow near K; the orientation gives the direction in which any point is mapped under the first return map of the flow.

In the rest of the paper, if not stated otherwise, when we talk about contact forms and their Reeb vector fields adapted to an open book we will always refer to them assuming the notations and the properties explained in this subsection. In particular the M-B tori T_1 and T_2 will be always assumed to be perturbed into the respective pairs of simple orbits.

Observation 1.20. *In the case of open books with non-connected binding L, the Reeb vector field of an adapted contact form satisfies the same properties above near each component of L.*

We saw that to any open book decomposition (L, S, ϕ) of Y it is possible to associate an adapted contact form. Let us now say something about the inverse map of the Giroux correspondence.

Theorem 1.21 (Giroux)**.** *Given a contact three-manifold (Y, ξ), there exists an open book decomposition (L, S, ϕ) of Y and an adapted contact form α such that* $\ker(\alpha) = \xi$.

Sketch of the proof. Given any contact structure ξ on Y, in [19] Giroux explicitly constructs an open book decomposition (L, S, ϕ) of Y for which there exists a compatible contact form α such that $\ker(\alpha) = \xi$. Following [6, Section 3], the proof can be carried on in three main steps.

The first step consists in providing a cellular decomposition $\mathcal{D}$ of Y that is, in a precise sense, "*compatible with* ξ". It is important to remark that, up to take a refinement (in a way that each 3-cell is contained in a Darboux ball) any cellular decomposition of Y can be isotoped to make it compatible with ξ.

In the second step, $\mathcal{D}$ is used to explicitly build (L, S, ϕ). We describe now some of the properties of S, seen as the embedded 0-page of the open book. Let $\mathcal{D}^i$ be the i-skeleton of $\mathcal{D}$ and let $\mathcal{N}(\mathcal{D}^1)$ be a tubular neighborhood of $\mathcal{D}^1$. Suppose that $\mathcal{N}(\mathcal{D}^0) \subset \mathcal{N}(\mathcal{D}^1)$ is a tubular neighborhood of $\mathcal{D}^0$ such that $\mathcal{N}(\mathcal{D}^1) \setminus \mathcal{N}(\mathcal{D}^0)$ is homeomorphic to a tubular neighborhood of $\mathcal{D}^1 \setminus \mathcal{N}(\mathcal{D}^0)$. Then:

1. $S \subset \mathcal{N}(\mathcal{D}^1)$, $L := \partial S \subset \partial \mathcal{N}(\mathcal{D}^1)$ and $\mathcal{D}^1 \subset int(S)$;
2. $S \cap (\mathcal{N}(\mathcal{D}^1) \setminus \mathcal{N}(\mathcal{D}^0))$ is a disjoint union of strips which are diffeomorphic to $(\mathcal{D}^1 \setminus \mathcal{N}(\mathcal{D}^0)) \times [-1, 1]$ with $\mathcal{D}^1 \setminus \mathcal{N}(\mathcal{D}^0)$ corresponding to $(\mathcal{D}^1 \setminus \mathcal{N}(\mathcal{D}^0)) \times \{0\}$;

The fact that $\mathcal{D}$ is compatible with ξ implies that L intersects each 2-simplex exactly twice and it is possible to use this fact to prove that the complement of L in Y fibers in circles over S, which implies that L is the binding of an open book with 0-page the complement in S of a small neighborhood of S.

The third step consists finally in defining the contact form α with the required properties. $\square$

Theorem 1.22 (Giroux correspondence)**.** *Let α and α' be contact structures on Y that are adapted to the open books (L, S, ϕ) and, respectively, (L', S', ϕ'). Then α and α' are isotopic if and only if (L', S', ϕ') can be obtained from (L, S, ϕ) by a sequence of Giroux stabilizations and destabilizations.*

A *Giroux stabilization of an open book* is an operation that associates to an open book decomposition (L, S, ϕ) of Y another open book decomposition (L', S', ϕ') of Y, obtained as follows. Choose two points P_1 and P_2 in ∂S (not necessarily in the same connected component) and let γ be an oriented embedded path in S from P_1 to P_2. Let now S' be the oriented surface obtained by attaching a 1-handle to S along the attaching sphere (P_1, P_2). Consider the closed oriented loop $\bar{\gamma} \subset S'$ defined by $\bar{\gamma} := \gamma \sqcup c$, where c is the core curve of the 1-handle, oriented from P_2 to P_1, and the gluing is done along the common boundary (P_1, P_2) of the two paths.

By the definition of monodromy of open book that we gave, the ϕ is the identity along ∂S. So ϕ extends to the identity map on the handle: we keep calling ϕ the resulting diffeomorphism on S'. If $\tau_{\bar{\gamma}}$ is a positive Dehn twist along $\bar{\gamma}$, define $\phi' = \tau_{\bar{\gamma}} \circ \phi$.

It results that $N' := N(S', \phi')$ embeds in Y and that $Y \setminus N'$ is a disjoint union of solid tori. Then, if L' is the set of the core curves of these tori, (L', S', ϕ') is an open book decomposition of Y, which is said to be obtained by *Giroux stabilization* of (L, S, ϕ) along γ.

There is an obvious inverse operation of the stabilization: with the notations above, we say that (L, S, ϕ) is obtained by *Giroux destabilization* of (L', S', ϕ') along γ'.

Note that a Giroux stabilization does not change the components of L that do not intersect the attaching sphere. Moreover it is not difficult to see that the number of connected components of L and L' differs by 1: if P_1 and P_2 are chosen in the same component then L' has one component more than L; otherwise L' has one component less then L.

2

Embedded contact homology

This chapter is devoted to recalling some basic facts about embedded contact homology theory.

In the first section we briefly remind the Hutchings' original definition of $ECH(Y, \alpha)$ and $\widehat{ECH}(Y, \alpha)$ for a closed contact three-manifold (Y, α). In Section 2.2 we summarize some definitions and results given in [9]. We present in particular the definition of the embedded contact homology groups $ECH(N, \alpha)$ and $\widehat{ECH}(N, \alpha)$ for contact three-manifolds (N, α) with torus boundary. Moreover, if N is the complement of a neighborhood of a knot K in a closed three manifold Y, we recall the definition of the relative versions $ECH(N, \partial N, \alpha)$ and $\widehat{ECH}(N, \partial N, \alpha)$, which are proved (still in [9]) to be isomorphic to the homologies $ECH(Y, \alpha)$ and $\widehat{ECH}(Y, \alpha)$ respectively.

In Section 2.3 we remind the definition of the periodic Floer homology groups for open books. As we will see their definition is closely related to that of ECH.

Finally in Section 2.4 we remind the definition of the version of $\widehat{ECH}$ for homologically trivial knots. We will not give the original definition in terms of sutures as appears in [13] but the reinterpretation given in the end of Chapter 9 of [8].

2.1 ECH for closed three-manifolds

Let (Y, α) be a closed contact three-manifold and assume that α is non-degenerate (i.e., that any Reeb orbit of α is non-degenerate).

For a fixed $\Gamma \in H_1(Y)$, define $ECC(Y, \alpha, \Gamma)$ to be the free $\mathbb{Z}_2$-module generated by the orbit sets of Y in the homology class Γ and pose

$$ECC(Y, \alpha) = \bigoplus_{\Gamma \in H_1(Y)} ECC(Y, \alpha, \Gamma).$$

This is the *ECH chain group of* (Y, α).

The *ECH-differential* ∂^{ECH} (called simply ∂ when no risk of confusion occurs) is defined in [27] in terms of holomorphic curves in the symplectization $(\mathbb{R} \times Y, d\alpha, J)$ of (Y, α) as follows.

Given $\gamma, \delta \in \mathcal{O}(Y)$, let $\mathcal{M}(\gamma, \delta)$ be the set of (possibly disconnected) holomorphic curves $u : (\dot{F}, j) \to (\mathbb{R} \times Y, J)$ from γ to δ, where $(\dot{F}, j)$ is a punctured compact Rieamannian surface. It is clear that u determines a relative homology class $[\mathrm{Im}(u)] \in H_2(\mathbb{R} \times Y; \gamma, \delta)$ and that if such a curve exists then $[\gamma] = [\delta] \in H_1(Y)$.

If $\xi = \ker(\alpha)$ and a trivialization τ of $\xi|_{\gamma \cup \delta}$ is given, to any surface $C \subset \mathbb{R} \times Y$ with $\partial C = \gamma - \delta$ it is possible to associate an *ECH-index*

$$I(C) := c_\tau(C) + Q_\tau(C) + \mu_\tau^I(\gamma, \delta),$$

which depends only on the relative homology class of C. Here

– $c_\tau(C) := c_1(\xi|_C, \tau)$ is the *first relative Chern class* of C;
– $Q_\tau(C)$ is the *τ-relative intersection paring* of $\mathbb{R} \times Y$ applied to C;
– $\mu_\tau^I(\gamma, \delta) := \sum_i \sum_{j=1}^{k_i} \mu_\tau(\gamma_i^j) - \sum_i \sum_{j=1}^{k_i} \mu_\tau(\delta_i^j)$, where μ_τ is the Conley-Zehnder index defined in Section 1.1.

We refer the reader to [28] for the details about these quantities. If u is a holomorphic curve from γ to δ set $I(u) = I(\mathrm{Im}(u))$ (well defined up to approximating $\mathrm{Im}(u)$ with a surface in the same homology class).

Define $\mathcal{M}_k(\gamma, \delta) := \{u \in \mathcal{M}(\gamma, \delta) \mid I(u) = k\}$. The ECH-differential is then defined on the generators of $ECC(Y, \alpha)$ by

$$\partial^{ECH}(\gamma) = \sum_{\delta \in \mathcal{O}(Y)} \sharp \left(\frac{\mathcal{M}_1(\gamma, \delta)}{\mathbb{R}} \right) \cdot \delta \tag{2.1}$$

where the fraction means that we quotient $\mathcal{M}_1(\gamma, \delta)$ by the $\mathbb{R}$-action on the curves given by the translation in the $\mathbb{R}$-direction in $\mathbb{R} \times Y$. In [28, Section 5] Hutchings proves that $\frac{\mathcal{M}_1(\gamma,\delta)}{\mathbb{R}}$ is a compact 0-dimensional manifold, so that $\partial^{ECH}(\gamma)$ is well defined.

The *(full) embedded contact homology of* (Y, α) is

$$ECH_*(Y, \alpha) := H_*(ECC(Y, \alpha), \partial^{ECH}).$$

It turns out that these groups do not depend either on the choices J in the symplectization or the contact form for ξ.

The index $*$ denotes a relative index induced on the generators by I. On the other hand it is possible to endow $ECH(Y,\xi)$ also with a canonical absolute $\mathbb{Z}/2$-grading as follows. If $\gamma = \prod_i \gamma_i^{k_i}$ set

$$\epsilon(\gamma) = \prod_i \epsilon(\gamma_i)^{k_i},$$

where $\epsilon(\gamma_i)$ is the Lefschetz sign of the simple orbit γ_i. Note that $\epsilon(\gamma)$ is given by the parity of the number of positive hyperbolic simple orbits in γ.

If u is a holomorphic curve from γ to δ, by simple computations it is possible to prove the following *index parity formula* (see for example Section 3.4 in [28]):

$$(-1)^{I(u)} = \epsilon(\gamma)\epsilon(\delta). \tag{2.2}$$

It follows then that the Lefschetz sign endows embedded contact homology with a well defined absolute grading.

Fix now a generic point $(0,z) \in \mathrm{Im}(u) \subset \mathbb{R} \times Y$. Given two orbit sets γ and δ, let

$$U_z : \quad ECC_*(Y,\alpha) \quad \longrightarrow \quad ECC_{*-2}(Y,\alpha)$$

be the map defined on the generators by

$$U_z(\gamma) = \sum_{\delta \in \mathcal{O}(Y)} \# \{u \in \mathcal{M}_2(\gamma,\delta) \mid (0,z) \in \mathrm{Im}(u)\} \cdot \delta.$$

Hutchings proves that U_z is a chain map that counts only a finite number of holomorphic curves and that this count does not depend on the choice of z. So it makes sense to define the map $U := U_z$ for any z as above. This is called the *U-map*.

The *hat version of embedded contact homology of* (Y,α) is defined as the homology $\widehat{ECH}(Y,\alpha)$ of the mapping cone of the U-map. By this we mean that $\widehat{ECH}(Y,\alpha)$ is defined to be the homology of the chain complex

$$ECC_{*-1}(Y,\alpha) \oplus ECC_*(Y,\alpha)$$

with differential defined by the matrix

$$\begin{pmatrix} -\partial_{*-1} & 0 \\ U & \partial_* \end{pmatrix}$$

where the element of the complex are thought as columns. Also $\widehat{ECH}(Y,\alpha)$ has the relative and the absolute gradings above.

Observation 2.1. *Note that ∂^{ECH} and U respect the homology class of the generators of $ECC_*(Y,\alpha)$. This implies that there are natural splits:*

$$\begin{array}{rcl} ECH(Y,\xi) & = & \bigoplus_{\Gamma \in H_1(Y)} ECC(Y,\xi,\Gamma); \\ \widehat{ECH}(Y,\xi) & = & \bigoplus_{\Gamma \in H_1(Y)} \widehat{ECH}(Y,\xi,\Gamma). \end{array} \tag{2.3}$$

We end this section by stating the following result (see for example [28]).

Theorem 2.2. *Let $\emptyset$ be the empty orbit. Then $[\emptyset] \in ECH(Y, \xi)$ is an invariant of the contact structure ξ.*

The class $[\emptyset]$ is called ECH *contact invariant* of ξ.

2.2 ECH for manifolds with torus boundary

In order to define ECH for contact three-manifolds (N, α) with nonempty boundary, some compatibility between α and ∂N should be assumed. In this paper we are particularly interested in three-manifolds whose boundary is a collection of disjoint tori.

In [9, Section 7] Colin, Ghiggini and Honda analyze this situation when ∂N is connected. If $\mathcal{T} = \partial N$ is homeomorphic to a torus, then they prove that the ECH-complex and the differential can be defined almost as in the closed case, provided that $R = R_\alpha$ is tangent to $\mathcal{T}$ and that α is non-degenerate in $int(N)$.

If the flow of $R|_\mathcal{T}$ is irrational they define $ECH(N, \alpha) = ECH(int(N), \alpha)$ while, if it is rational, they consider the case of $\mathcal{T}$ Morse-Bott and do a M-B perturbation of α near $\mathcal{T}$; this gives two Reeb orbits h and e on $\mathcal{T}$ and, since α is now a M-B contact form, the ECH-differential counts special holomorphic curves, called *M-B buildings*.

Definition 2.3. *Let α be a Morse-Bott contact form on the three manifold Y and J a regular almost complex structure on $\mathbb{R} \times Y$. Suppose that any M-B torus T in (Y, α) comes with a fixed a Morse function f_T. Let $\mathcal{P}(Y)$ be the set of simple Reeb orbits in Y minus the set of the orbits which correspond to some regular point of some f_T.*

A nice Morse-Bott building in *(Y, α) is a disjoint union of objects u of one of the following two types:*

1. *u is the submanifold of a M-B torus T corresponding to a gradient flow line of f_T: in this case the positive and negative end of u are the positive and, respectively, the negative end of the flow line;*
2. *u is a union of curves $\widetilde{u} \cup u_1 \cup \ldots \cup u_n$ of the following kind. $\widetilde{u}$ is a J-holomorphic curve in $\mathbb{R} \times Y$ with n ends $\{\delta_1, \ldots, \delta_n\}$ corresponding to regular values of some $\{f_{T_1}, \ldots, f_{T_n}\}$. Then, for each i, $\widetilde{u}$ is augmented by a gradient flow trajectory u_i of f_{T_i}: u_i goes from the maximum ϵ_i^+ of f_{T_i} to δ_i if δ_i is a positive end and goes from δ_i to the minimum ϵ_i^- of f_{T_i} if δ_i is a negative end. The ends of u are obtained from the ends of $\widetilde{u}$ by substituting each δ_i with the respective ϵ_i^+ or ϵ_i^-.*

Suppose now that Y is closed and $\mathcal{N} \cong D^2 \times S^1$ is a solid torus embedded in Y. If $N = Y \setminus int(\mathcal{N})$, under some assumption on the behaviour of α in a neighborhood of $\mathcal{N}$, in [9] the authors define relative versions $ECH(N, \partial N, \alpha)$ and $\widehat{ECH}(N, \partial N, \alpha)$ of embedded contact homology groups and prove that

$$ECH(N, \partial N, \alpha) \cong ECH(Y, \alpha); \tag{2.4}$$
$$\widehat{ECH}(N, \partial N, \alpha) \cong \widehat{ECH}(Y, \alpha). \tag{2.5}$$

The notation suggests that these new homology groups are obtained by counting only orbits in N and quotienting by orbits on ∂N. Let us see the definition of these homologies in more details.

As mentioned above, to define these versions of embedded contact homology and prove the isomorphisms above, some compatibility between α and $\mathcal{N}$ is required. We refer the reader to [9, Section 6] for the details. Essentially two conditions are required. The first one fixes α near $\mathcal{N}$ in a way that R behaves similarly to the Reeb vector field defined in Section 1.4 near $\mathcal{N}(K)$, where K was the binding of an open book decomposition of Y.
Briefly, this means that there exists a smaller closed solid torus $V \subset \mathcal{N}$ and a neighborhood $T^2 \times [0, 2]$ of $\partial\mathcal{N} = T^2 \times \{1\}$ in Y such that:

1. $T^2 \times [0, 1] \subset N$, $\mathcal{N} = (T^2 \times [1, 2]) \cup V$ and $\partial V = T^2 \times \{2\}$;
2. $T^2 \times \{y\}$ is foliated by Reeb trajectories for any $y \in [0, 2]$;
3. if $K = \{0\} \times S^1 \subset \mathcal{N}$, then K is a Reeb orbit and $\mathrm{int}(V) \setminus K$ is foliated by concentric tori, which in turn are linearly foliated by Reeb trajectories that intersect positively a meridian disk for K in V.
4. $T_1 := T^2 \times \{1\}$ and $T_2 := T^2 \times \{2\}$ are negative and, respectively, positive M-B tori foliated by Reeb orbits which are meridians of K.

As in Section 1.4, the families of Reeb orbits in T_1 and T_2 are perturbed into two pairs of Reeb orbits (e, h) and, respectively, (e_+, h_+): here e and e_+ are elliptic and h and h_+ are positive hyperbolic (see figure 1.1). If α satisfies the conditions above we say that α *is adapted to* K.

The second condition of compatibility is that there must exist a Seifert surface $S \subset Y$ for K such that R is positively transverse to $int(S)$. In this case we say that α is *adapted to* S.

Lemma 2.4. *(see Theorem 10.3.2 in [9]) Given a null-homologous knot K and a contact structure ξ on Y there exists a contact form α for ξ and a genus minimizing Seifert surface S for K such that:*

1. *α is adapted to K;*
2. *α is adapted to S.*

Proof. We give here only the proof of 1), referring the reader to [9] for 2).

Up to isotopy, we can assume that K is transverse to ξ and let α' be any contact form for ξ. Up to isotopy of α' we can suppose that K is a Reeb orbit.

Since the compatibility condition with K can be arranged on a neighborhood of K, by the Darboux-Weinstein neighborhood theorem (see for example [20]) there exists a contact form α which is compatible with K and contactomorphic to α'. □

Example 2.5. *If (K, S, ϕ) is an open book decomposition of Y and α is a contact form adapted to (K, S, ϕ), then it is adapted also to K and to any page of (K, S, ϕ).*

In [9] the authors prove that it is possible to define the ECH-chain groups without taking into account the orbits in $int(V)$ and in $T^2 \times (1, 2)$, so that the only interesting orbits in $\mathcal{N}(K)$ are the four orbits above (plus, obviously, the empty orbit). Moreover the only curves counted by the (restriction of the) ECH-differential ∂ have projection on Y as depicted in figure 2.1. These curves give the following relations:

$$\begin{array}{rcl} \partial(e) & = & 0 \\ \partial(h) & = & 0 \\ \partial(h_+) & = & e + \emptyset \\ \partial(e_+) & = & h. \end{array} \tag{2.6}$$

Note that the two holomorphic curves from h to e, as well as the two from e_+ to h_+, cancel one each other since we work with coefficients in $\mathbb{Z}/2$.

Observation 2.6. *The compactification of the projection of the holomorphic curve that limits to the empty orbit is topologically a disk with boundary h_+, which should be seen as a cylinder closing on some point of K. This curve contribute to the "$\emptyset$ part" of the third of the equations above, which gives $[e] = [\emptyset]$ in ECH-homology. In the rest of this manuscript the fact that this disk is the only ECH index 1 connected holomorphic curve that crosses K will be essential.*

Notation. *From now on we will use the following notation. If (Y, α) is understood, given a submanifold $X \subset Y$ and a set of Reeb orbits $\{\gamma_1, \dots, \gamma_n\} \subset \mathcal{P}(Y \setminus X)$, we will denote $ECC^{\gamma_1,\dots,\gamma_n}(X, \alpha)$ the free $\mathbb{Z}/2$-module generated by orbit sets in $\mathcal{O}(X \sqcup \{\gamma_1, \dots, \gamma_n\})$.*

Unless stated otherwise, the group $ECC^{\gamma_1,\dots,\gamma_n}(X, \alpha)$ will come with the natural restriction, still denoted ∂^{ECH}, of the ECH-differential of $ECC(Y, \alpha)$: if this restriction is still a differential the associated homology is

$$ECH^{\gamma_1,\dots,\gamma_n}(X, \alpha) := H_*(ECC^{\gamma_1,\dots,\gamma_n}(X, \alpha), \partial^{ECH}).$$

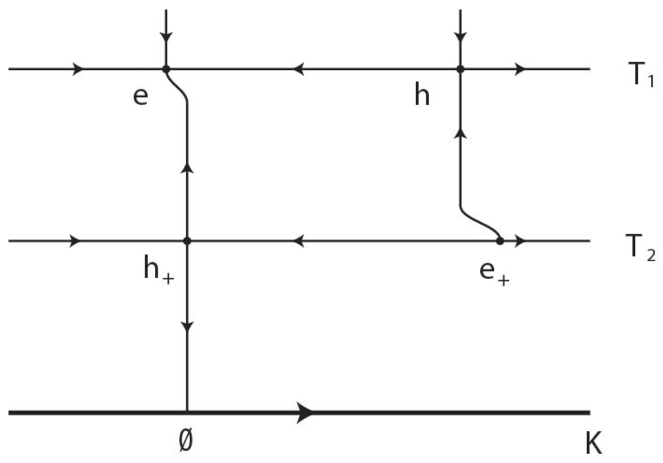

Figure 2.1: Orbits and holomorphic curves near K. Here the marked points denote the simple Reeb orbits and the flow lines represent projections of the holomorphic curves counted by ∂^{ECH}. The two flow lines arriving from the top on e and h are depicted only to remember that, by the Trapping Lemma, holomorphic curves can only arrive to T_1.

This notation is not used in [9], where the authors introduced a specific notation for each relevant ECH-group. In particular with their notation:

$$\begin{aligned} ECC^{\flat}(N,\alpha) &= ECC^{e}(int(N),\alpha); \\ ECC^{\sharp}(N,\alpha) &= ECC^{h}(int(N),\alpha); \\ ECC^{\natural}(N,\alpha) &= ECC^{h_+}(N,\alpha). \end{aligned}$$

As mentioned before, even if in $\mathcal{N}$ there are other Reeb orbits, it is possible to define chain complexes for the ECH homology of (Y,α) only taking into account the orbits $\{e,h,e_+,h_+\}$.

The Blocking and Trapping lemmas and the relations above imply that the restriction of the full ECH-differential of Y to the ECH-chain group $ECH^{e_+,h_+}(N,\alpha)$ is given by:

$$\partial(e_+^a h_+^b \gamma) = e_+^{a-1} h_+^b h\gamma + e_+^a h_+^{b-1}(1+e)\gamma + e_+^a h_+^b \partial\gamma, \tag{2.7}$$

where $\gamma \in \mathcal{O}(N)$ and a term in the sum is meant to be zero if it contains some elliptic orbit with negative total multiplicity or a hyperbolic orbit with total multiplicity not in $\{0,1\}$ (see [9, Section 9.5]). We remark that the Blocking Lemma implies also that $\partial\gamma \in \mathcal{O}(N)$.

The further restriction of the differential to $ECH^{h_+}(N,\alpha)$ is then given by

$$\partial(h_+^b \gamma) = h_+^{b-1}(1+e)\gamma + h_+^b \partial\gamma. \tag{2.8}$$

Combining the computations of sections 8 and 9 of [9] the authors get the following result.

Theorem 2.7. *Suppose that α is adapted to K and there exists a Seifert surface S for K such that α is adapted to S. Then*

$$\begin{aligned} ECH(Y,\alpha) &\cong ECH^{e_+,h_+}(N,\alpha); && (2.9)\\ \widehat{ECH}(Y,\alpha) &\cong ECH^{h_+}(N,\alpha). && (2.10)\end{aligned}$$

Observation 2.8. *It is important to remark that the empty orbit is always taken into account as a generator of the groups above. This implies that if orbit sets with h_+ are considered, ∂^{ECH} counts also the holomorphic "plane" that contributes to the third of relations 2.6. Later we will give the definition of another differential, that we will call ∂^{ECK}, which is obtained from ∂^{ECH} by simply deleting that plane.*

Define now the *relative embedded contact homology groups of* $(N, \partial N)$ by

$$\begin{aligned} ECH(N,\partial N,\alpha) &= \frac{ECH^{e}(int(N),\alpha)}{[e\gamma]\sim[\gamma]}\\ \widehat{ECH}(N,\partial N,\alpha) &= \frac{ECH(N,\alpha)}{[e\gamma]\sim[\gamma]}.\end{aligned}$$

Since here h_+ is not counted in the definition of the chain complexes, the ECH-differentials count only holomorphic curves in N. This "lack" is balanced by the quotient by the equivalence relation

$$[e\gamma]\sim[\gamma]. \tag{2.11}$$

The reason behind this claim lie in the third of the relations 2.6. Indeed we can prove the following:

Lemma 2.9.

$$ECH^{e_+,h_+}(N,\alpha)\cong\frac{ECH^{e_+}(N,\alpha)}{[e\gamma]\sim[\gamma]}.$$

Proof. Using the fact that h_+ can have multiplicity at most 1, it is not difficult to see that the long exact homology sequence associated to the pair

$$\big(ECC^{e_+}(N,\alpha), ECC^{e_+,h_+}(N,\alpha)\big)$$

is

$$\begin{aligned} \dots \longrightarrow ECH^{e_+}(N,\alpha) \xrightarrow{i_*} ECH^{e_+,h_+}(N,\alpha) \xrightarrow{\pi_*}\\ \xrightarrow{\pi_*} H(h_+ECC^{e_+}(N,\alpha),\partial) \xrightarrow{d} ECH^{e_+}(N,\alpha) \xrightarrow{i_*} \dots\end{aligned}$$

where:

- $i : ECC^{e_+}(N,\alpha) \hookrightarrow ECC^{e_+,h_+}(N,\alpha)$ is the inclusion map;
- $h_+ECC^{e_+}(N)$ is the module generated by orbit sets of the form $h_+\gamma$ with $\gamma \in \mathcal{O}(N \sqcup e_+)$;

– $\pi : ECC^{e_+,h_+}(N,\alpha) \twoheadrightarrow h_+ECC^{e_+}(N)$ is the quotient map sending to 0 all generators having no contributions of h_+;
– d is the standard connecting morphism, that in this case is defined by

$$d([h_+\gamma]) = [\gamma + e\gamma].$$

We can then extract the short exact sequence

$$0 \longrightarrow \mathrm{coker}(d) \xrightarrow{i_*} ECH^{e_+,h_+}(N,\alpha) \xrightarrow{\pi_*} \ker(d) \longrightarrow 0$$

where

$$\mathrm{coker}(d) = \frac{ECH^{e_+}(N,\alpha)}{[e\gamma] \sim [\gamma]}.$$

Since $\ker(d) = \{0\}$, the map i_* is an isomorphism.

□

Similarly, the fourth line of Equation 2.6 "explains" why we can avoid considering h in the full $ECH(Y,\alpha)$. In fact with similar arguments of the proof of last lemma, we can prove:

Lemma 2.10 ([9], Section 9)**.** $ECH^{e_+}(N,\alpha) \cong ECH^{e}(int(N),\alpha)$.

Observe that since $\partial(e\gamma) = e\partial(\gamma)$, the differential is compatible with the equivalence relation. So, instead of take the quotient by $[e\gamma] \sim [\gamma]$ of the homology, we could take the homology of the quotient of the chain groups under the relation $e\gamma \sim \gamma$, and we would obtain the same homology groups. We will use this fact later. Note moreover that for every k, $[e^k] = [\emptyset]$.

Equations 2.4 and 2.5 follow then from last two lemmas and Theorem 2.7.

2.2.1 ECH and $\widehat{\text{ECH}}$ from open books

An important example of the situation depicted above is when K is the binding of an open book decomposition (K,S,ϕ) of a closed three manifold Y, and N is the associated mapping torus considered in section 1.4. Using the same notations, define the *extended pages* of (S,ϕ) to be the surfaces

$$S' \times \{t\} := (S \times \{t\}) \sqcup_{\partial S \times \{t\}} (S^1 \times \{t\} \times [1,3)),\ t \in \frac{[0,2]}{0 \sim 2}.$$

Let α be a contact form on Y compatible with (K,S,ϕ). In particular α is adapted to both K and any page of (K,S,ϕ).

Definition 2.11. *If γ is a Reeb orbit in $Y \setminus K$, define the* degree *of γ by*

$$\deg(\gamma) = \langle \gamma, S' \times \{0\} \rangle$$

If $\gamma = \prod_i \gamma_i^{k_i}$ is some orbit set, we define $\deg(\gamma) = \sum_i k_i \deg(\gamma_i)$. If $X \subset (Y \setminus K)$, we indicate by $\mathcal{O}_i(X)$ (resp. $\mathcal{O}_{\leq i}(X)$) the set of multiorbits in X with degree equal (resp. less or equal) to i.

Note that $\deg(\gamma)$ depends only on the homology class of γ in $Y \setminus K$. In this context the relative embedded contact homology groups can also be defined in terms of limits as follows.

Define $ECC^e_j(int(N), \alpha)$ to be the free $\mathbb{Z}_2$-module generated by orbit sets in $\mathcal{O}_j(int(N) \cup \{e\})$. Similarly let $ECC_j(N, \alpha)$ be generated by orbit sets in $\mathcal{O}_j(N)$. Define the inclusions

$$\begin{aligned} \mathcal{I}^e_j : &\quad ECC^e_j(int(N), \alpha) \to ECC^e_{j+1}(int(N), \alpha) \\ \mathcal{I}_j : &\quad ECC_j(N, \alpha) \to ECC_{j+1}(N, \alpha) \end{aligned}$$

given by the map $\gamma \mapsto e\gamma$. Each of these chain groups can be endowed with (the restriction of) the ECH-differential, which counts M-B buildings in N. Let $ECH^e_j(N, \alpha)$ and $ECH_j(N, \alpha)$ be the associated homology groups. Then the relative embedded contact homology groups above can be defined also by

$$\begin{aligned} ECH(N, \partial N, \alpha) &= \lim_{j \to \infty} ECH^e_j(int(N), \alpha); \\ \widehat{ECH}(N, \partial N, \alpha) &= \lim_{j \to \infty} ECH_j(N, \alpha). \end{aligned}$$

Observation 2.12. *If $ECC_{\leq k}(N, \alpha) := \bigoplus_{j=0}^{k} ECC_j(N, \alpha)$, let $ECH_{\leq k}(N, \alpha)$ be the homology of $ECC_{\leq k}(N, \alpha)$ with the ECH-boundary map. The "stabilization" Theorem 1.0.2 of [11] implies that for the definition of $\widehat{ECH}(N, \partial N, \alpha)$ it is sufficient to take into account just orbit sets in $\mathcal{O}_{\leq 2g}(N)$. Then:*

$$\widehat{ECH}(N, \partial N, \alpha) \cong \frac{ECH_{\leq 2g}(N, \alpha)}{[e\gamma] \sim [\gamma]}. \tag{2.12}$$

2.3 Periodic Floer homology for open books

Another Floer homology theory closely related to ECH is the *periodic Floer homology*, denoted by PFH, and defined by Hutchings (see [26]). Given a symplectic surface (S, ω) (here with possibly empty boundary) and a symplectomorphism $\phi : S \to S$, consider the mapping torus

$$N(S, \phi) = \frac{S \times [0, 2]}{(x, 2) \sim (\phi(x), 0)}.$$

Then $PFH(N(S, \phi))$ is defined in an analogous way than ECH for an open book but replacing the Reeb vector field with a stable Hamiltonian vector field R parallel to ∂_t, where t is the coordinate of $[0, 2]$: we refer the reader to [26] or [31] for the details.

The chain group $PFC(N(S, \phi))$ is the free $\mathbb{Z}_2$ module generated by orbit sets of R and the boundary map counts index 1 holomorphic curves in the symplectization; then, under some condition on ϕ, the associated homology $PFH(N(S,\phi))$ is well defined. Homology groups $PFH_i(Y(S,\phi))$ associated to the chain groups $PFC_i(N(S,\phi))$ generated by degree-i multiorbits are also well defined.

If (S,ϕ) is an open book as in the sections above, ∂S is connected and N is the associated mapping torus, in [10] the following is proved:

Theorem 2.13. *If α is a contact form adapted to (S,ϕ) then there exists a stable Hamiltonian structure such that for any $i \geq 0$,*

$$PFH_i(N) \quad \cong \quad ECH_i(N,\alpha) \tag{2.13}$$

(here we are using a simplified notation which is different from that used in [10]).

Defining

$$\widehat{PFH}(N,\partial N) = \frac{PFH_{\leq 2g}(N)}{[e\gamma] \sim [\gamma]},$$

then

$$\widehat{PFH}(N,\partial N) \cong \widehat{ECH}(N,\partial N,\alpha),$$

where contact form and stable Hamiltonian structure are as in the last theorem.

It is interesting to remark that $PFC_1(N(S,\phi))$ is generated by orbits of period 1, which are in bijective correspondence with the set $\mathrm{Fix}(\phi)$ of the fixed points of ϕ via the map

$$\begin{array}{ccc} \mathcal{O}_1(int(N)) & \longrightarrow & \mathrm{Fix}(\phi) \\ \gamma & \longmapsto & \gamma \cap S, \end{array} \tag{2.14}$$

which moreover evidently respects the Lefschetz signs. Then this correspondence induces an isomorphism between $PFC_1(N(S,\phi))$ and the chain complex $SC(S,\phi)$ of the standard *symplectic Floer homology* $SH(S,\phi)$ of (S,ϕ) (see for example [7] and [17]). Indeed the following holds (see for example [31]):

Proposition 2.14. *The correspondence above induces an isomorphism*

$$PFH_1(N(S,\phi)) \cong SH(S,\phi).$$

2.4 $\widehat{\text{ECH}}$ for knots

Let K be a homologically trivial knot in a contact three-manifold (Y,α). In this section we recall the definition of a hat version of contact homology for the triple (K,Y,α). This was first defined in [13, Section 7] as a particular case of *sutured* contact homology. On the other hand, following [9, Section 10], it is possible to proceed without dealing directly with sutures: we follow here this approach.

Let S be a Seifert surface for K. By standard arguments in homology, it is easy to compute that

$$\begin{array}{ccc} H_1(Y \setminus K) & \longrightarrow & H_1(Y) \times \mathbb{Z} \\ [a] & \longmapsto & (i_*[a]\,, \langle a, S\rangle) \end{array} \tag{2.15}$$

is an isomorphism. Here $i : Y \setminus K \to Y$ is the inclusion and $\langle a, S\rangle$ denotes the intersection number between a and S: this is a homological invariant of the pair (a, S) and is well defined up to a slight perturbation of S (to make it transverse to a). Note that a preferred generator of $\mathbb{Z}$ is given by the homology class of a meridian for K, positively oriented with respect to the orientations of S and Y.

Example 2.15. *If Y is a homology three-sphere, the number $\langle a, S\rangle$ depends only on a and K. This is the* linking number between a and K *and it is usually denoted by $lk(a, K)$.*

If $\gamma = \prod_i \gamma_i^{k_i}$ is a finite formal product of closed curves in $Y \setminus K$, then $\langle \gamma, S\rangle = \sum_i k_i \langle \gamma_i, S\rangle$.

Example 2.16. *If (K, S, ϕ) is an open book decomposition of Y and α is an adapted contact form, then $\langle \gamma, S\rangle = \deg(\gamma)$ for any orbit set $\gamma \in \mathcal{O}(Y \setminus K)$, where $\deg$ is given in Definition 2.11.*

Proposition 2.17 (See Proposition 7.1 in [13])**.** *Suppose that K is an orbit of R_α and let S be any Seifert surface for K. If γ and δ are two orbit sets in $Y \setminus K$ and $u : (F, j) \to (\mathbb{R} \times Y, J)$ is a holomorphic curve from γ to δ, then*

$$\langle \gamma, S\rangle \geq \langle \delta, S\rangle.$$

Proof. Let $\widehat{u}$ be the compactification of u in $[-1, 1] \times Y$. Since u has no limits in K, then

$$\langle \widehat{u}, [-1, 1] \times K\rangle = \langle u, \mathbb{R} \times K\rangle \geq 0, \tag{2.16}$$

where the inequality follows by the positivity of intersection in dimension 4 (since K is a Reeb orbit, $\mathbb{R} \times K$ is holomorphic). Consider the two surfaces

$$L_{-1} = \{-1\} \times S \;\text{ and }\; L_1 = \{1\} \times -S$$

and define the closed surface

$$L \;=\; L_{-1} \cup ([-1, 1] \times K) \cup L_1$$

where the first gluing is made along $\{-1\} \times K$ and the second along $\{1\} \times -K$. Since $0 = [L] \in H_2([-1, 1] \times Y)$, then:

$$\begin{array}{rcll} 0 & = & \langle \widehat{u}, L\rangle & = \\ & = & \langle \widehat{u}, L_{-1}\rangle + \langle \widehat{u}, [-1, 1] \times K\rangle + \langle \widehat{u}, L_1\rangle & = \\ & = & \langle \delta, S\rangle + \langle \widehat{u}, [-1, 1] \times K\rangle - \langle \gamma, S\rangle. & \end{array}$$

The result then follows by observing that the last equation implies that

$$\langle \gamma, S\rangle - \langle \delta, S\rangle = \langle \widehat{u}, [-1,1]\times K\rangle \geq 0. \tag{2.17}$$

$\square$

Suppose that α is adapted to K in the sense of Section 2.2; a choice of (a homology class for) the Seifert surface S for the orbit K defines a *knot filtration* on the chain complex $(ECC^{h_+}(N,\alpha), \partial^{ECH})$ for $\widehat{ECH}(Y,\alpha)$, where, recall, N is the complement of a neighborhood $\mathcal{N}(K)$ of K in which the only "interesting" orbits and holomorphic curves are the ones represented in Figure 2.1.

Let $ECC_d^{h_+}(N,\alpha)$ be the free sub-module of $ECC^{h_+}(N,\alpha)$ generated by orbit sets γ in $\mathcal{O}(N \sqcup \{h_+\})$ such that $\langle \gamma, S\rangle = d$. Define moreover

$$ECC_{\leq d}^{h_+}(N,\alpha) := \bigoplus_{j\leq d} ECC_j^{h_+}(N,\alpha).$$

Observation 2.18. *The direct sum above is not in general finite. On the other hand if α is adapted to S then $\langle \gamma, S\rangle \geq 0$ for any γ and the sum is finite for any d.*

Even if α is not adapted to S, the intersection number induces an exhaustive filtration

$$\ldots \subseteq ECC_{\leq d-1}^{h_+}(N,\alpha) \subseteq ECC_{\leq d}^{h_+}(N,\alpha) \subseteq ECC_{\leq d+1}^{h_+}(N,\alpha) \subseteq \ldots$$

on $ECC^{h_+}(N,\alpha)$.

Definition 2.19. *The filtration above is the* knot filtration *induced by K. If γ is a generator of $ECC_d^{h_+}(N,\alpha)$, the integer d is the* filtration degree *of γ.*

Corollary 2.20. *The differential ∂^{ECH} of $ECC^{h_+}(N,\alpha)$ respects the knot filtration.*

Proof. Proposition 2.17 applied to the M-B buildings counted by ∂^{ECH} implies immediately that

$$\partial^{ECH}\left(ECC_{\leq d}^{h_+}(N,\alpha)\right) \subseteq ECC_{\leq d}^{h_+}(N,\alpha)$$

for any d and the result follows. $\square$

Suppose now that α is adapted to S. By standard arguments in algebra, the filtration above induces a spectral sequence whose page ∞ is isomorphic to $ECH^{h_+}(N,\alpha) \cong \widehat{ECH}(Y,\alpha)$ and whose page 0 is the chain complex

$$\bigoplus_d \left(ECC_d^{h_+}(N,\alpha), \partial_d^{ECK}\right) \tag{2.18}$$

where $ECC_d^{h+}(N,\alpha)$ should be seen as $\frac{ECC_{\leq d}^{h+}(N,\alpha)}{ECC_{\leq d-1}^{h+}(N,\alpha)}$ and

$$\partial_d^{ECK} : ECC_d^{h+}(N,\alpha) \to ECC_d^{h+}(N,\alpha)$$

is the map induced by ∂^{ECH} on the quotient, i.e, it is the part of $\partial^{ECH}|_{ECC_d^{h+}(N,\alpha)}$ that strictly preserves the filtration degree.

Observation 2.21. *The proof of Proposition 2.17 implies that the holomorphic curves counted by ∂^{ECH} that strictly decrease the degree are exactly the curves that intersect K. So we can interpret ∂^{ECK} as the restriction of ∂^{ECH} (given by Equation 2.7) to the count of curves that do not cross a thin neighborhood of K. This is indeed the proper ECH-differential of the manifold $Y \setminus int(V(K))$ (and not the restriction of the ECH-differential of Y to the orbit sets in $Y \setminus int(V(K))$).*

Note that, by definition of $ECC^{h+}(N,\alpha)$, all the holomorphic curves contained in $\mathbb{R} \times N$ strictly preserve the filtration degree. In fact the only holomorphic curve that contributes to $\partial^{ECH}|_{ECC^{h+}(N,\alpha)}$ and decreases the degree (by 1) is the disk from h_+ to $\emptyset$. Equation 2.8 gives then

$$\partial(h_+^d \gamma) = h_+^{d-1} e\gamma + h_+^d \partial\gamma. \tag{2.19}$$

where $\gamma \in \mathcal{O}(N)$ and any term is meant to be zero if it contains some orbit with total multiplicity that is negative or not in $\{0,1\}$ if the orbit is hyperbolic

Definition 2.22. *The* hat version of embedded contact (knot) homology *of the triple (K,Y,α) is*

$$\widehat{ECK}_*(K,Y,\alpha) := H_*\left(ECC^{h+}(N,\alpha), \partial^{ECK}\right).$$

Observation 2.23. *In [9] $\widehat{ECK}(K,Y,\alpha)$ is called $ECH(M(K),\alpha)$ and in Theorem 10.3.2 it is proved that*

$$\widehat{ECK}(K,Y,\alpha) = ECH^\sharp(N,\alpha)$$

where, recall, with our notation $ECH^\sharp(N,\alpha) = ECH^h(int(N),\alpha)$. On the other hand, by using exactly the same arguments of Lemma 2.9, it is easy to see that

$$ECH^h(int(N),\alpha) \cong H_*\left(ECC^{h+}(N,\alpha), \partial^{ECK}\right).$$

Observation 2.24. *Note that in order to define $\widehat{ECK}(K,Y,\alpha)$, we supposed that α is compatible with S. This hypothesis is not present in the original definition (via sutures) in [13]. Indeed, without this condition we can still apply all the arguments above and define the knot filtration on $ECC^{h+}(N,\alpha)$ exactly* *in the same way. The page* 1 *of the spectral sequence is again the well defined homology in the definition above, and the page* ∞ *is still isomorphic to* $ECH^{h+}(N,\alpha)$*.*

The only difference is that now we do not know that $ECH^{h+}(N,\alpha) \cong \widehat{ECH}(Y,\alpha)$*, since in Theorem 2.7 the hypothesis that* α *is adapted to* S *is assumed.*

This homology comes naturally with a further relative degree, inherited by the filtered degree: if $\widehat{ECK}_{*,d}(K,Y,\alpha) := H_*\left(ECC_d^{h+}(N,\alpha), \partial_d^{ECK}\right)$ then

$$\widehat{ECK}_*(K,Y,\alpha) = \bigoplus_d \widehat{ECK}_{*,d}(K,Y,\alpha).$$

Sometimes, in analogy with Heegaard Floer, we will call this degree the *Alexander degree*.

Example 2.25. *Suppose that* (K,S,ϕ) *is an open book decomposition of* Y *and that* α *is an adapted contact form. Since any non-empty Reeb orbit in* $Y \setminus K$ *has strictly positive intersection number with* S*,*

$$\widehat{ECK}_{*,0}(K,Y,\alpha) \cong \langle[\emptyset]\rangle_{\mathbb{Z}/2}.$$

This is the ECH*-analogue of the fact that if* K *is fibered, then*

$$\widehat{HFK}_{*,-g}(K,Y) \cong \langle[c]\rangle_{\mathbb{Z}/2},$$

where g *is the genus of* K *and* c *is the associated contact element (see [45]).*

Observation 2.26. *We remark that the Alexander degree can be considered as an absolute degree only once a relative homology class in* $H_2(Y,K)$ *for* S *has been fixed, since the function* $\langle\cdot,S\rangle$ *defined on* $H_1(Y\setminus K)$ *changes if* $[S]$ *varies.*

On the other hand, suppose that $[\gamma]=[\delta]\in H_1(Y\setminus K)$ *and let* $F\subset Y$ *be a surface such that* $\partial F=\gamma-\delta$*. Computations analogue to that in the proof of Proposition 2.17 imply that*

$$\langle\gamma,S\rangle - \langle\delta,S\rangle = \langle F,K\rangle, \tag{2.20}$$

and the Alexander degree, considered as a relative degree, does not depend on the choice of a homology class for S*.*

Obviously if $H_2(Y)=0$*, the Alexander degree can be lifted to an absolute degree.*

In [13] the authors conjectured that their sutured embedded contact homology is isomorphic to sutured Heegaard-Floer homology. Both the hat version of embedded contact knot homology and of Heegaard Floer knot homology can be defined in terms of sutures. In this case their conjecture becomes

Conjecture 2.27. *For any knot K in Y:*

$$\widehat{ECK}(K, Y, \alpha) \cong \widehat{HFK}(-K, -Y),$$

where α is a contact form on Y adapted to K.

3

Heegaard Floer homology and Alexander polynomial

Heegaard-Floer homology was developed by Ozsváth and Szabó in an attempt to provide a more combinatorial version of Kronheimer and Mrowka's Seiberg-Witten-Floer homology ([33]). Heegaard-Floer theory has been able to yield powerful invariants for closed three and four manifolds, as well as for knots and links in three-manifolds.

In section 3.1 we introduce Heegaard-Floer theory for three-manifolds. Because of the abundance of literature about the argument we will show only some of the aspects of the construction. We refer the reader to the original papers by Ozsváth and Szabó ([42], [43]) for all the details. Other presentations of the subject can be found in [48] and [53].

In addition to the original definition , there exists also another possible definition of Heegaard Floer homology. This is the "cylindrical formulation" of HF, which is due to Lipshitz ([36]). This has been used by Colin, Ghiggini and Honda to prove the equivalence between ECH and HF. Only in Chapter 4 we will shortly recall this alternative construction in the special case of Heegaard diagrams arising from open books, as presented in [10].

In Section 3.2 we briefly recall definition and basic properties of Heegaard Floer homology for knots and links in three-manifolds. Some details will be provided about the relations with the Alexander polynomial.

Finally, in Section 3.3 we recall the interpretation of the multivariable Alexander polynomial Δ_L of a link $L \subset S^3$ in terms of the dynamics of suitable vector fields in $S^3 \setminus L$. This characterisation of Δ_L originates from the work of Franks (see e.g. [15]), later generalised by Fried ([16]). This will be a key ingredient for the results in Chapter 6.

Even if Heegaard Floer homologies can be defined using integer coefficients, in this chapter, as well as in the rest of the paper, we will always use coefficients in $\mathbb{Z}/2$.

3.1 HF for three manifolds

Let Y be a closed, compact and oriented three-manifold. Heegaard-Floer theory assigns to Y four homology groups

$$HF^{\infty}(Y), \quad HF^{+}(Y), \quad HF^{-}(Y), \quad \widehat{HF}(Y).$$

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ be a *pointed Heegaard diagram* for Y. This means that:

- Σ is an oriented compact and closed surface of genus g, called "Heegaard surface for Y";
- there exists a *Heegaard decomposition* $Y = Y_1 \cup Y_2$ of Y, where Y_1 and Y_2 are handlebodies such that $\partial Y_1 = \partial Y_2 = \Sigma$;
- $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_g\}$ and $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_g\}$ are sets of *attaching circles* for Y_1 and, respectively, Y_2;
- w is a point in $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$.

Let $\mathrm{Sym}^g(\Sigma)$ be the g-fold symmetric product of Σ. If Σ is endowed with a fixed almost complex structure, $\mathrm{Sym}^g(\Sigma)$ inherits a product complex structure J. The two g-dimensional submanifolds

$$\mathbb{T}_\alpha := \alpha_1 \times \ldots \times \alpha_g \qquad \mathbb{T}_\beta := \beta_1 \times \ldots \times \beta_g.$$

of $\mathrm{Sym}^g(\Sigma)$ are Lagrangian (see [49]). Any point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ can be seen as a g-tuple of points $\{x_1, \ldots, x_g\}$ in Σ for which there exists a permutation $\sigma \in \mathcal{S}_g$ such that $x_i \in \alpha_i \cap \beta_{\sigma(i)}$.

Define the associated *(infinite version of) Heegaard Floer chain group* by

$$CF^{\infty}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w) := \langle \{[\mathbf{x}, i] \mid \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta,\ i \in \mathbb{Z}\} \rangle_{\mathbb{Z}/2}.$$

This chain group can be endowed with a differential ∂^{HF} defined by

$$\partial^{HF}([\mathbf{x}, i]) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \ \sum_{[u] \in \mathcal{M}_1([\mathbf{x}, \mathbf{y}])/\mathbb{R}} ([\mathbf{y}, i - n_w(u)]),$$

where $n_w(u)$ is the intersection number $\langle u, \{w\} \times \mathrm{Sym}^{g-1}(\Sigma) \rangle$ and $\mathcal{M}_1(\mathbf{x}, \mathbf{y})/\mathbb{R}$ is the set of equivalence classes (modulo $\mathbb{R}$-translations) of holomorphic strips $u : [0,1] \times \mathbb{R} \to \mathrm{Sym}^g(\Sigma)$ from x to y such that:

- $u(\{1\} \times \mathbb{R}) \subset \mathbb{T}_\alpha$ and $u(\{0\} \times \mathbb{R}) \subset \mathbb{T}_\beta$;
- $\lim_{s \to -\infty} u(t, s) = \mathbf{x}$ and $\lim_{s \to \infty} u(t, s) = \mathbf{y}$;
- u has Maslov index 1 (see [42] for the details).

Since we are working with $\mathbb{Z}/2$ coefficients, the sum above is understood to be taken modulo 2.

The associated homology group is the *(infinite version of) Heegaard Floer homology* $HF^\infty(Y)$ of Y.

Since u and $\{w\} \times \mathrm{Sym}^{g-1}$ are both holomorphic, the positivity of intersection implies $n_w(u) \geq 0$ and

$$CF^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w) := \langle \{[\mathbf{x}, i] \mid \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta,\ i \in \mathbb{Z}_{<0}\} \rangle_{\mathbb{Z}/2}$$

endowed with the (restriction of) ∂^{HF} is a subcomplex of the infinite version. The associated homology $HF^-(Y)$ is the *minus version of Heegaard Floer homology of* Y.

The quotient

$$CF^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w) := \frac{CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)}{CF^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)}$$

gives rise to the *plus version of the Heegaard Floer homology of* Y, denoted by $HF^+(Y)$.

Finally, define

$$\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w) := \langle \{\mathbf{x} \mid \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta\} \rangle_{\mathbb{Z}/2}$$

and endow it with the restriction of the differential ∂^{HF} that counts only holomorphic curves u with $n_w(u) = 0$. The associated homology group $\widehat{HF}(Y)$ is the *hat version of Heegaard Floer homology of* Y.

A priori all homologies defined depend on the choice of the pointed Heegaard diagram and on the almost complex structure J on $\mathrm{Sym}^g(\Sigma)$, but Ozsváth and Szabó proved in fact the following:

Theorem 3.1 ([42])**.** *All the Heegaard Floer homology groups of* Y *do not depend on any of the choices made and are topological invariants of* Y.

Observation 3.2. *It is important to say that to any* $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ *it is possible to associate a well defined* $\mathfrak{s}_\mathbf{x} \in \mathrm{Spin}^c(Y)$. *It is possible to prove that if there exists a surface like the ones counted by* ∂^{HF} *and whose limits are* $\mathbf{x}$ *and* $\mathbf{y}$, *then* $\mathfrak{s}_\mathbf{x} = \mathfrak{s}_\mathbf{y}$. *This is a kind of Poincaré-dual version of the fact that* ∂^{ECH} *respects the first homology of the generators of the* ECH *chain groups.*

If $CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, \mathfrak{s})$ *is the submodule of* $CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ *generated by the elements* $\mathbf{x}$ *such that* $\mathfrak{s}_\mathbf{x} = \mathfrak{s}$, *then its homology* $HF^\infty(Y, \mathfrak{s})$ *is well defined. Moreover there is a natural splitting*

$$HF^\infty(Y) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} HF^\infty(Y, \mathfrak{s}). \tag{3.1}$$

Analogous splittings exist also in the other versions of $HF(Y)$.

3.2 Heegaard Floer homology for knots and links

In this section we briefly recall the definition of Heegaard-Floer homology for knots and links in a three manifolds Y. These can be seen as the first page of a spectral sequence arising from a filtration defined by the knot (or link) on a suitable complex for $HF(Y)$.

Heegaard-Floer homology for knots has been defined independently in [44] by Ozsváth and Szabó and by Rasmussen in his Ph.D. thesis ([50]). The link version has been defined later in [46].

3.2.1 The knot filtration

Let us begin with the case of knots. Let K be a homologically trivial knot in a three-manifold Y. We say that a doubly pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ is *compatible with* K if:

– $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ is a pointed Heegaard diagram for Y;

– $z \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ is a second marked point, different from w and such that, if a_1 (resp. a_2) is an oriented arc in $\Sigma \setminus \boldsymbol{\alpha}$ (resp. $\Sigma \setminus \boldsymbol{\beta}$) from z to w (resp. from w to z), then the oriented loop $a_1 \cup a_2$ is isotopic in Y to K.

As recalled in last section, a Heegaard-Floer chain complex for Y splits into a direct sum over the set $\mathrm{Spin}^c(Y)$ of the Spin^c-structures of Y, and the differential ∂^{HF} respects this splitting.

Now the corresponding Heegaard Floer chain groups also split into direct sums over $\mathrm{Spin}^c(Y_0(K))$, where $Y_0(K)$ is the 3-manifold obtained by 0-surgery of Y along K. In fact, given a doubly pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ compatible with K, to any generator $\mathbf{x} = (x_1, \ldots, x_g)$ of $CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ it is possible to associate (almost canonically) a generator $\mathbf{x}'$ of a Heegaard-Floer chain group for $Y_0(K)$ and a well defined Spin^c-structure $\underline{\mathfrak{s}}_w(\mathbf{x}) := \mathfrak{s}_w(\mathbf{x}') \in \mathrm{Spin}^c(Y_0(K))$ depending only on $\mathbf{x}$. One can check that

$$\mathrm{Spin}^c(Y_0(K)) \cong \mathrm{Spin}^c(Y) \times \mathbb{Z}. \tag{3.2}$$

and the projection on the second factor is the integer $\frac{1}{2}\langle c_1(\underline{\mathfrak{s}}_w(\mathbf{x})), [\widehat{F}]\rangle$ where c_1 denotes the first Chern class, F is a Sifert surface for K and $\widehat{F}$ is the surface obtained by capping off F along K in $Y_0(K)$.

This integer can be computed as follows. Let $\{R_i\}_i$ be the set of the connected components (called *regions*) of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$. If a projection of F to Σ is the *domain* $\mathcal{P} = \sum_i m_i R_i$, then (see [43, Proposition 7.5]):

$$\langle c_1(\underline{\mathfrak{s}}_w(\mathbf{x})), [\widehat{F}]\rangle = \chi(\mathcal{P}) + 2\sum_{i=1}^{g} n_{x_i}(\mathcal{P}) \tag{3.3}$$

where:

– $\chi(\mathcal{P})$ is the *Euler measure* of $\mathcal{P}$ (see [43, Section 7.1] or [36, Section 4.1]);
– $n_y(\mathcal{P}) = \sum_i m_i n_y(R_i)$ with $n_y(R_i) = \frac{n}{4}$, where n is the number of vertices of R_i which are identified with y.

Observation 3.3. *Equation 3.2 can be viewed just as a Poincaré-dual of the version in $Y_0(K)$ of Equation 2.15.*

Lemma 3.4 ([44], Lemma 2.5)**.** *Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ be a doubly pointed Heegaard diagram for (Y, K). Given $\mathbf{x}$, $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, for any holomorphic strip u from x to y counted by ∂^{HF}:*

$$\mathfrak{s}_w(\mathbf{x}) - \mathfrak{s}_w(\mathbf{y}) = (n_z(u) - n_w(u))PD([\mu]) \tag{3.4}$$

where $[\mu]$ is the homology class in $Y_0(K)$ of the (positively oriented) meridian μ and PD denotes the Poincaré dual.

Note that $n_z(u), n_w(u) \geq 0$ by the positivity of intersection.
Let $CFK^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ be the free $\mathbb{Z}/2$-module generated by triples $[\mathbf{x}, i, j]$ with $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $i, j \in \mathbb{Z}$. The HF-differential is then given by

$$\partial^{HF}([\mathbf{x}, i, j]) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{[u] \in \mathcal{M}_1(\mathbf{x},\mathbf{y})/\mathbb{R}} ([\mathbf{y}, i - n_w(u), j - n_z(u)]),$$

where $\mathcal{M}_1(\mathbf{x}, \mathbf{y})/\mathbb{R}$ is the set of equivalence classes (modulo $\mathbb{R}$-translations) of holomorphic disks with Maslov-index 1 from $\mathbf{x}$ to $\mathbf{y}$.

Let ∂^{HFK} be the part of ∂^{HF} that preserves the $PD([\mu])$ component in Equation 3.4, that is, the map that restricts the sum in ∂^{HF} to the holomorphic curves u such that $n_z(u) - n_w(u) = 0$. The *(full) Heegaard Floer knot homology of K* is then

$$HFK^\infty(K, Y) = H\left(CFK^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z), \partial^{HFK}\right).$$

This homology naturally inherits a relative degree induced by the difference between the $PD([\mu])$-components of the Spin^c-structures in $Y_0(K)$ associated to the generators. Then $HFK^\infty_{*,*}(K, Y)$ is a bigraded homology: the first degree is the usual homological degree, while we will call *Alexander degree* the further $\mathbb{Z}$-degree given by the filtration.

Suppose now that u is a holomorphic curve counted by ∂^{HF} in the hat version. Then $n_w(u) = 0$ and Equation 3.4 becomes

$$\mathfrak{s}_w(\mathbf{x}) - \mathfrak{s}_w(\mathbf{y}) = n_z(u)PD[\mu], \tag{3.5}$$

where $n_z(u) \geq 0$ by positivity of intersection. Define the *hat version of Heegaard Floer knot homology of K* as follows. Let $\widehat{CFK}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ be the free $\mathbb{Z}/2$-module generated by g-tuples $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ where, as usual, g is the genus of Σ. Let now ∂^{HFK} be the part of ∂^{HF} that preserves the filtration given by Equation 3.5, which is in this case the map that restricts the count of ∂^{HF} to the holomorphic curves u with $n_z(u) = n_w(u) = 0$. We call the associated homology $\widehat{HFK}(K, Y)$.

A further version of HFK is obtained by taking the homology of the subcomplex $CFK^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ of $CFK^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ freely generated by the triples $[\mathbf{x}, i, j]$ with $i < 0$. Endowed with the restriction of the differential, this gives the *minus version of Heegaard Floer knot homology of K* $HFK^-(K, Y)$.

Finally, like in the case of closed three-manifolds, the *plus version of Heegaard Floer knot homology* $HFK^+(K, Y)$ of K is defined to be the homology of the quotient $CFK^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ of $CFK^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ by the submodule $CFK^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$.

Obviously also these homologies inherit the additional Alexander degree. This degree induces a splitting of $\widehat{HFK}(K, Y)$:

$$\widehat{HFK}_*(K, Y) = \bigoplus_d \widehat{HFK}_{*,d}(K, Y) \tag{3.6}$$

where $\widehat{HFK}_{*,d}(K, Y)$ is the homology of the subcomplex $\widehat{CFK}_d(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ of $\widehat{CFK}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ generated by the $\mathbf{x}$ such that $\underline{\mathfrak{s}}_w(x)$ is sent to $(\mathfrak{s}_w(x), d)$ by the isomorphism 3.2, i.e., the $\mathbf{x}$ with

$$\frac{1}{2}\langle c_1(\underline{\mathfrak{s}}_w(\mathbf{x})), [\widehat{F}]\rangle = d.$$

Similar splits hold also for the other versions of HFK.

Observation 3.5. *Like in ECK, also here the Alexander degree is in general only a relative degree and is defined only up an overall shift.*

Theorem 3.6 ([44],[50]). *$HFK^\infty(K, Y)$, $HFK^-(K, Y)$, $HFK^+(K, Y)$ and $\widehat{HFK}(K, Y)$ are topological invariants of the pair (K, Y).*

In [46], using the Lipshitz's cylindrical reformulation of HF, Ozsváth and Szabó generalize the filtration above to the case of links in S^3 with a generic number of components. We give here just the idea of the construction. Given an n-component oriented link $L = K_1 \sqcup \ldots \sqcup K_n \subset Y$, let μ_i be a positively oriented meridian of K_i. Then there is an isomorphism

$$\begin{array}{ccc} H_1(S^3 \setminus L) & \longrightarrow & \mathbb{Z}_{[\mu_1]} \oplus \ldots \oplus \mathbb{Z}_{[\mu_n]} \\ [a] & \longmapsto & (lk(a, K_1), \ldots, lk(a, K_n)) \, , \end{array} \tag{3.7}$$

where $\mathbb{Z}_{[\mu_i]}$ is the direct summand generated by $[\mu_i] \in H_1(Y \setminus L)$. This is just the generalization, for $Y = S^3$, of Equation 2.15 and, by taking the Poincaré dual, of Equation 3.2.

By arguments similar to the case of knots, Equation 3.7 induces a $\mathbb{Z}^n$-filtration on suitable Heegaard-Floer complexes $CF^-(S^3)$ and $\widehat{CF}(S^3)$ defined using special Heegaard diagrams for S^3 *compatible with* L. The first pages of the spectral sequences in the two versions are the *Heegaard-Floer link homologies* $HFL^-(L, S^3)$ and $\widehat{HFL}(L, S^3)$.

Now these homology groups inherit (from Equation 3.7) a $\mathbb{Z}^n$ grading, or, analogously, n $\mathbb{Z}$-gradings, one for each component of L. We will keep to call this $\mathbb{Z}^n$-degree the *Alexander degree*.

Theorem 3.7 ([46]). *$HFL^-(L, S^3)$ and $\widehat{HFL}(L, S^3)$ are topological invariants of the couple (L, S^3). Moreover if $n = 1$ and we write $L = K_1 = K$, then:*

$$HFL^-(K, S^3) \cong HFK^-(K, S^3) \text{ and } \widehat{HFL}(L, S^3) \cong \widehat{HFK}(K, S^3)$$

as bigraded modules.

Knot and link Floer homologies enjoy many interesting properties. For example, under a suitable choice of a lifting of the relative degree:

- if g is the genus of K, $\widehat{HFK}_{*,d}(K, S^3) \neq \{0\}$ only if $d \in \{-g, \dots, g\}$ and in particular $\widehat{HFK}(K, S^3) \cong \mathbb{Z}$ if and only if K is the unknot ([44]);
- $\widehat{HFK}_{*,g}(K, S^3) \cong \mathbb{Z}$ if and only if K is fibered; the generator is the homology class of the *contact element* on HF (see [45] for the "if" part, [18] for the "only if" in the case $g = 1$ and [41] in general);
- $\widehat{HFK}(K, S^3)$ gives a bound for the slice genus of knots in S^3 ([47]).

3.2.2 HFL and Alexander polynomial

Another beautiful property of Heegaard Floer knot homology is that it *categorifies the Alexander polynomial* of knots and links in S^3.

Given a collection of chain complexes

$$(C, \partial) = \{(C_{*,(i_1,\dots,i_n)}, \partial_{(i_1,\dots,i_n)})\}_{(i_1,\dots,i_n)\in\mathbb{Z}^n},$$

where $*$ denotes a relative homological degree, its *graded Euler characteristic* is

$$\chi(C) = \sum_{i_1,\dots,i_n} \chi\left(C_{*,(i_1,\dots,i_n)}\right) t_1^{i_1} \cdots t_n^{i_n} \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

where $\chi\left(C_{*,(i_1,\dots,i_n)}\right)$ is the standard Euler characteristic of $C_{*,(i_1,\dots,i_n)}$ and the t_j's are formal variables. By definition, $\chi(C)$ is a Laurent polynomial and the properties of the standard Euler characteristic imply

$$\chi(C) = \chi\left(H(C, \partial)\right).$$

In this case the homology $H(C, \partial)$ is a *categorification* of the polynomial $\chi(C)$.

Given any link $L = K_1 \sqcup \ldots \sqcup K_n$ in S^3 we can associate to it its *multivariable Alexander polynomial*

$$\Delta_L(t_1, \ldots, t_n) \in \frac{\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]}{\pm t_1^{a_1} \cdots t_n^{a_n}}.$$

with $a_i \in \mathbb{Z}$. The quotient means that the Alexander polynomial is well defined only up to multiplication by monomials of the form $\pm t_1^{a_1} \cdots t_n^{a_n}$.

A slightly simplified version is the *(classical) Alexander polynomial* $\Delta_L(t)$, defined by setting $t_1 = \ldots = t_n = t$, i.e.:

$$\Delta_L(t) := \Delta_L(t, \ldots, t).$$

If L is a knot the two notions obviously coincide.

Observation 3.8. *The fact that the Alexander polynomial is defined up to multiplication of terms of the form $\pm t_1^{a_1} \cdots t_n^{a_n}$ depends on the choice of a lifting of a basis of $H_1(S^3 \setminus L)$ to a basis for the homology of the universal abelian cover of $S^3 \setminus L$. An equivalent ambiguity appears also in HFL and HFK when a lifting of the relative degrees to absolute degrees must be chosen.*

From now on we will use the equivalence symbol "$\doteq$" to indicate that two polynomials coincide up to a factor of the form $\pm t_1^{a_1} \cdots t_n^{a_n}$, $a_i \in \mathbb{Z}$.

Alexander polynomial is a "classical" invariant, and was first introduced by Alexander in 1928 ([2]). It enjoys a quantity of beautiful properties and admits many possible definitions. In the next section we will remind a definition of Δ_L in terms of the dynamic of certain vector fields defined in the link complement. We also refer the reader to [51] for a beautiful treatment of Δ_L.

Theorem 3.9 (Ozsváth, Szabó). *For any link $L = K_1 \sqcup \ldots \sqcup K_n$ in S^3:*

$$\chi\left(HFL^-(L, S^3)\right) \doteq \begin{cases} \Delta_L(t_1, \ldots, t_n) & \text{if } n > 1 \\ \Delta_L(t)/(1-t) & \text{if } n = 1 \end{cases} \tag{3.8}$$

and

$$\chi\left(\widehat{HFL}(L, S^3)\right) \doteq \begin{cases} \Delta_L \cdot \prod_{i=1}^{n} (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}}) & \text{if } n > 1 \\ \Delta_L(t) & \text{if } n = 1. \end{cases} \tag{3.9}$$

This theorem has been proved in [44] and [50] in the case $n = 1$: this came from a direct application of a *skein exact sequence* in HFK, analogous to the skein relation for Δ_K. The result has been then generalized in [46] to any link: in this case the proof utilizes the *Reidemeister-Franz torsion* $\tau(L)$ of the universal abelian cover of the link complement (see for example [56]). Indeed, for links in S^3

$$\tau(L) \doteq \begin{cases} \Delta_L(t_1, \dots, t_n) & \text{if } n > 1 \\ \\ \Delta_L(t)/(1-t) & \text{if } n = 1 \end{cases} \tag{3.10}$$

and Equation 3.8 can be restated as

$$\chi\left(HFL^-(L, S^3)\right) \doteq \tau(L). \tag{3.11}$$

3.3 A dynamical formulation of $\Delta(t)$

As remarked before, given a link $L \in S^3$, there are many possible definitions of Δ_L. In this section we give a formulation of Δ_L in terms of the dynamics of suitable vector fields in $S^3 \setminus L$. The details about the proof of the statements can be found in the references.

The fact that the Alexander polynomial is related to dynamical properties of its complement in S^3 origins with the study of fibrations of S^3. For example in [1] A'Campo studied the *twisted Lefschetz zeta function* of the monodromy of an open book decomposition (S, ϕ) of S^3 associated to a Milnor fibration of a complex algebraic singularity. More in general, if (K, S, ϕ) is any open book decomposition of S^3, one can easily prove (see for example [51]) that

$$\Delta_K(t) \doteq \det(\mathbb{1} - t\phi^1_*),$$

where $\mathbb{1}$ and ϕ^1_* are the identity map and, respectively, the application induced by ϕ, on $H_1(S, \mathbb{Z})$. The basic idea in this context is to express the right-hand side of equation above in terms of traces of iterations of ϕ^1_*; then to apply the Lefschetz fixed point theorem to get expressions in terms of periodic points, (i.e. periodic orbits) for the flow of some vector field in $S^3 \setminus K$ whose first return on a page is ϕ.

Suppose now that L is not a fibered link, so that its complement is not globally fibered over S^1 and let R be a vector field in $S^3 \setminus L$. If one wants to apply arguments like above, it is necessary to decompose $S^3 \setminus L$ in "fibered-like" pieces with respect to R, in which it is possible to define at least a local first return map of the flow ϕ_R of R. Obviously some condition on R is required. For example, in his beautiful paper [15], Franks consider *Smale vector fields*, that is, vector fields whose chain recurrent set is one-dimensional and hyperbolic (cf. [52]).

Here we are more interested in the approach used by Fried in [16]. Consider a three-dimensional manifold X. Any *abelian cover* $\widetilde{X} \xrightarrow{\pi} X$ with deck transformations group isomorphic to a fixed abelian group G is uniquely determined by the choice of a class $\rho = \rho(\pi) \in H^1(X, G) \cong \mathrm{Hom}\,(H_1(X, \mathbb{Z}), G)$. Here ρ is determined by the following property: for any $[\gamma] \in H_1(X)$, if

$\widetilde{\gamma} : [0,1] \to \widetilde{X}$ is any lifting of the loop $\gamma : [0,1] \to X$, then $\rho([\gamma])$ is determined by $\rho([\gamma])(\widetilde{\gamma}(0)) = \widetilde{\gamma}(1)$.

Since the correspondence between Abelian covers and cohomology classes is bijective, with abuse of notation sometimes we will refer to an abelian cover directly by identifying it with the corresponding ρ.

Example 3.10. *The* universal abelian cover *of X is the abelian cover with deck transformation group $G = H_1(X,\mathbb{Z})$ and corresponding to $\rho = id$.*

Example 3.11. *Let $L = K_1 \sqcup \ldots \sqcup K_n$ be an n-components link in a three manifold Y such that K_i is homologically trivial for any i and fix a Seifert surface S_i for K_i. Let moreover μ_i be a positive meridian for K_i. If $i : Y \setminus L \hookrightarrow Y$ is the inclusion, the isomorphism*

$$\begin{array}{ccc} H_1(Y \setminus L) & \longrightarrow & H_1(Y) \oplus \mathbb{Z}_{[\mu_1]} \oplus \ldots \oplus \mathbb{Z}_{[\mu_n]} \\ [\gamma] & \longmapsto & (i_*([\gamma]), \langle\gamma, S_1\rangle, \ldots, \langle\gamma, S_n\rangle) \end{array} \tag{3.12}$$

gives rise naturally to the abelian cover

$$\rho_L \in \mathrm{Hom}\,(H_1(Y \setminus L, \mathbb{Z}), \mathbb{Z}^n)$$

of $Y \setminus L$ defined by

$$\rho_L([\gamma]) = (\langle\gamma, S_1\rangle, \ldots, \langle\gamma, S_n\rangle)\,.$$

Setting $t_i = [\mu_i] \in H_1(Y \setminus L, \mathbb{Z})$, we can regard $\rho_L([\gamma])$ as a monomial in the variables t_i:

$$\rho_L([\gamma]) = t_1^{\langle\gamma,S_1\rangle} \cdots t_n^{\langle\gamma,S_n\rangle}.$$

In the rest of the paper we will often use this notation.

Note finally that if Y is a homology three-sphere, ρ_L coincides with the universal abelian cover of $Y \setminus L$.

If R is a vector field on X satisfying some compatibility condition with ρ (and with ∂X if this is non-empty), the author relates the Reidemeister-Franz torsion of $(X, \partial X)$ with the *twisted Lefschetz zeta function* of the flow ϕ_R.

3.3.1 Twisted Lefschetz zeta function of flows

Let R be a vector field on X and γ a closed isolated orbit of ϕ_R. Pick any point $x \in \gamma$ and let D be a small disk transverse to γ such that $D \cap \gamma = \{x\}$. With this data it is possible to define the Lefschetz sign of γ exactly like we did in Section 1.1 for orbits of Reeb vector fields associated to a contact structure ξ, but using now T_xD instead of ξ_x. Indeed it is possible to prove that the Lefschetz sign of γ does not depend on the choice of x and D and it is an invariant $\epsilon(\gamma) \in \{-1, 1\}$ of ϕ_R near γ.

Definition 3.12. *The* local Lefschetz zeta function *of ϕ_R near γ is the formal power series $\zeta_\gamma(t) \in \mathbb{Z}[[t]]$ defined by*

$$\zeta_\gamma(t) := \exp\left(\sum_{i\geq 1} \epsilon(\gamma^i)\frac{t^i}{i}\right).$$

Let now $\widetilde{X} \xrightarrow{\pi} X$ be an abelian cover with deck transformation group G and let $\rho = \rho(\pi) \in H^1(X, G)$. Suppose that all the periodic orbits of ϕ_R are isolated.

Definition 3.13. *We define the ρ-*twisted Lefschetz zeta function of ϕ_R *by*

$$\zeta_\rho(\phi_R) := \prod_\gamma \zeta_\gamma\left(\rho([\gamma])\right),$$

where the product is taken over the set of simple periodic orbits of ϕ_R.

When ρ is understood we will write directly $\zeta(\phi_R)$ and we will call it twisted Lefschetz zeta function of ϕ_R.

We remark that in [16] the author defines $\zeta_\rho(\phi_R)$ in a slightly different way and then he prove (Theorem 2) that, under some assumptions that we will state in next subsection, the two definitions coincide.

Notation. *Suppose that $\rho \in H^1(X, \mathbb{Z}^n)$ is an abelian cover of X and chose a generator $(t_1, \ldots, t_n)$ of $\mathbb{Z}^n$. Then, with a similar notation to that of Example 3.11, we can see $\zeta_\rho(\phi_R)$ as an element of $\mathbb{Z}[[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]]$.*

3.3.2 Torsion and flows

In [16] Fried relates the Reidemeister torsion of an abelian cover ρ of a (non necessarily closed) three-manifold X with the twisted Lefschetz zeta function of certain flows. In particular in Section 5 he considers a kind of torsion that he calls *Alexander quotient* and denotes by $\mathrm{ALEX}_\rho(X)$: the reason for the "quotient" comes from the fact that Fried uses a definition of the Reidemeister torsion only up to the choice of a sign (this is the "refined Reidemeister torsion" of [56]), while $\mathrm{ALEX}_\rho(X)$ is defined up to an element in the Abelian group of deck transformations of ρ (see also [5]).

In fact one can check that $\mathrm{ALEX}_\rho(X)$ is exactly the Reidemister-Franz torsion τ considered in [46]. In particular, when X is the complement of an n-component link L in S^3 and ρ is the universal abelian cover of X, then

$$\mathrm{ALEX}(S^3 \setminus L) \doteq \begin{cases} \Delta_L(t_1, \ldots, t_n) & \text{if } n > 1 \\ \Delta_L(t)/(1-t) & \text{if } n = 1 \end{cases}. \tag{3.13}$$

where we removed $\rho = id_{H_1(S^3\setminus L,\mathbb{Z})}$ from the notation (see [16, Section 8] and [56]).

Since the notation "τ" is ambiguous, we follow [16] and we refer to the Reidemeister-Franz as the Alexander quotient, that will be indicated $\mathrm{ALEX}_\rho(X)$.

In order to relate $\mathrm{ALEX}_\rho(X)$ to the twisted Lefschetz zeta function of the flow ϕ_R of a vector field R, Fried assumes some hypothesis on R.

The first condition that R must satisfy is the *circularity*.

Definition 3.14. *A vector field R on X is* circular *if there exists a C^1 map $\theta : X \to S^1$ such that $d\theta(R) > 0$.*

If $\partial X = \emptyset$ this is equivalent to say that R admits a global cross section. Intuitively, the circularity condition on R allows to define a kind of first return map of ϕ_R.

Suppose R circular and consider $S^1 \cong \frac{\mathbb{R}}{\mathbb{Z}}$ with $\mathbb{R}$-coordinate t. The cohomology class

$$u_\theta := \theta^*([dt]) \in H^1(X, \mathbb{Z})$$

is then well defined.

Definition 3.15. *Given an abelian cover $\widetilde{X} \xrightarrow{\pi} X$ with deck transformations group G, let $\rho = \rho(\pi) \in H^1(X, G)$ be the corresponding cohomology class. A circular vector field R on X is* compatible *with ρ if there exists a homomorphism $v : G \to \mathbb{R}$ such that $v \circ \rho = u_\theta$, where θ and u_θ are as above.*

Example 3.16. *The universal abelian cover corresponds to $\rho = id : H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z})$, so it is automatically compatible with any circular vector field on X.*

The following theorem is not the most general result in [16] but it will be enough for our purposes:

Theorem 3.17 (Theorem 7, [16])**.** *Let X be a three manifold and $\rho \in H^1(X, G)$ an abelian cover. Let R be a non-singular, circular and non degenerate vector field on X compatible with ρ. Suppose moreover that, if $\partial X \neq \emptyset$, then R is transverse to ∂X and pointing out of X. Then*

$$\mathrm{ALEX}_\rho(X) \doteq \zeta_\rho(\phi_R),$$

where the symbol $\doteq$ denotes the equivalence up to multiplication for an element $\pm g$, $g \in G$.

An immediate consequence is the following

Corollary 3.18. *If L is any n-component link in S^3, let $\mathcal{N}(L)$ be a tubular neighborhood of L and pose $N = S^3 \setminus \mathcal{N}(L)$. Let R be a non-singular circular* *vector field on* N*, transverse to* ∂N *and pointing out of* N*. Then*

$$\zeta(\phi_R) \doteq \begin{cases} \Delta_L(t_1, \ldots, t_n) & \text{if } n > 1 \\ \\ \Delta_L(t)/(1-t) & \text{if } n = 1 \end{cases} . \tag{3.14}$$

4

The equivalence between $\widehat{\text{ECH}}$ and $\widehat{\text{HF}}$

In their series of papers [8]-[12], Colin, Ghiggini and Honda proved an equivalence between Heegaard Floer homology and embedded contact homology for three manifolds.

Theorem 4.1 (Colin, Ghiggini, Honda)**.** *Given a closed, oriented, three dimensional contact manifold* (Y, ξ)*,*

$$HF^+(-Y) \cong ECH(Y, \alpha)$$
$$\widehat{HF}(-Y) \cong \widehat{ECH}(Y, \alpha),$$

where $-Y$ *is the manifold* Y *with the inverted orientation and* α *is a suitable contact form for* ξ*.*

In this thesis we are mostly interested in the second line. A key ingredient in the proof is the Giroux equivalence between contact structures and open book decompositions. In Section 4.1 we present how to define the $\widehat{HF}(Y)$ using an open book decomposition of Y. In Section 4.2 we remind the definition of some symplectic cobordisms defined in [10]. Finally in Section 4.3 we recall the definition of the chain map Φ that induces an isomorphism from $\widehat{HF}(-Y)$ to $\widehat{ECH}(Y, \alpha)$: this chain map is defined by a certain count of holomorphic curves in one of the symplectic cobordisms defined in the preceding section.

4.1 $\widehat{\mathrm{HF}}$ for open books

As shown in [25], to an open book decomposition (S, ϕ) of Y it is possible associate a particular Heegaard diagram. Let us recall the slightly different construction given in [10]. Another construction can be found in [45].

Let us denote by S_t the page $S \times \{t\}$ of the open book. Take a basis of arcs $\mathbf{a} = \{a_1, \ldots, a_{2g}\}$ in the page $S_{\frac{1}{2}}$, that is, a set of properly embedded arcs in $S_{\frac{1}{2}}$ such that $S_{\frac{1}{2}} \setminus \{a_1, \ldots, a_{2g}\}$ is a topological disk (see figure 4.1 for a $g = 1$ example).

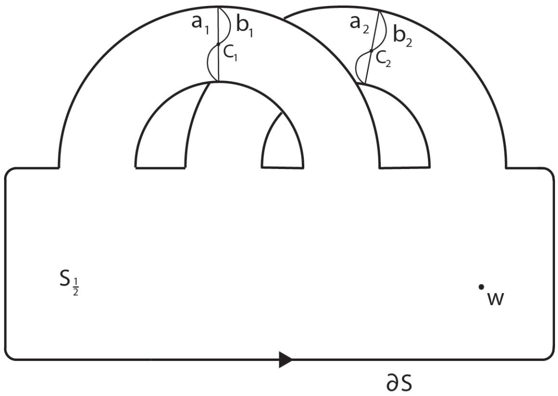

Figure 4.1: a_i's and b_i's in $S_{\frac{1}{2}}$.

Let $\iota : S_{\frac{1}{2}} \to S_0$ be the map $(x, \frac{1}{2}) \mapsto (x, 0)$ where $(x, t) \in S \times S^1$ and define the surface

$$\Sigma = S_{\frac{1}{2}} \sqcup_{\partial S_0} S_0$$

and the set $\boldsymbol{\alpha}$ of $2g$ closed curves in Σ by

$$\alpha_i = a_i \amalg_{\partial a_i} \iota(a_i), \ i = 1, \ldots, 2g.$$

Consider now the set of arcs $\phi(\iota(a_i)) \subset S_0$, for $i = 1, \ldots, 2g$ and define new arcs b_i in $S_{\frac{1}{2}}$ as obtained by modifying a_i by a little isotopy relative to the boundary, such that

- $a_i \pitchfork b_i = \{C_i\}$ in the interior of $S_{\frac{1}{2}}$;
- $a_i \cap b_j = \emptyset$ for $i \neq j$;
- if we orient a_i and b_i has the orientation induced from the one of a_i then $\{C_i\}$ has negative sign;
- in a neighborhood of ∂S_0 in Σ, b_i is a smooth extension of $\phi(\iota(a_i))$ to $S_{\frac{1}{2}}$.

Note that since ϕ is the identity map on ∂S, for every i, $\alpha_i \cap \beta_i \cap \partial S$ consists of a pair of points, that we will call x_i and x_i'.

Define the set of $2g$ curves $\boldsymbol{\beta}$ by $\beta_i = b_i \amalg_{\partial b_i} \phi(\iota(a_i))$ and choose a basepoint $w \in S_{\frac{1}{2}}$ outside of the little strips given by the isotopies from a_i's to b_i's. $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ is the required Heegaard diagram for Y.

For us it will be convenient to work on $-Y$ and so we will use the diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, w)$.

It is easy to see that this diagram is weakly admissible, that is, every periodic domain has both positive and negative components of $\Sigma \setminus \{\boldsymbol{\alpha} \cup \boldsymbol{\beta}\}$ (this condition is required in the definition of $\widehat{HF}$, see [42, Section 4]). Indeed if a periodic domain involves α_i, then the sign in the thin strip between α_i and β_i given by isotopy must change when the domain crosses C_i.

When it is clear from the context that we are working only on S_0 we will omit the use of ι and we will refer to a_i and $\phi(a_i)$ as arcs in S_0.

The Heegaard Floer chain complex $\left(\widehat{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, w), \partial^{HF}\right)$ in the hat version is then defined as in section 3.1.

From now on we will often switch to the Lipshitz's four-dimensional definition of HF (see [36] for a complete dissertation or directly [10, Section 4] for the case we are treating here). About this reformulation we recall only a few things in the setting we have in hand.

If g is the genus of S, then in Lipshitz's formulation of our situation the auxiliary manifold $\mathrm{Sym}^{2g}(\Sigma)$ is replaced by the the four-dimensional manifold $\mathbb{R} \times [0,1] \times \Sigma$. First of all recall that a point $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ can be seen as a $2g$-tuple of points in $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$, where now g is the genus of S (so that $2g$ is the genus of Σ). In the new formulation $\mathbf{y}$ is identified with the set of $2g$ *chords* $[0,1] \times \mathbf{y} \subset [0,1] \times \Sigma$: these are the new generators (over $\mathbb{Z}/2$) of the complex $\widehat{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, w)$.

Endow $\mathbb{R} \times [0,1] \times \Sigma$ with an *admissible* almost complex structure J (see [10, Definition 4.2.1]) and the symplectic form

$$ds \wedge dt + \omega$$

where s and t are the coordinates of $\mathbb{R}$ and, respectively, $[0,1]$ and ω is a symplectic form on Σ. From now on we will assume that $\mathbb{R} \times [0,1] \times \Sigma$ comes with these data.

For every $i \in \{1, \dots, 2g\}$, call L_{α_i} and L_{β_i} the Lagrangian submanifolds $\mathbb{R} \times \{1\} \times \alpha_i$ and, respectively, $\mathbb{R} \times \{0\} \times \beta_i$ of $\mathbb{R} \times [0,1] \times \Sigma$. Define moreover $L_{\boldsymbol{\alpha}} = \bigsqcup_{i=1}^{2g} L_{\alpha_i}$ and $L_{\boldsymbol{\beta}} = \bigsqcup_{i=1}^{2g} L_{\beta_i}$.

Let (F, j) be a compact (possibly disconnected) Riemann surface with two sets of punctures $\mathbf{p}^+ = \{p_1^+, \dots, p_k^+\}$ and $\mathbf{p}^- = \{p_1^-, \dots, p_k^-\}$ on ∂F such that (i) every component of F has nonempty boundary, (ii) every component C of ∂F contains at least one element of $\mathbf{p}^+$ and one of $\mathbf{p}^-$ in a way that these alternate along C. Let $\dot{F}$ denote F with the sets of punctures removed.

Definition 4.2. *Let $\mathbf{y} = \{y_1, \dots, y_k\}$ and $\mathbf{y}' = \{y_1', \dots, y_k'\}$ be two k-tuple ($k \leq 2g$) of points in Σ with $y_i \in \alpha_i \cap \beta_{\sigma(i)}$ and $y_i' \in \alpha_i \cap \beta_{\sigma'(i)}$ for some permutations $\sigma, \sigma' \in \mathcal{S}_k$.*

A degree-k multisection *of* $\mathbb{R}\times[0,1]\times\Sigma$ *from* y *to* y' *is a holomorphic map*

$$u:(\dot F,j)\longrightarrow(\mathbb{R}\times[0,1]\times\Sigma,J)$$

satisfying the following conditions:

1. $(\dot F,j)$ *is a punctured Riemann surface as above;*
2. u *is a multisection of degree* k *of the fibration* $\pi:\mathbb{R}\times[0,1]\times\Sigma\to\mathbb{R}\times[0,1]$*;*
3. $u(\partial\dot F)\subset L_\alpha\cup L_\beta$ *and maps each connected component of* $\partial\dot F$ *to a different* L_{α_i} *or* L_{β_i}*;*
4. $\lim_{w\to p_i^+}u_{\mathbb{R}}(w)=+\infty$ *and* $\lim_{w\to p_i^-}u_{\mathbb{R}}(w)=-\infty$*;*
5. *near* p_i^+ *(resp.* p_i^-*),* u *converges to the strip over* $[0,1]\times\{y_i\}$ *(resp.* $[0,1]\times\{y_i'\}$*);*
6. *the energy of* u *given by Equation 4.1 below is finite.*

Definition 4.3. *Let* $(\dot F,j)$ *be as above. Define the energy of the holomorphic multisection* $u:(\dot F,j)\to(\mathbb{R}\times[0,1]\times\Sigma,J)$ *by*

$$E(u)=\int_{\dot F}u^*\omega+\sup_{\zeta\in\mathcal{C}}\int_{\dot F}u^*d(\zeta(s)dt),\tag{4.1}$$

where $\mathcal{C}$ *is the set of non-decreasing smooth functions* $\zeta:\mathbb{R}\to[0,1]$.

If J is generic, the HF-differential (in the hat version) is then defined by

$$\partial^{HF}(\mathbf{y})=\sum_{\mathbf{y}'}\sum_{[u]\in\widehat{\mathcal{M}}_1([\mathbf{y},\mathbf{y}'])/\mathbb{R}}\mathbf{y}'\mod(2),$$

where the first sum is taken over the set of generators of $\widehat{CF}(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha},w)$ and $\widehat{\mathcal{M}}_1(\mathbf{y},\mathbf{y}')/\mathbb{R}$ is the set of equivalence classes (modulo $\mathbb{R}$-translations) of holomorphic multisections $u:(\dot F,j)\longrightarrow(\mathbb{R}\times[0,1]\times\Sigma,J)$ of degree $2g$ from y to y' such that:

1. $n_w(u):=\langle u(\dot F),\mathbb{R}\times[0,1]\times\{w\}\rangle=0$;
2. u has ECH-index 1.

See [10] for details.

Observation 4.4. *The positivity of intersection in dimension* 4 *and the condition 1 above imply that if* u *is a holomorphic curve counted by* ∂^{HF} *with a chord at the positive end associated to* x_i *or* x_i'*, then a connected component of* u *is a trivial strip over that chord. See Section 4.9 of [10].*

If $N = N(S, \phi)$, in order to define a map from a chain complex for $\widehat{HF}(Y)$ to a chain complex for $\widehat{PFH}(N, \partial N)$, in [10] the authors redefine $\widehat{HF}(Y)$ using only the 0-page S_0 of (S, ϕ) (roughly speaking this is the half of Σ containing the information about ϕ).

Let $CF'(S, \mathbf{a}, \phi(\mathbf{a}))$ be the submodule of $\widehat{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, w)$ generated by the $2g$-tuples of intersection points contained in S_0 and endow it with the restriction of ∂^{HF}. By Observation 4.4 and the property 1 that $\widehat{\mathcal{M}}_1(\mathbf{y}, \mathbf{y}')/\mathbb{R}$ must satisfy, this is a subcomplex of $\left(\widehat{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}), \partial^{HF}\right)$. In particular, if u is a holomorphic curve counted by this restriction of ∂^{HF} to $CF'(S, \mathbf{a}, \phi(\mathbf{a}))$, the projection of $\mathrm{Im}(u)$ on Σ gives a domain completely contained in S_0.

Let $\sim$ be the equivalence relation on $CF'(S, \mathbf{a}, \phi(\mathbf{a}))$ induced on the generators by the relation

$$\mathbf{y} \sim \mathbf{y}' \text{ if } \exists i \mid \mathbf{y} = \{y_1, \ldots, x_i, \ldots, y_{2g}\} \text{ and } \mathbf{y}' = \{y_1, \ldots, x_i', \ldots, y_{2g}\}, \quad (4.2)$$

where, recall, x_i and x_i' are the intersection points in $\alpha_i \cap \beta_i \cap \partial S_0$. Define

$$\widehat{CF}(S, \mathbf{a}, \phi(\mathbf{a})) = \frac{CF'(S, \mathbf{a}, \phi(\mathbf{a}))}{\sim}.$$

In [10, Section 4.9] the authors prove that $\widehat{CF}(S, \mathbf{a}, \phi(\mathbf{a}))$ is a chain complex if endowed with the differential induced by the one of $CF'(S, \mathbf{a}, \phi(\mathbf{a}))$. Call $\widehat{HF}(S, \mathbf{a}, \phi(\mathbf{a}))$ its homology.

Theorem 4.5. *(see [10, Theorem 4.9.4])*

$$\widehat{HF}(S, \mathbf{a}, \phi(\mathbf{a})) \cong \widehat{HF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, w).$$

We end this section by recalling the following theorem, which is an adaptation to this context of the results in [45] and [25].

Theorem 4.6. *Let* $\mathbf{x}$ *be (the equivalence class under* $\sim$ *of) the* $2g$*-tuple* $\{x_1, \ldots, x_{2g}\}$. *Then* $[\mathbf{x}] \in \widehat{HF}(S, \mathbf{a}, \phi(\mathbf{a}))$ *depends only on the contact structure* ξ *compatible with* (S, ϕ).

The generator $\mathbf{x}$ is in fact the HF*-contact element* mentioned in the end of Subsection 3.2.1 and $[\mathbf{x}]$ is the HF*-contact invariant* of ξ.

4.2 Symplectic cobordisms

In this section we recall the definitions of some symplectic cobordisms useful to define the chain map Φ giving the isomorphism from $\widehat{HF}(S, \mathbf{a}, \phi(\mathbf{a}))$ to $\widehat{PFH}(N, \partial N)$, where N is the mapping torus

$$N = \frac{S \times [0, 2]}{(x, 2) \sim (\phi, 0)}.$$

Let ω be a symplectic form on S as the one defined in [10, section 3] and consider the stable Hamiltonian structure (dt, ω) on N; let us indicate R the associated Hamiltonian vector field.

Consider now the following trivial cobordisms:

$$W = \mathbb{R} \times [0,1] \times S \qquad \text{and} \qquad W' = \mathbb{R} \times N$$

and define

$$B = \mathbb{R} \times [0,1] \qquad \text{and} \qquad B' = \mathbb{R} \times S^1.$$

The first cobordism can be viewed as a (trivial) fibration $\pi_B : B \times S \to B$, while the second as a fibration $\pi_{B'} : \mathbb{R} \times N \to B'$, naturally extending to the $\mathbb{R}$-component the fibration $N \to S^1$ defined by $(x, t) \mapsto t$.

Note that ∂^{HF} is defined by counting ECH-index 1 holomorphic multisections of the fibration π_B, while ∂^{ECH} is defined by counting ECH-index 1 holomorphic curves in W'.

We will indicate by $\pi_{\mathbb{R}}$ the projection on the first component of the cobordisms above.

Consider now the subset $B_+^c := [2, \infty) \times (1, 2)$ of $B' \cong \mathbb{R} \times \frac{[0,2]}{0 \sim 2}$ with all the corners smoothed and define $B_+ = (\mathbb{R} \times S^1) \setminus B_+^c$ (see figure 4.2).

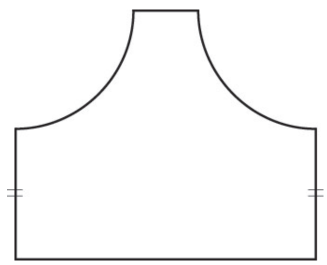

Figure 4.2: B_+.

The surface B_+ can be seen as the union of its *cylinder-part* $\{s < 2\}$ and its *strip-part* $\{s > 2\}$.

We can then define the cobordism:

$$W_+ = \pi_{B'}^{-1}(B_+)$$

Like before, W_+ can be viewed as a fibration with base B_+ and fiber S. Obviously W_+ is a submanifold with boundary of W'. The *cylinder-part* and the *strip-part* of W_+ are the counter-images under $\pi_{B'}$ of the cylinder-part and, respectively, the strip-part of B_+. We will continue to indicate by $\pi_{\mathbb{R}}$ the restriction to W_+ of the projection to the $\mathbb{R}$-component of W'.

If $\mathbb{R}$ is parametrized by s, the 2-form $\Omega = ds \wedge dt + \omega$ on W' is symplectic. Moreover this induces, by restriction, a symplectic form Ω_+ on W_+.

Endowed with this symplectic structures, the cobordisms defined can be seen as symplectic fibrations over their bases. Consider now in particular the symplectic fibration

$$\pi_{B_+} : (W_+, \Omega_+) \longrightarrow (B_+, ds \wedge dt).$$

This defines the symplectic connection given by the Ω_+-orthogonal of the tangent space of the fibers: on $W_+ \cap \{s > 2\}$ this is then spanned by ∂_s and ∂_t.

Take a copy of $\mathbf{a}$ in $\pi_{B_+}^{-1}(3,1)$ and call $L_{\mathbf{a}}^+$ the trace of the parallel transport of $\mathbf{a}$ along ∂B_+ using the symplectic connection; $L_{\mathbf{a}}^+$ is Lagrangian and

$$\begin{array}{rcl} L_{\mathbf{a}}^+ \cap \{s \geq 3,\ t = 0\} & = & \{s \geq 3\} \times \{t = 0\} \times \phi(\mathbf{a}); \\ L_{\mathbf{a}}^+ \cap \{s \geq 3,\ t = 1\} & = & \{s \geq 3\} \times \{t = 1\} \times \mathbf{a}. \end{array} \tag{4.3}$$

Note that $L_{\mathbf{a}}^+$ has $2g$ connected components $L_{a_i}^+$, one for each component a_i of $\mathbf{a}$.

The following definitions will be useful later.

Definition 4.7. *Given a point P in S, $\chi_P := [0,1] \times \{P\}$ will denote the Reeb chord in $[0,1] \times \overline{S}$ passing through P. Moreover $\sigma_P := [0,1] \times \mathbb{R} \times \{P\}$ will denote the trivial section of $W \to \mathbb{R} \times [0,1]$ on the chord χ_P.*

Definition 4.8. *Given a simple orbit δ in N, $\sigma_\delta = \mathbb{R} \times \delta$ will denote the trivial section of W' over δ and σ_δ^+ its restriction to W_+.*

4.3 The chain map Φ

Because of the huge amount of notations and results necessary, we give here only a rough explanation and refer again the reader to sections 5 and 6 of [10] for any detail.

Let (K, S, ϕ) be an open book decomposition of a three-manifold Y and consider the mapping torus $N = N(S, \phi)$. In [10], the authors define two chain maps

$$\begin{array}{rcccl} \Phi & : & \widehat{CF}(S, \mathbf{a}, \phi(\mathbf{a})) & \longrightarrow & PFC_{2g}(N) \\ \Psi & : & PFC_{2g}(N) & \longrightarrow & \widehat{CF}(S, \mathbf{a}, \phi(\mathbf{a})) \end{array} \tag{4.4}$$

and in [11] they prove that they induce isomorphisms in homology, one the inverse of the other. In this thesis we will be mostly interested in Φ.

The chain map Φ is defined by counting *multisections of the symplectic fibration $\pi_{B_+} : (W_+, \Omega_+) \longrightarrow (B_+, ds \wedge dt)$.*

Let J_+ be a suitable almost complex structure on W_+ (see [10, section 5]). Let (F, j) be a compact (possibly disconnected) Riemann surface with two sets of punctures $\mathbf{p} = \{p_1, \ldots, p_l\}$ in the interior and $\mathbf{q} = \{q_1, \ldots, q_k\}$ in the boundary of F such that (i) every connected component of F contains at least an element of $\mathbf{p}$ and a connected component of ∂F and (ii) every connected component of ∂F contains at least an element of $\mathbf{q}$. We will set $\dot{F} = F \setminus \{\mathbf{p} \cup \mathbf{q}\}$.

Definition 4.9. *Let* $\mathbf{y} = \{y_1, \ldots, y_k\}$ *be a* k*-tuple* ($k \leq 2g$) *of* $\mathbf{a} \cap \phi(\mathbf{a})$ *and* $\gamma = \prod_j \gamma^{m_j} \in \mathcal{O}_k(N)$*. A degree* k *multisection of* (W_+, J_+) *is one between:*

(i) a holomorphic multisection

$$u : (\dot{F}, j) \longrightarrow (W_+, J_+)$$

of degree k *of the fibration* $\pi_{B_+} : W_+ \to B_+$*, where* (F, j) *is a Riemann surface endowed with two sets of punctures* $\mathbf{p}$ *and* $\mathbf{q}$ *as above, and* u *is such that*

1. $u(\partial \dot{F}) \subset L^+_{\mathbf{a}}$ *and maps each connected component of* $\partial \dot{F}$ *to a different* $L^+_{a_i}$*;*
2. $\lim_{w \to q_i} \pi_{\mathbb{R}} \circ u(w) = +\infty$ *and* $\lim_{w \to p_i} \pi_{\mathbb{R}} \circ u(w) = -\infty$*;*
3. *near* q_i*,* u *converges to a strip over* $[0, 1] \times \{y_i\}$*;*
4. *near each* p_i*,* u *converges to a cylinder over a multiple of some* γ_j *so that the total multiplicity of* γ_j *over all the* p_i *is* m_i*;*
5. *the energy of* u *given by Equation 4.1 is finite ;*

(ii) a M-B building in W_+ *that, after a perturbation of* R *and* J_+*, becomes a degree* k *multisection of* π_+ *satisfying 1-5 of (i).*

A (W_+, J_+)-curve *is a degree* $2g$ *multisection of* (W_+, J_+)*.*

In practice holomorphic multisections in W_+ interpolates between multisections in W and W'. Moreover in [10, Section 5], the authors define an ECH *index for holomorphic multisections of* W_+ which interpolates between the Lipshitz's index for holomorphic curves in W and the ECH-index in W'. See Observation 4.10 below

As in [10], we will call *irreducible component of* u a connected component of $\mathrm{Im}(u(\dot{F}))$.

Define the chain map

$$\begin{array}{rccc} \Phi' : & CF'(S, \boldsymbol{a}, \phi(\boldsymbol{a})) & \longrightarrow & PFC_{2g}(N) \\ & y & \longmapsto & \sum_{\gamma \in \mathcal{O}_{2g}(N)} \langle \Phi'(y), \gamma \rangle \cdot \gamma \end{array} \tag{4.5}$$

where $\langle \Phi'(y), \gamma \rangle$ is the modulo 2 count of degree $2g$, ECH-index 0 multisections of (W_+, J_+) from y to γ.

Observation 4.10. *Intuitively* Φ' *counts holomorphic curves that:*

1. *"start" in a collection of* $2g$ *chords and in the strip-like part* $\{s > 2\}$ *of* W_+ *is topologically like a holomorphic multisection of* π_B *(with boundary) counted by* ∂^{HF}*;*
2. *when the curve "arrives" in* $\{s = 2\} \subset W_+$ *the components of its boundary (contained in* $L^+_{\mathbf{a}}$*) are glued together using the map* ϕ*;*

3. *the curve in the cylinder-like part is topologically like a holomorphic multisection of $\pi_{B'}$ counted by ∂^{ECH}, which limits to a degree $2g$ multiorbit in N.*

Given $P \in \partial S$, let $\delta_P = \frac{[0,2]\times\{x\}}{(2,P)\sim(0,P)}$ be the simple Reeb orbit containing P.

In the following theorem we summarize some of the results about Φ proved in [10]:

Theorem 4.11. *The following hold:*

1. *if u is a holomorphic curve counted by Φ' which has a x_i (resp. x'_i) at the positive end, then it must have $\sigma^+_{\delta_{x_i}}$ (resp. $\sigma^+_{\delta_{x'_i}}$) as an irreducible component, so that at the negative end u must have a copy of e for each x_i or x'_i lying at the positive end;*
2. *Φ' respects the equivalence relation 4.2 and the passage to the quotient induces a map*
$$\Phi : \widehat{CF}(S, \boldsymbol{a}, \phi(\boldsymbol{a})) \to PFC_{2g}(N),$$
which send the HF-contact element to the ECH-contact element;
3. *let $\mathfrak{s}_\xi$ be the Spin^c-structure of the plane field ξ; then Φ splits into a direct sum of chain maps*
$$\Phi_\Gamma : \widehat{CF}(S, \boldsymbol{a}, \phi(\boldsymbol{a});\ \mathfrak{s}_\xi + PD(\Gamma)) \to PFC_{\leq 2g}(N, \partial N, \Gamma),$$
for $\Gamma \in H_1(Y;\mathbb{Z})$, where PD indicates the Poincaré dual.

II

5

Generalizations of $\widehat{\text{ECK}}$

Let K be a homologically trivial knot in a contact three-manifold (Y, α). As recalled in 2.4, if α is adapted to K, a choice of a Seifert surface S for K induces a filtration on the chain complex $\left(ECC^{h_+}(N, \alpha), \partial^{ECH}\right)$, where $int(N)$ is homeomorphic to $Y \setminus K$. Moreover if α is also adapted to S, the homology of $\left(ECC^{h_+}(N, \alpha), \partial^{ECH}\right)$ is isomorphic to $\widehat{ECH}(Y, \alpha)$, and the first page of the spectral sequence associated to the filtration is the hat version of embedded contact knot homology $\widehat{ECK}(K, Y, \alpha)$.

In this chapter we generalise the knot filtration in two natural ways.

In Section 5.1 we extend the filtration induced by K on the chain complex $\left(ECC^{h_+, e_+}(N, \alpha), \partial^{ECH}\right)$. This filtration is defined in a way completely analogue to the hat case. We define the *full version of embedded contact knot homology* of (K, Y, α) to be the first page $ECK(K, Y, \alpha)$ of the associated spectral sequence. Moreover we remove the condition that α must be compatible with S, in order to consider a wider class of contact forms: the knot spectral sequence is still well defined, but at the price of renouncing to a proof of the existence of an isomorphism between $ECH(Y, \alpha)$ and the page ∞ of the spectral sequence.

In Section 5.2 we generalise the knot filtration to n-components links L. The resulting homologies, defined in a way analogue to the case of knots, are the *full and hat versions of embedded contact knot homologies of* (L, Y, α), which will be still denoted $ECK(L, Y, \alpha)$ and, respectively, $\widehat{ECK}(L, Y, \alpha)$. Similarly to Heegaard-Floer link homology, these homologies come endowed with an *Alexander (relative)* $\mathbb{Z}^n$*-degree*.

5.1 The full ECK

Let K be a homologically trivial knot in a contact three-manifold (Y, α) and suppose that α is adapted to K in the sense of Section 2.2. Recall in particular that there exist two concentric neighborhoods $V(K) \subset \mathcal{N}(K)$ of K whose boundaries are M-B tori $T_1 = \partial\mathcal{N}(K)$ and $T_2 = \partial V(K)$ foliated by orbits of R_α in the homology class of meridians for K. These two families of orbits are modified into the two couples of orbits $\{e, h\}$ and, respectively, $\{e_+, h_+\}$. Let moreover $N = Y \setminus int(\mathcal{N}(K))$.

Consider the chain complex $\left(ECC^{e_+,h_+}(N, \alpha), \partial^{ECH}\right)$ where, recall, the chain group is freely generated on $\mathbb{Z}/2$ by the orbit sets γ in $\mathcal{O}(N) \sqcup \{h_+, e_+\}$ and ∂^{ECH} is the ECH-differential (obtained by restricting the differential on $ECC(Y, \alpha)$) given by Equation 2.7.

A Seifert surface S for K induces an *Alexander degree* $\langle\cdot, S\rangle$ on the generators of $ECC^{h_+,e_+}(N, \alpha)$ exactly like in the case of $ECC^{h_+}(N, \alpha)$. Let $ECC_d^{h_+,e_+}(N, \alpha)$ be the submodule of $ECC^{h_+,e_+}(N, \alpha)$ generated by the $\gamma \in \mathcal{O}(N) \sqcup \{h_+, e_+\}$ with $\langle\gamma, S\rangle = d$. If

$$ECC_{\leq d}^{h_+,e_+}(N, \alpha) := \bigoplus_{j \leq d} ECC_j^{h_+,e_+}(N, \alpha),$$

we have the exhaustive filtration

$$\ldots \subseteq ECC_{\leq d-1}^{h_+,e_+}(N, \alpha) \subseteq ECC_{\leq d}^{h_+,e_+}(N, \alpha) \subseteq ECC_{\leq d+1}^{h_+,e_+}(N, \alpha) \subseteq \ldots$$

of $ECC^{h_+,e_+}(N, \alpha)$. Proposition 2.17 again implies that ∂^{ECH} preserves the filtration. Let

$$\partial_d^{ECK} : ECC_d^{h_+,e_+}(N, \alpha) \longrightarrow ECC_d^{h_+,e_+}(N, \alpha)$$

be the part of ∂^{ECH} that strictly preserves the filtration degree d, that is, the differential induced by $\partial^{ECH}|_{ECC_{\leq d}^{h_+,e_+}(N,\alpha)}$ on the quotient

$$\frac{ECC_{\leq d}^{h_+,e_+}(N, \alpha)}{ECC_{\leq d-1}^{h_+,e_+}(N, \alpha)} = ECC_d^{h_+,e_+}(N, \alpha).$$

Set

$$\partial^{ECK} := \bigoplus_d \partial_d^{ECK} : ECC^{e_+,h_+}(N, \alpha) \longrightarrow ECC^{e_+,h_+}(N, \alpha).$$

Definition 5.1. *We define the* full embedded contact knot homology of (K, Y, α) *by*

$$ECK(K, Y, \alpha) := H_*\left(ECC^{e_+,h_+}(N, \alpha), \partial^{ECK}\right).$$

Note that, as in the hat case, the only holomorphic curves counted by ∂^{ECH} that do not strictly respect the filtration degree are the curves that contain the plane from h_+ to $\emptyset$ (see Observation 2.21). Recalling the expression of ∂^{ECH} given in Equation 2.7, it follows that ∂^{ECK} is given by

$$\partial^{ECK}(e_+^a h_+^b \gamma) = e_+^{a-1} h_+^b h\gamma + e_+^a h_+^{b-1} e\gamma + e_+^a h_+^b \partial\gamma, \tag{5.1}$$

where $\gamma \in \mathcal{O}(N)$ and any term is meant to be 0 if it contains an orbit with total multiplicity that is negative or not in $\{0,1\}$ if the orbit is hyperbolic.

Again the homology comes with an *Alexander degree*, which is well defined once the an homology class for S is fixed. In fact we have the natural splitting:

$$ECK_*(K,Y,\alpha) \cong \bigoplus_{d\in\mathbb{Z}} ECK_{*,d}(K,Y,\alpha) \tag{5.2}$$

where

$$ECK_{*,d}(K,Y,\alpha) := H_*(ECC_d^{h_+,e_+}(N,\alpha), \partial_d^{ECK}).$$

Recalling that $Y \setminus \mathcal{N}(K)$ is homeomorphic to $Y \setminus K$, it is interesting to state the following:

Lemma 5.2. *If $\mathcal{N}(K)$ is a neighborhood of K as above then*

$$ECK(K,Y,\alpha) \cong ECH(Y \setminus \mathcal{N}(K), \alpha).$$

Proof. By arguments similar to those in the proof of Lemma 2.9 it is easy to prove that:

$$\begin{aligned} ECK(K,Y,\alpha) &\cong H_*\left(ECC^{e_+,h_+}(N,\alpha), \partial^{ECK}\right) \\ &\cong H_*\left(ECC^{e,h_+}(int(N),\alpha), \partial^{ECK}\right) \\ &\cong H_*\left(ECC(int(N),\alpha), \partial^{ECK}\right) \\ &\cong ECH(int(N),\alpha), \end{aligned}$$

where the last comes from the fact that $\partial^{ECK}(\gamma) = \partial^{ECH}(\gamma)$ for any $\gamma \in \mathcal{O}(N)$. □

Observation 5.3. *Note that so far we only assumed that α is compatible with K, while we did not suppose the condition*

(♠) α is compatible with a Seifert surface S for K.

As remarked in Observation 2.24, without ♠ we can not prove theorem 2.7, and so we do not know if the spectral sequence whose 0-page is the ECK-chain complex limits to $ECH(Y,\alpha)$. On the other hand, this spectral sequence is in any case well defined, as well as $ECK(K,Y,\alpha)$.

Even if, in light of Lemma 2.4 we could assume ♠ here without restrictions on K, we prefer to avoid it in the general definition of $ECK(K,Y,\alpha)$ in order to consider a wider class of contact forms.

We remark that, reading carefully [9], we feel that the requirement ♠ could be not really necessary to prove Theorem 2.7. We try to roughly motivate our feeling.

By direct limit arguments the orbits in the no man's land $int(\mathcal{N}(K))\setminus V(K)$ can be avoided also if ♠ is not assumed, so that we can still write

$$ECC(Y,\alpha) \cong ECC(V,\alpha) \otimes ECC(N,\alpha)$$

(up to some restriction on the action of the orbits, see [9, Section 9]). The computations for $ECH(V,\alpha)$ in [9, Section 8] do not use ♠, and in fact here the hypothesis is not even assumed. Similarly, $ECH(N,\alpha)$ is still well defined as in [9, Subsection 7.1] and does not depend on the choice of S. Moreover the Blocking Lemma still implies that holomorphic curves with positive limit in N can not cross ∂N, so that $ECC(N,\alpha)$ is again a subcomplex of $ECC(Y,\alpha)$. This suggests that what happens in N should not influence the direct limits computations in V.

An even more basic motivation behind our perception comes from the intuitive approach to Theorem 1.1.1 presented in Subsection 9.1 of [9]: this argument is evidently local near K and ♠ is not used.

In analogy with Conjecture 2.27 we state the following:

Conjecture 5.4. *For any knot K in Y:*

$$ECK(K,Y,\alpha) \cong HFK^+(-K,-Y),$$

where α is any contact form on Y adapted to K.

5.2 The generalization to links

In this section we extend the definitions of ECK and $\widehat{ECK}$ to the case of homologically trivial links with more than one component. For us a (*strongly*) *homologically trivial n-link* in Y is a disjoint union of n knots, each of which is homologically trivial in Y.

Suppose that

$$L = K_1 \sqcup \ldots \sqcup K_n$$

is a homologically trivial n-link in Y. We say that a contact form α on Y is *adapted to L* if it is adapted to K_i for each i.

Lemma 5.5. *For any link L and contact structure ξ on Y there exists a contact form compatible with ξ which is adapted to L.*

Proof. The proof of part 1) of Lemma 2.4 is local near the knot K and can then be applied recursively to each K_i. □

Fix $L = K_1 \sqcup \ldots \sqcup K_n$ homologically trivial and α an adapted contact form. Since α is adapted to each K_i, there exist pairwise disjoint tubular neighborhoods

$$V(K_i) \subset \mathcal{N}(K_i)$$

of K_i where α behaves exactly like in the neighborhoods $V(K) \subset \mathcal{N}(K)$ in Section 2.2.

In particular, for each i, the tori $T_{i,1} := \partial\mathcal{N}(K_i)$ and $T_{i,2} := \partial V(K_i)$ are M-B and foliated by families of orbits of R_α in the homology class of a meridian of K_i. We will consider these two families as perturbed into two pairs $\{e_i, h_i\}$ and $\{e_i^+, h_i^+\}$ in the usual way.

Let

$$V(L) := \bigsqcup_i V(K_i) \text{ and } \mathcal{N}(L) := \bigsqcup_i \mathcal{N}(K_i)$$

and set

$$N := Y \setminus int(\mathcal{N}(L)).$$

Define moreover $\bar{e} := \bigsqcup_i e_i$ and let $\bar{h},\ \bar{e}_+$ and $\bar{h}_+$ be similarly defined.

Consider now $ECC^{\bar{e}_+,\bar{h}_+}(N,\alpha)$ endowed with the restriction ∂^{ECH} of the ECH differential of (Y,α) and let $ECH^{\bar{e}_+,\bar{h}_+}(N,\alpha)$ be the associated homology.

Lemma 5.6. *$ECH^{\bar{e}_+,\bar{h}_+}(N,\alpha)$ is well defined and the curves counted by ∂^{ECH} inside each $\mathcal{N}(K_i)$ are given by expressions analogue to those in 2.6.*

Proof. The Blocking and Trapping lemmas can be applied locally near each component of ∂N and the proofs of lemmas 7.1.1 and 7.1.2 in [9] work immediately in this context too. This imply that the homology of $\left(ECC(N,\alpha), \partial^{ECH}\right)$ is well defined.

Again the Blocking and Trapping lemmas together with the local homological arguments in lemmas 9.5.1 and 9.5.3 in [9], imply that the only holomorphic curves counted by ∂^{ECH} inside each $\mathcal{N}(K_i)$ are as required (see Figure 2.1), and so that $ECH^{\bar{e}_+,\bar{h}_+}(N,\alpha)$ is well defined. $\square$

An explicit formula for ∂^{ECH} can be obtained by generalizing Equation 2.7 in the obvious way.

For each $i \in \{1,\ldots,n\}$, fix now a (homology class for a) Seifert surface S_i for K_i. These surfaces are not necessarily pairwise disjoint and it is even possible that $S_i \cap K_j \neq \emptyset$ for some $i \neq j$.

Consider then the *Alexander $\mathbb{Z}^n$-degree* on $ECC^{\bar{e}_+,\bar{h}_+}(N,\alpha)$ given by the function

$$\begin{array}{ccc} ECC^{\bar{e}_+,\bar{h}_+}(N,\alpha) & \longrightarrow & \mathbb{Z}^n \\ \gamma & \longmapsto & (\langle\gamma,S_1\rangle,\ldots,\langle\gamma,S_n\rangle). \end{array} \tag{5.3}$$

Observation 5.7. *Last expression is in fact the generalization to any manifold of Equation 3.7 for S^3 and inducing the Alexander filtration in Heegaard-Floer.*

Define the partial ordering on $\mathbb{Z}^n$ given by

$$(a_1, \dots, a_n) \leq (b_1, \dots, b_n) \iff a_i \leq b_i \; \forall \, i.$$

Proposition 2.17 applied to each K_i implies that if γ and δ are two orbit sets in $\mathcal{O}(N \sqcup \{\bar{e}_+, \bar{h}_+\})$, then for any k

$$\frac{\mathcal{M}_k(\gamma, \delta)}{\mathbb{R}} \neq 0 \implies (\langle \delta, S_1 \rangle, \dots, \langle \delta, S_n \rangle) \leq (\langle \gamma, S_1 \rangle, \dots, \langle \gamma, S_n \rangle) \, .$$

This implies that ∂^{ECH} does not increase the Alexander degree, which induces than a $\mathbb{Z}^n$-filtration on $\left(ECC^{\bar{e}_+, \bar{h}_+}(N, \alpha), \partial^{ECH} \right)$.

Reasoning as in the previous section, we are interested in the part of ∂^{ECH} that strictly respects the filtration degree. This can be defined again in terms of quotients as follows.

Let $d \in \mathbb{Z}^n$ and let $ECC_d^{\bar{e}_+, \bar{h}_+}(N, \alpha)$ be the submodule of $ECC^{\bar{e}_+, \bar{h}_+}(N, \alpha)$ freely generated by orbit sets $\gamma \in \mathcal{O}(N \sqcup \{\bar{e}_+, \bar{h}_+\})$ such that

$$(\langle \gamma, S_1 \rangle, \dots, \langle \gamma, S_n \rangle) = d.$$

Define

$$ECC_{\leq d}^{\bar{e}_+, \bar{h}_+}(N, \alpha) := \bigoplus_{j \leq d} ECC_d^{\bar{e}_+, \bar{h}_+}(N, \alpha)$$

and let $ECC_{<d}^{\bar{e}_+, \bar{h}_+}(N, \alpha)$ be similarly defined.

Define the *full ECK-differential in degree d* to be the map

$$\partial_d^{ECK} \colon ECC_d^{\bar{e}_+, \bar{h}_+}(N, \alpha) \longrightarrow ECC_d^{\bar{e}_+, \bar{h}_+}(N, \alpha)$$

induced by $\partial^{ECH}|_{ECC_{\leq d}^{\bar{e}_+, \bar{h}_+}(N, \alpha)}$ on the quotient

$$\frac{ECC_{\leq d}^{\bar{e}_+, \bar{h}_+}(N, \alpha)}{ECC_{<d}^{\bar{e}_+, \bar{h}_+}(N, \alpha)} \cong ECC_d^{\bar{e}_+, \bar{h}_+}(N, \alpha).$$

Define then the *full ECK-differential* by

$$\partial^{ECK} := \bigoplus_d \partial_d^{ECK} \colon ECC^{\bar{e}_+, \bar{h}_+}(N, \alpha) \longrightarrow ECC^{\bar{e}_+, \bar{h}_+}(N, \alpha).$$

Observation 5.8. *Observing the form of ∂^{ECH}, it is easy again to see that the only holomorphic curves that are counted by ∂^{ECH} and not by ∂^{ECK} are the ones containing a holomorphic plane from some h_i^+ to $\emptyset$.*

Definition 5.9. *The* full embedded contact knot homology of (L, Y, α) *is*

$$ECK(L, Y, \alpha) := H_*\left(ECC^{\bar{e}_+, \bar{h}_+}(N, \alpha), \partial^{ECK}\right).$$

The fact that $ECK(L, Y, \alpha)$ is well defined is a direct consequence of the good definition of $ECH^{\bar{e}_+, \bar{h}_+}(N, \alpha)$ and the fact that ∂^{ECH} respects the Alexander filtration.

Note that also for links we have a natural splitting

$$ECK_*(L, Y, \alpha) = \bigoplus_{d \in \mathbb{Z}^n} ECK_{*,d}(L, Y, \alpha) \tag{5.4}$$

where

$$ECK_{*,d}(L, Y, \alpha) = H_*\left(ECC_d^{\bar{e}_+, \bar{h}_+}(N, \alpha), \partial_d^{ECK}\right).$$

The proof of the following lemma is the same of that of the analogous Lemma 5.2 for knots applied to each component of L.

Lemma 5.10. *If $\mathcal{N}(L)$ is a neighborhood of L as above then*

$$ECK(L, Y, \alpha) \cong ECH(Y \setminus \mathcal{N}(L), \alpha).$$

Consider now the submodule $ECC^{\bar{h}_+}(N, \alpha)$ of $ECC^{\bar{e}_+, \bar{h}_+}(N, \alpha)$ endowed with the restriction of ∂^{ECH}. Again its homology $ECH^{\bar{h}_+}(N, \alpha)$ is well defined.

Proceeding exactly like above, the choice of a Seifert surface S_i for each component K_i of L gives (up to small perturbations of S) an Alexander degree on the orbit sets defined by Equation 5.3. This induces a $\mathbb{Z}^n$-filtration on the chain complex $\left(ECC^{\bar{h}_+}(N, \alpha), \partial^{ECH}\right)$.

For any $d \in \mathbb{Z}^n$, define $ECC_d^{\bar{h}_+}(N, \alpha)$ and

$$\partial_d^{ECK} \colon ECC_d^{\bar{h}_+}(N, \alpha) \longrightarrow ECC_d^{\bar{h}_+}(N, \alpha)$$

exactly as above.

Definition 5.11. *The* hat version of embedded contact knot homology of (L, Y, α) *is*

$$\widehat{ECK}(L, Y, \alpha) := H_*\left(ECC^{\bar{h}_+}(N, \alpha), \partial^{ECK}\right).$$

Observation 5.8 and a splitting like the one in equation 5.4 hold also for $\widehat{ECK}(L, Y, \alpha)$. Moreover it is easy to see that if L has only one connected component we get the same theories of sections 2.4 and 5.1.

We state the following

Conjecture 5.12. *If L is a link in Y:*

$$\begin{aligned} ECK(L,Y,\alpha) &\cong HFK^{-}(L,Y), \\ \widehat{ECK}(L,Y,\alpha) &\cong \widehat{HFK}(L,Y), \end{aligned}$$

where α is any contact form on Y adapted to L.

Observation 5.13. *Note that the analogous conjectures stated before, as well as Theorem 4.1, suggest that we should use the plus version of HFL and not the minus one. The problem is that in [46] the authors define Heegaard-Floer homology for links only in the hat and minus versions.*

On the other hand this switch is not really significant. Indeed one could define Heegaard-Floer cohomology *groups by taking the duals, with coefficients in $\mathbb{Z}/2$, of the chain groups $\widehat{CF}_*(Y)$, $CF_*^+(Y)$ and $CF_*^-(Y)$ in the usual way and get cohomology groups (for the three-manifold Y)*

$$\widehat{HF}^*(Y), \quad HF_+^*(Y), \quad HF_-^*(Y).$$

Since we are working in $\mathbb{Z}/2$ we have that each of this cohomology group is isomorphic to its respective homology group.

On the other hand one can prove also that (see Proposition 2.5 in [43]):

$$\widehat{HF}^*(Y) \cong \widehat{HF}_*(-Y) \quad \text{and} \quad HF_\pm^*(Y) \cong HF_*^{\mp}(-Y).$$

Analogous formulae hold for knots also. The conjecture above is then consistent with those stated in the previous sections.

Observation 5.14. *As in the definition of $\widehat{ECK}(K,Y,\alpha)$ and $ECK(K,Y,\alpha)$ also here we used the hypothesis that α is adapted to L, while we dropped condition ♠ of last section. One could wonder if it is possible to further relax the assumptions and get still a good definition of the ECK homology groups.*

The onjectures above suggest indeed that $ECK(L,Y,\alpha)$ (as well as the other homologies) would be independent from α and so, in particular, that we could be able to define it simply as the ECH homology of the complement of (any neighborhood of) L, provided that L is a disjoint union of Reeb orbits of α. Indeed, even if we could not have an easy description of the curves counted by ∂^{ECH} that cross L, Proposition 2.17 still holds in this more general case.

On the other hand, technical aspects about contact flows and holomorphic curves suggest that the components of L should be at least elliptic orbits. This property will be necessary even in computing Euler characteristics in next chapter, where we will need a circularity property of R_α near L that cannot be assumed in an evident way if a component of L is hyperbolic.

Notations. *In order to simplify the notation, in the rest of the paper we will indicate the ECH chain groups for the knot embedded contact homology groups of links and knots by:*

$$\begin{aligned} ECC(L, Y, \alpha) &:= ECC^{\bar{e}_+, \bar{h}_+}(N, \alpha), \\ \widehat{ECC}(L, Y, \alpha) &:= ECC^{\bar{h}_+}(N, \alpha), \end{aligned}$$

where N and α are as above. In particular, if not stated otherwise, we will always assume that the contact form α is adapted to L. These groups will implicitly come endowed with the differential ∂^{ECK}.

We end this chapter by saying some word about a further generalization of ECK to *weakly homologically trivial* links. We say that $L \subset Y$ is a weakly homologically trivial (or simply *weakly trivial*) n-component link if there exist surfaces with boundary $S_1, \ldots, S_m \subset Y$, with $m \leq n$ and such that $\partial S_i \cap \partial S_j = \emptyset$ if $i \neq j$ and $\bigsqcup_{i=1}^m \partial S_i = L$. Also here we do not require that S_i or even ∂S_i is disjoint from S_j for $j \neq i$.

Clearly L is a strongly trivial link if and only if it is weakly trivial with $m = n$.

In this case we cannot in general define a homology with a filtered n-degree. If L is a weakly trivial link with $m \lneq n$ and α is an adapted contact form, then there exists $S \in \{S_1, \ldots, S_m\}$ such that ∂S has more then one connected component. Suppose for instance that $\partial S = K_1 \sqcup K_2$. The arguments of proposition 2.17 say then that if $u : (F, j) \to (\mathbb{R} \times Y, J)$ is a holomorphic curve from γ to δ, then

$$\langle \gamma, S \rangle - \langle \delta, S \rangle = \langle \mathrm{Im}(u), \mathbb{R} \times (K_1 \sqcup K_2) \rangle \geq 0.$$

So in this case we can still apply the arguments above and get well defined ECH invariants for L. However this time they will come only with a filtered (relative) $\mathbb{Z}^m$-degree on the generators γ of an ECH complex of Y, which is given by the m-tuple $(\langle \gamma, S_1 \rangle, \ldots, \langle \gamma, S_m \rangle)$.

Example 5.15. *Let (L, S, ϕ) be an open book decomposition of Y with, possibly, disconnected boundary. Using a (connected) page of (L, S, ϕ) to compute the Alexander degree and, with the notations of Subsection 2.2.1, we get*

$$ECK_d(L, Y, \alpha) \cong ECH_d(int(N), \alpha)$$

for any $d \in \mathbb{Z}$.

6

Euler characteristics

In this chapter we compute the graded Euler characteristics of the embedded contact homology groups for knots and links in homology three spheres Y with respect to suitable contact forms. The computations will be done in terms of the Lefschetz zeta function of the flow of the Reeb vector field.

In the particular case of $Y = S^3$ we relate the resulting expressions to the corresponding multivariable Alexander polynomial:

Theorem 6.1. *Let L be any n-link in S^3. Then there exists a contact form α adapted to L such that:*

$$\chi\left(ECK(L, S^3, \alpha)\right) \doteq \begin{cases} \Delta_L(t_1, \ldots, t_n) & \text{if } n > 1 \\ \Delta_L(t)/(1-t) & \text{if } n = 1 \end{cases} \tag{6.1}$$

and

$$\chi\left(\widehat{ECK}(L, S^3, \alpha)\right) \doteq \begin{cases} \Delta_L \cdot \prod_{i=1}^n (1 - t_i) & \text{if } n > 1 \\ \Delta_L(t) & \text{if } n = 1. \end{cases} \tag{6.2}$$

An immediate consequence of theorem above and Theorem 3.9 is:

Corollary 6.2. *For any link L in S^3 there exists a contact form α such that:*

$$\chi(ECK(L, S^3, \alpha)) \doteq \chi(HFL^-(L, S^3)), \tag{6.3}$$
$$\chi(\widehat{ECK}(L, S^3, \alpha)) \doteq \chi(\widehat{HFL}(L, S^3)). \tag{6.4}$$

The last corollary implies that conjecture 5.12 (which generalizes conjectures 2.27 and 5.4) holds for links in S^3 at least at the level of Euler characteristic.

As recalled in Section 3.2, graded Euler characteristics are polynomials: when we want to highlight the variables of these polynomials we will indicate them as subscripts of the symbol χ. For example if L is an n-link and we want to express its Euler characteristic by a polynomial in the n variables $t_1, \ldots, t_n$, we will write $\chi\,(ECK(L, Y, \alpha)) = \chi_{t_1,\ldots,t_n}\,(ECK(L, Y, \alpha))$.

Theorem 6.1 is in effect a consequence of the following more general result. Recall that an n-link $L \subset Y$ determines the abelian cover $\rho_L \in H^1(Y \setminus L, \mathbb{Z}^n)$ of $Y \setminus L$ given in Example 3.11. When Y is a homology three-sphere, we have

$$\rho_L \equiv \mathbb{1} \colon H_1(Y \setminus L) \longrightarrow H_1(Y \setminus L) \cong \mathbb{Z}^n.$$

In order to simplify the notations, we remove ρ_L from the notations of the Alexander quotient and of the twisted Lefschetz zeta function:

$$\begin{aligned} \mathrm{ALEX}(Y \setminus L) &:= \mathrm{ALEX}_{\mathbb{1}}(Y \setminus L); \\ \zeta(\phi) &:= \zeta_{\mathbb{1}}(\phi). \end{aligned}$$

Let $(t_1, \ldots, t_n)$ be a basis for $H_1(Y \setminus L)$, where $[\mu_i] = t_i$ for μ_i positively oriented meridian of K_i.

Theorem 6.3. *Let L be an n-link in a homology three-sphere Y. Then there exists a contact form α such that*

$$\chi_{t_1,\ldots,t_n}(ECK(L, Y, \alpha)) \doteq \mathrm{ALEX}(Y \setminus L).$$

Last two theorems imply that *the homology ECK categorifies the Alexander quotient of knots and links in homology three-spheres*. This is the third known categorification of this kind, after the ones in Heegaard-Floer homology and in Seiberg-Witten-Floer homology (see [34] and [35]).

The proofs of theorems 6.1 and 6.3 will be carried on in two main steps: in Section 6.1 we will prove the theorems in the case of fibered links, while the general case will be treated in Section 6.2.

6.1 Fibered links

In this section we prove theorems 6.1 and 6.3 for fibered links. Let (L, S, ϕ) be an open book decomposition of a homology three-sphere Y and let α be an adapted contact form on Y. In particular, with our definition, α is also adapted to L.

In order to prove the theorems above we want to express the Euler characteristic $\chi_{t_1,\dots,t_n}(ECK(L,Y,\alpha))$ in terms of the twisted Lefschetz zeta function of the Reeb flow ϕ_R of $R = R_\alpha$ and then apply Theorem 3.17. The first thing that one should do is then to check if ϕ_R and ρ_L satisfy the hypothesis of that theorem. Unfortunately this is not the case. The needed properties are in fact the following:

1. R is non-singular and circular;
2. R is compatible with ρ_L;
3. R is non-degenerate;
4. R is transverse to $\partial V(L)$ and pointing out of $Y \setminus \mathring{V}(L)$,

where $\mathring{V}(L) = int(V(L))$.

In our situation only properties 1 and 2 are satisfied. Indeed, by the definition of open book decomposition, there is a natural fibration $\theta : Y \setminus \mathring{V}(L) \to S^1 \cong \frac{\mathbb{R}}{\mathbb{Z}}$ such that the surfaces $\theta^{-1}(t)$ are the pages of the open book. The fact that α is adapted to (L, S, ϕ) implies that R is always positively transverse to the pages. This evidently implies that $d\theta(R) > 0$ so that R is circular.

The fact that R is compatible with ρ_L (that coincides with the universal abelian cover of $Y \setminus \mathring{V}(L)$) comes from Example 3.16.

However properties 3 and 4 above are not satisfied. Indeed, after the M-B perturbation of T_2, R is tangent to $\partial V(L)$ on $\bar{e}_+$ and $\bar{h}_+$. Moreover, as observed in Section 1.3, the M-B perturbations near the two tori T_1 and T_2 may create degenerate orbits.

What we will do is then to perturb R to get a new vector field R'. This vector field will be defined in $Y \setminus V'(L)$, where $V'(L) \subset \mathring{V}(L)$ is an open tubular neighborhood of L defined by $V'(L) = V'(K_1) \sqcup \ldots \sqcup V'(K_n)$, where, using the coordinates of Section 1.4, $\partial(V'(K_i)) = \{y = 2.5\}$.

Lemma 6.4. *There exists a (non-contact) vector field R' such that:*

(i) *R' coincides with R outside a neighborhood of $\mathcal{N}(L)$;*

(ii) *R' satisfies properties 1-4 above with $V(L)$ replaced by $V'(L)$;*

(iii) *the only periodic orbits of R' in $\mathcal{N}(V) \setminus V'(L)$ are the four sets of non-degenerate orbits $\bar{e}, \bar{h}, \bar{e}_+, \bar{h}_+$.*

Observe that Property (i) implies that the twisted Lefschetz zeta functions of the restrictions of the flows ϕ_R and $\phi_{R'}$ to $Y \setminus \mathcal{N}(K)$ coincide, while Property (ii) allows to apply Theorem 3.17 to $\phi_{R'}$.

Proof. A perturbation of R into an R' satisfying the conditions (i)-(iii) can be obtained in more than one way. An example is pictured in Figure 6.1 (cf. also Figure 1.1). We briefly explain how it is obtained. Since the modification of R is non trivial only inside disjoint neighborhoods of each K_i, we will describe it only for a fixed component K of L. The characterization of the perturbation will be presented in terms of perturbation of the lines in a page S of (L, S, ϕ) that are invariant under the first return map ϕ of ϕ_R: we will refer to these curves as to *ϕ-invariant lines on S*. Note that these curves are naturally oriented by the flow.

Outside a neighborhood of $\partial V'$ one can see this perturbation in terms of a perturbation of ϕ into another monodromy ϕ', and R' is the vector field ∂_t in $Y \setminus V'(L) \cong \frac{S\times[0,1]}{(x,1)\sim(\phi'(x),0)}$, where t is the coordinate of $[0, 1]$.

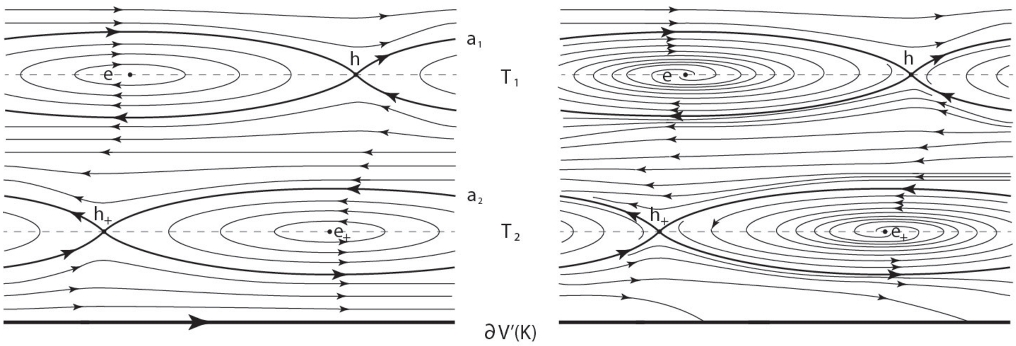

Figure 6.1: The dynamics of the vector fields R and R' near $\mathcal{N}(V) \setminus V'(L)$. Each oriented line represents an invariant subset of a page of (L, S, ϕ) under the first return map ϕ at the left and ϕ' at the right (the invariant lines a_1 and a_2 are stressed). The situation at the left is the same depicted in Figure 1.1.

Observe first that the only periodic orbit in the (singular) ϕ-invariant line a_1 containing h (in correspondence to the singularity) is exactly h. Similarly, the only periodic orbit in the ϕ-invariant singular flow line a_2 containing h_+ is precisely h_+. Denote $A_i \subset Y$ the mapping torus of $(a_i, \phi|_{a_i})$, $i = 1, 2$. We modify R separately inside the regions of $(Y \setminus V'(K)) \setminus (A_1 \sqcup A_2)$ as follows.

In the region containing e (and with boundary A_1), the set of ϕ-invariant lines (the elliptic lines in the picture at left) is perturbed in a set of ϕ'-invariant spiral-kind lines (at right), each of which is negatively asymptotic to a_1 and positively asymptotic to e. It is easy to see that after the perturbation the only periodic orbit in the interior of this region is e. Moreover, we can arrange the perturbation in a way that the differential $\mathfrak{L}_e^{R'}$ of the first return map on S of $\phi_{R'}$ along e, coincides, up to a positive factor smaller then 1, with $\mathfrak{L}_e^R$, so that the Lefschetz sign $\epsilon(e)$ of e is still $+1$.

A similar perturbation is done in the region of $(Y \setminus V'(K)) \setminus (A_1 \sqcup A_2)$ containing e_+, in a way that e_+ is the only periodic orbit of the perturbed vector field R', with still $\epsilon(e_+) = +1$.

The perturbation in the region between A_1 and A_2 is done by slightly pushing the monodromy in the positive y-direction in a way that the set of ϕ-invariant lines is perturbed into a set of ϕ'-invariant lines, each of which is negatively asymptotic to a_1 and positively asymptotic to a_2 (and so in particular there can not exist periodic orbits in this region).

A similar perturbation is done also inside the region between A_2 and $\partial V'(K)$, but in this case each ϕ'-invariant line is negatively asymptotic to a_2 and intersects $\partial V'(K)$ pointing out of the three-manifold.

Finally we leave $R' = R$ in the rest of the manifold, where R was supposed having only isolated and non degenerate periodic orbits.

Note that the two basis of eigenvectors of $\mathfrak{L}_h^R$ and $\mathfrak{L}_{h+}^R$ are contained in the tangent spaces of the curves a_1 and, respectively, a_2. Since on these curves $\phi_R = \phi_{R'}$, the Lefschetz signs of the two orbits are not changed by the perturbation.

It is easy to convince ourselves that R' satisfies the properties i-iii above. $\square$

Call $\zeta = \zeta_{\mathbb{1}}$. Since the Lefschetz zeta function of a flow depends only on its periodic orbits and their signs, we have the following:

Corollary 6.5. *If R' is obtained from R as above, then*

$$\begin{aligned} \zeta(\phi_{R'}) &= \zeta(\phi_{R'}|_{(Y\setminus\mathcal{N}(K)\sqcup\{\bar{e},\bar{h},\bar{e}_+,\bar{h}_+\})}) = \\ &= \zeta(\phi_{R|_{(Y\setminus\mathcal{N}(K)}}) \cdot \prod_{\gamma\in\{\bar{e},\bar{h},\bar{e}_+,\bar{h}_+\}} \zeta_\gamma([\gamma])). \end{aligned}$$

where $[\gamma]$ is the homology class of γ in $H_1(Y \setminus \mathcal{N}(K))$.

Now we want to compute more explicitly the twisted Lefschetz zeta function $\zeta(\phi_{R'})$. Let us begin with the local Lefschetz zeta function of the simple orbits (see Definition 3.12).

Lemma 6.6. *Let γ be an orbit of R or R'. Then:*

$$\zeta_\gamma(t) = \begin{cases} (1-t)^{-1} \quad = 1 + t + t^2 + \dots & \text{if } \gamma \text{ elliptic;} \\ \quad 1-t & \text{if } \gamma \text{ positive hyperbolic;} \\ \quad 1+t & \text{if } \gamma \text{ negative hyperbolic;} \end{cases} \tag{6.5}$$

Proof. Remember that the Lefschetz number of γ is $\varepsilon(\gamma) = +1$ if γ is elliptic or negative hyperbolic and $\varepsilon(\gamma) = -1$ if γ is positive hyperbolic.

γ elliptic: all the iterated are elliptic, so that $\epsilon(\gamma^i) = +1$ for every $i > 0$. Then:

$$\begin{aligned} \zeta_\gamma(t) &= \exp\left(\sum_{i\geq 1} \frac{t^i}{i}\right) &= \\ &= \exp\left(-\sum_{i\geq 1}(-1)^{i+1}\frac{(-t)^i}{i}\right) &= \\ &= \exp\left(-\log(1-t)\right) &= \\ &= (1-t)^{-1}. \end{aligned}$$

γ positive hyperbolic: all the iterated are positive hyperbolic, so that $\epsilon(\gamma^i) = -1$ for every $i > 0$. Then:

$$\begin{aligned} \zeta_\gamma(t) &= \exp\left(\sum_{i\geq 1} -\frac{t^i}{i}\right) &= \\ &= \exp\left(\sum_{i\geq 1}(-1)^{i+1}\frac{(-t)^i}{i}\right) &= \\ &= \exp\left(\log(1-t)\right) &= \\ &= 1-t. \end{aligned}$$

γ negative hyperbolic: the odd iterated are negative hyperbolic while the even iterated are positive hyperbolic, so that $\epsilon(\gamma^i) = (-1)^{i+1}$ for any $i > 0$. Then:

$$\begin{aligned} \zeta_\gamma(t) &= \exp\left(\sum_{i\geq 1}(-1)^{i+1}\frac{t^i}{i}\right) &= \\ &= \exp\left(\log(1+t)\right) &= \\ &= 1+t \end{aligned}$$

□

Observation 6.7. *Note that the equations above are exactly the generating functions given by Hutchings in [28, Chapter 2].*

Let μ_i be a positive meridian of K_i for $i \in \{1, \dots, n\}$ and set $t_i = [\mu_i] \in H_1(Y \setminus K)$; fix moreover a Seifert surface S_i for each K_i. Recall that, for a given $X \subset Y$, $\mathcal{P}(X)$ denotes the set of simple Reeb orbits contained in X.

Corollary 6.8. *The twisted Lefschetz zeta function of $\phi_{R|_{(Y\setminus\mathcal{N}(L))}}$ is*

$$\zeta(\phi_{R|_{(Y\setminus\mathcal{N}(L))}}) = \prod_{\gamma\in\mathcal{P}(Y\setminus\mathcal{N}(L))} \zeta_\gamma([\gamma]),$$

where $\zeta_\gamma([\gamma])$ is determined as follows:

- *if γ is elliptic then:*

$$\zeta_\gamma(\rho_L(\gamma)) = \left(1 - \prod_{i=1}^n t_i^{\langle\gamma,S_i\rangle}\right)^{-1} = \sum_{l=0}^\infty \left(\prod_{i=1}^n t_i^{\langle\gamma,S_i\rangle}\right)^l;$$

- *if γ is positive hyperbolic then:*

$$\zeta_\gamma(\rho_L(\gamma)) = 1 - \prod_{i=1}^n t_i^{\langle\gamma,S_i\rangle};$$

- *if γ is negative hyperbolic then:*

$$\zeta_\gamma(\rho_L(\gamma)) = 1 + \prod_{i=1}^n t_i^{\langle\gamma,S_i\rangle}.$$

Proof. This is an easy computation. It suffices to substitute the monomial representation of $\rho_L([\gamma)] = [\gamma]$ given in Example 3.11 in the expression of the Lefschetz zeta function of Lemma 6.6. □

Proof. (of Theorem 6.3 for fibered links). To finish the proof it remains essentially to prove that

$$\chi_{t_1,\dots,t_n}(ECC(L,Y,\alpha)) = \zeta(\phi_{R|_{(Y\setminus\mathcal{N}(L)}}) \cdot \prod_{\gamma\in\{\bar{e},\bar{h},\bar{e}_+,\bar{h}_+\}} \zeta_\gamma([\gamma])). \tag{6.6}$$

This is easy to verify recursively on the set of simple orbits. Suppose $\delta = \prod_j \delta_j^{k_j}$ is an orbit set and let γ be an orbit such that $\gamma \neq \delta_j$ for any j. Then the set of all multiorbits that we can build using δ and γ can be expressed via the product formulae:

$$\begin{array}{ll} \delta\cdot\{\emptyset,\gamma,\gamma^2,\dots\} & \text{if } \gamma \text{ is elliptic;} \\ \delta\cdot\{\emptyset,\gamma\} & \text{if } \gamma \text{ is hyperbolic.} \end{array} \tag{6.7}$$

As remarked in Section 2.1, the index parity formula 2.2 implies that the Lefschetz sign endows the ECH-chain complex with an absolute degree and it coincides with the parity of the ECH-index. Then the contribution to the graded Euler characteristic of $\delta\cdot\gamma^l$, for any l ($l \in \mathbb{N}$ if γ is elliptic and $l \in \{0,1\}$ if γ is hyperbolic) is:

$$\epsilon(\delta)\prod_{i=1}^n t_i^{\langle\delta,S_i\rangle}\cdot\left(\epsilon(\gamma)\prod_{i=1}^n t_i^{\langle\gamma,S_i\rangle}\right)^l.$$

Substituting the last formula in Expressions 6.7, the total contribution of the product formulae to the Euler characteristic are:

- $\epsilon(\delta)\prod_{i=1}^n t_i^{\langle\delta,S_i\rangle}\cdot\sum_{l=0}^\infty\left(\prod_{i=1}^n t_i^{\langle\gamma,S_i\rangle}\right)^l$ if γ is elliptic,
- $\epsilon(\delta)\prod_{i=1}^n t_i^{\langle\delta,S_i\rangle}\cdot\left(1-\prod_{i=1}^n t_i^{\langle\gamma,S_i\rangle}\right)$ if γ is positive hyperbolic,
- $\epsilon(\delta)\prod_{i=1}^n t_i^{\langle\delta,S_i\rangle}\cdot\left(1+\prod_{i=1}^n t_i^{\langle\gamma,S_i\rangle}\right)$ if γ is negative hyperbolic,

that is

$$\epsilon(\delta)\prod_{i=1}^n t_i^{\langle\delta,S_i\rangle}\cdot\zeta_\gamma([\gamma]).$$

Starting from $\delta = \emptyset$, Equation 6.6 follows by induction on the set of $\gamma \in \mathcal{P}\left((Y\setminus\mathcal{N}(L))\sqcup\{\bar{e},\bar{h},\bar{e}_+,\bar{h}_+\}\right)$.

The theorem follows then by applying Corollary 6.5 and Theorem 3.17 to the flow of R'. □

Proof. (of Theorem 6.1 for fibered links). Theorem 6.3 and Equation 3.13 immediately imply Equation 6.1.

To prove the result in the hat version we reason again at the level of chain complexes. Recall that, if $N := Y \setminus \mathring{\mathcal{N}}(L)$, by the definition of the ECK-chain complexes:

$$\begin{aligned} ECC(L, Y, \alpha) &= ECC^{\bar{e}_+, \bar{h}_+}(N, \alpha) = \\ &= ECC^{\bar{h}_+}(N, \alpha) \bigotimes_{i=1}^{n} \langle \emptyset, e_i^+, (e_i^+)^2, \ldots \rangle = \\ &= \widehat{ECC}(L, Y, \alpha) \bigotimes_{i=1}^{n} \langle \emptyset, e_i^+, (e_i^+)^2, \ldots \rangle \end{aligned}$$

where the second line comes from the product formula 6.7 and the fact that e_i^+ is elliptic for any i. Taking the graded Euler characteristics as above we have:

$$\begin{aligned} \chi(ECC(L, Y, \alpha)) &= \chi(\widehat{ECC}(L, Y, \alpha)) \cdot \prod_{i=1}^{n} \zeta_{e_i^+}([e_i^+]) = \\ &= \chi(\widehat{ECC}(L, Y, \alpha)) \cdot \prod_{i=1}^{n} \frac{1}{1 - t_i}, \end{aligned}$$

where the last equality comes from the fact that $[e_i^+] = [\mu_i] = t_i \in H_1(Y \setminus L)$. If $Y = S^3$, last equation and Equation 6.1 evidently imply Equation 6.2. $\square$

Note that if (L, S, ϕ) is an open book decomposition of Y, one can think of $ECK(L, Y, \alpha)$ and $\widehat{ECK}(L, Y, \alpha)$ as invariants of the pair (S, ϕ) and the adapted α. It is interesting to note that the Euler characteristic of $ECK_1(L, Y, \alpha)$ with respect to the surface S (see Example 5.15) coincides with the sum of the Lefschetz signs of the Reeb orbits of period 1 in the interior of S, i.e. the Lefschetz number $\Lambda(\phi)$ of ϕ.

In fact, given Y (not necessarily an homology three-sphere) we can say even more by relating $ECK_1(L, Y, \alpha)$ to the Hamiltonian Floer homology $SH(S, \phi)$, whose Euler characteristic is precisely $\Lambda(\phi)$. Here we are considering the version of $SH(S, \phi)$ for surfaces with boundary that is slightly rotated by ϕ in the positive direction, with respect to the orientation induced by S on ∂S (see for example [7] and [17]).

Proposition 6.9. *Let (L, S, ϕ) be an open book decomposition of a three-manifold Y and let α be an adapted contact form. Then*

$$ECK_1(L, Y, \alpha) \cong SH(S, \phi),$$

where the degree of $ECK(L, Y, \alpha)$ is computed using a page of the open book.

Proof. This is an easy consequence of the definitions and of some results in Chapter 2. By Lemma 5.10 and using the notations of Subsection 2.2.1,

$$ECK_1(L, Y, \alpha) \cong ECH_1(int(N), \alpha).$$

Observing that the proof of Theorem 2.13 in [10] (Theorem 3.6.1) works also if ∂S is disconnected we get

$$ECK_1(L, Y, \alpha) \cong PFH_1(N(S, \phi)).$$

The result then follows applying Proposition 2.14. □

We get an interesting consequence of this fact when also the Alexander degree of Heegaard-Floer knot homology of a fibered knot is computed with respect to (the homology class of) a page of the associated open book. Indeed, using the symmetrized degree adopted by Ozsváth and Szabó, we know that $HFK^-_{-g}(K, Y)$ is isomorphic to a copy of $\mathbb{Z}/2$ generated by the class of the corresponding contact element. Moreover, whenever $\chi(ECK(K, Y, \alpha)) = \chi(HFK^-(K, Y))$, we have also that $HFK^-_{-g+1}(K, Y)$ *categorifies* $\Lambda(\phi)$. Obviously, if the conjectures we stated in last chapter hold, then $HFK^-_{-g+1}(K, Y) \cong SH(S, \phi)$.

6.2 The general case

In this section we prove theorems 6.1 and 6.3 in the general case.

The first approach that one could attempt to apply Theorem 3.17 to a general link $L \subset Y$ is to look for a contact form on Y that is compatible with L and whose Reeb vector field is circular outside a neighborhood of L. Unfortunately we will not be able to find such a contact form. The basic idea to solve the problem consists in two steps:

Step 1. find a contact form α on Y which is compatible with L and for which there exists a finite decomposition $Y \setminus L = \bigsqcup_i X_i$ for which $R = R_\alpha$ is circular in each X_i;

Step 2. apply Theorem 3.17 separately in each X_i to get the result: this can be done using the (more general results) in Sections 6 of [16].

On the other hand the special decomposition of $Y \setminus L$ that we find in Step 1 will allow us to follow an easier way and we will substitute Step 2 by:

Step 2′. apply repeatedly the *Torres formula* for links to get the result.

Torres formula, first proved in [55], is a classical result about Alexander polynomial, which essentially explains how, starting from the Alexander polynomial of a given link L, to compute the Alexander polynomials of any sub-link of L. We will recall the formula in the next subsection.

6.2.1 Preliminary

The key ingredient to solve the Step 1 of our strategy is the following:

Proposition 6.10. *Let $L = K_1 \sqcup \ldots \sqcup K_n \subset Y$ be an n-components link and let ξ be any fixed contact structure on Y. Then there exists an m-components link $L' \subset Y$ with $m \geq n$ and such that:*

1. *$L' = L \sqcup K_{n+1} \sqcup \ldots \sqcup K_m$;*
2. *L' is fibered and the associated open book decomposition of Y supports ξ.*

This result has been proved in the case of knots by Guyard in his Ph.D. thesis (in preparation, [24]). Using part of his arguments, we give here a proof for the case of links.

Proof. As recalled in Section 1.4, given a contact structure ξ on Y, in [19] Giroux explicitly constructs an open book decomposition of Y that supports a contact form α such that $\ker(\alpha) = \xi$. In the proof of Theorem 1.22 we saw that such an open book decomposition is built starting from a cellular decomposition $\mathcal{D}$ of Y that is compatible with ξ. Moreover we recalled that, up to taking a refinement, any cellular decomposition of Y can be made compatible with ξ by an isotopy.

Using the simplicial approximation theorem, it is possible to choose a triangulation $\mathcal{D}$ of Y in a way that, up to isotopy, L is contained in the 1-skeleton $\mathcal{D}^1$ of $\mathcal{D}$. Up to take a refinement, we can suppose moreover that $\mathcal{D}$ is adapted to ξ.

Let S be the 0-page of the associated open book built via Theorem 1.21: properties 1 and 2 of S reminded during the proof of that theorem, imply that $L \subset \text{int}(S)$ and that, if $\mathcal{N}(\mathcal{D}^0)$ is a suitable neighborhood of $\mathcal{D}^0$, then it is possible to push $L \setminus \mathcal{N}(\mathcal{D}^0)$ inside S to make it contained in ∂S. Note that in each strip composing $S \setminus \mathcal{N}(\mathcal{D}^0)$ we have only one possible choice for the direction in which to push $L \setminus \mathcal{N}(\mathcal{D}^0)$ to ∂S in a way that the orientation of L coincides with that of ∂S.

We would like to extend this isotopy also to $L \cap \mathcal{N}(\mathcal{D}^0)$ to make the whole L contained in ∂S. Suppose that B is a connected component (homeomorphic to a ball) of $\mathcal{N}(\mathcal{D}^0)$. In particular we suppose that $B \cap S$ is connected. Then $L \cap \partial B$ consists of two points Q_1 and Q_2. The extension is done differently in the following two cases (see figure 6.2):

1. *Easy case*: this is when Q_1 and Q_2 belong to the same connected component of $\partial S \cap B$. The isotopy is then extended to B by pushing $L \cap B$ to $\partial S \cap B$ inside $S \cap B$ (figure at left);
2. *General case*: if Q_1 and Q_2 belong to (the boundary of) different connected components a_1 and a_2 of $\partial S \cap B$ we proceed as follows.
   Let P_i be a point in the interior of a_i, $i = 1, 2$. Let γ be a simple arc in $S \cap B$ from P_1 to P_2 (there exists only one choice for γ up to isotopy). Let

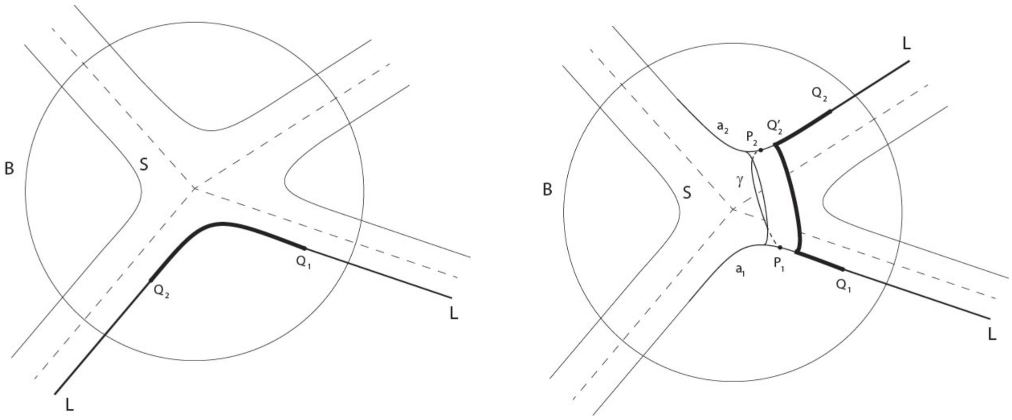

Figure 6.2: Making L contained in ∂S in $\mathcal{N}(\mathcal{D}^0)$: easy case at the left and general case at right. The dotted lines are 1-simplexes in $\mathcal{D}^1$, while the bold segments from Q_1 to Q_2 represent the push-offs of L in $\mathcal{N}(\mathcal{D}^0)$.

S' be obtained by positive Giroux stabilization of S along γ (see figure at the right).
Now we can connect Q_1 with a_2 by an arc in $\partial S'$ crossing once the belt sphere of the 1-handle of the stabilization; let Q'_2 be the end point of this arc. Since a Giroux stabilization is compatible with the orientation of ∂S, Q'_2 and Q_2 are in the same connected component of $a \setminus \{P_2\}$, so that we can connect them inside $\partial S \cap B$ and we are done.

Pushing L to ∂S (and changing L and S as before where necessary) gives a link $\overline{L}$ that is contained in ∂S. To see that $\overline{L}$ is isotopic to L we have to prove that, for any B as before, the two kinds of push-offs we use do no change the isotopy class of L.

Clearly the isotopy class of L is preserved in the easy case. For the general case, it suffices to show that substituting the arc $L \cap S \cap B$ from Q_1 to Q_2 with an arc crossing once the belt sphere of the handle does not change the isotopy class of L. This is equivalent to proving that, if γ is the path of the Giroux stabilization and $\bar{\gamma} = \gamma \cup c$, where c is the core curve of the handle, then $\bar{\gamma}$ bounds a disk in $Y \setminus L$. This can be proved for example by using the particular kind of Heegaard diagrams presented in Section 4.1. Observe that, if b is the co-core of the handle, then $\bar{\gamma}$ is isotopic in S to $b \cup \phi'(b)$, where ϕ' is the monodromy on S' given by the Giroux stabilization. We finish by observing that, up to a small perturbation near ∂S, $b \cup \phi'(b)$ is isotopic to an attaching curve of a Heegaard diagram of Y. $\square$

We recall now the Torres formula that we will use in the second step of our proof of Theorem 6.3. Since we need to consider the Alexander quotient as a polynomial, we will use the same convention adopted for the graded Euler characteristic and we will express the variables as subscripts of the symbol ALEX.

Theorem 6.11 (Torres formula). *Let $L = K_1 \sqcup \ldots \sqcup K_n$ be an n-link in a homology three-sphere Y, K_{n+1} a knot in $Y \setminus L$ and $L' = L \sqcup K_{n+1}$. Let S_i be a Seifert surface for K_i, $i \in \{1, \ldots, n+1\}$. Then*

$$\mathrm{ALEX}_{t_1,\ldots,t_n,1}(Y \setminus L') \doteq \mathrm{ALEX}_{t_1,\ldots,t_n}(Y \setminus L) \cdot \left(1 - \prod_{i=1}^{n} t_i^{\langle K_{n+1}, S_i\rangle}\right),$$

where $\mathrm{ALEX}_{t_1,\ldots,t_n,1}(Y \setminus L')$ indicates the polynomial $\mathrm{ALEX}_{t_1,\ldots,t_{n+1}}(Y \setminus L')$ evaluated in $t_{n+1} = 1$.

We refer the reader to [55] for the original proof. See also [15] for a proof making use of techniques of dynamics. We also mention that in [4] a proof of this theorem is provided making use only of elementary techniques about Seifert surfaces; moreover a generalization of the formula to links in any three-manifold is given in [56].

Sketch of the proof. Apply Theorem 3.17 to $\mathrm{ALEX}(Y \setminus L)$ using a flow ϕ for which

1. K_{n+1} is the only periodic orbit of ϕ contained in a neighborhood of K_{n+1};
2. K_{n+1} is elliptic.

The factor

$$1 - \prod_{i=1}^{n} t_i^{\langle K_{n+1}, S_i\rangle} = \left(\zeta_{K_{n+1}}(\rho_L(K_{n+1}))\right)^{-1} \tag{6.8}$$

expresses then the fact that K_{n+1} is the only orbit counted in $\mathrm{ALEX}(Y \setminus L)$ and not in $\mathrm{ALEX}(Y \setminus L')$.
The condition $t_{n+1} = 1$ comes from the fact that, if μ_{n+1} is a meridian for K_{n+1}, so that $t_{n+1} = [\mu_{n+1}]$, then $\zeta_{\mu_{n+1}}(\rho_L([\mu_{n+1}])) = 1$. □

Observation 6.12. *One can see the condition $t_{n+1} = 1$ also from a purely topological point of view. Image to take the manifold $Y \setminus L'$ and then to glue back K_{n+1}. The effect on $H_1(Y \setminus L')$ is that the generator $[\mu_{n+1}]$ is killed and now the homology class of a loop $\gamma \subset Y \setminus L'$ is determined only by the numbers $\langle \gamma, S_i\rangle$, $S_i \in \{1, \ldots, n\}$ (i.e. by $\rho_L(\gamma)$).*

6.2.2 Proof of the results

Proof of Theorem 6.3. Let $L = K_1 \sqcup \ldots \sqcup K_n$ be a given link in Y. Proposition 6.10 implies that there exists an open book decomposition (L', S, ϕ) of Y with binding

$$L' = L \sqcup K_{n+1} \sqcup \ldots \sqcup K_m$$

for some $m \geq n$. Let α be a contact form on Y adapted to (L', S, ϕ). Let $R = R_\alpha$ be its Reeb vector field. As remarked in Section 6.1, and using the same notations, R is circular in $Y \setminus \mathring{V}'(L')$ where, recall, $V'(L)$ is an union of tubular neighborhoods $V'(K_i) \subsetneq V(K_i)$, $i \in \{1, \ldots, m\}$ of L.

Since α is also adapted to L', then each $\mathring{V}(K_i)$ is, by definition, foliated by concentric tori, which in turn are linearly foliated by Reeb orbits that intersect positively a meridian disk for K_i in $V(K_i)$. Now, we can choose α in a way that for each $i \in \{n+1, \ldots, m\}$ the tori contained in $V'(K_i)$ are foliated by orbits of R with fixed irrational slope. This condition can be achieved by applying the Darboux-Weinstein theorem in $V(K_i)$ to make $\alpha|_{V'(K_i)}$ like in Example 6.2.3 of [9].

This implies that, for each $i \in \{n+1, \ldots, m\}$, the only closed orbit of R in $V'(K_i)$ is K_i. Define $U(L') = \bigsqcup_{i=1}^m U(K_i)$, where

$$U(K_i) = \begin{cases} V(K_i) & \text{if} \quad i \in \{1, \ldots, n\}; \\ V'(K_i) & \text{if} \quad i \in \{n+1, \ldots, m\}; \end{cases}$$

We have:

$$\begin{aligned}
\chi(ECC(L, Y, \alpha)) &= \zeta_{\rho_L}(\phi_R|_{Y \setminus V(L)}) \\
&= \zeta_{\rho_L}(\phi_R|_{Y \setminus U(L')})) \cdot \prod_{i=n+1}^{m} \prod_{\gamma \in \mathcal{P}(V'(K_i))} \zeta_\gamma(\rho_L([\gamma])) \\
&= \zeta_{\rho_L}(\phi_R|_{Y \setminus U(L')})) \cdot \prod_{i=n+1}^{m} \zeta_{K_i}(\rho_L([K_i])) \\
&= \zeta_{\rho_{L'}}(\phi_R|_{Y \setminus U(L')}))|_{t_1, \ldots, t_n, 1, \ldots, 1} \cdot \prod_{i=n+1}^{m} \zeta_{K_i}(\rho_L([K_i])) \\
&\doteq \mathrm{ALEX}_{t_1, \ldots, t_n, 1, \ldots, 1}(Y \setminus L') \cdot \prod_{i=n+1}^{m} \zeta_{K_i}(\rho_L([K_i])) \\
&= \mathrm{ALEX}_{t_1, \ldots, t_n, 1, \ldots, 1}(Y \setminus L') \cdot \prod_{i=n+1}^{m} \left(1 - \prod_{j=1}^{n} t_j^{\langle K_i, S_j \rangle}\right)^{-1} \\
&= \mathrm{ALEX}_{t_1, \ldots, t_n}(Y \setminus L),
\end{aligned}$$

where:

lines 1 and 2 follow reasoning like in the proof of Equation 6.6;

line 3 hold since K_i, for $i \in \{n+1, \ldots, m\}$, is the only Reeb orbit of α in $V'(K_i)$;

line 4 comes from the idea in Observation 6.12: ρ_L and $\rho_{L'}$ coincide on the generators t_i of $H_1(Y \setminus L)$ for $i \in \{1, \ldots, n\}$ and $t_i = [\mu_i] = 1 \in H_1(Y \setminus L)$ for $i \in \{n+1, \ldots, m\}$;

line 5 $\rho_{L'}$ and $R|_{Y \setminus U(L')}$ satisfy hypothesis of Theorem 3.17, up to slightly perturb R near $\partial U(K_i)$, $i \in \{1, \ldots, n\}$, to make it non degenerate transverse to the boundary like in the proof in Section 6.1;

line 6 is due to the fact that the K_i's are elliptic;

line 7 is obtained by applying repeatedly the Torres formula on the components $K_{n+1}, \ldots, K_m$.

□

Observation 6.13. *As mentioned at the beginning of this section, we could also apply the more general results in [16] using the fact that the Reeb vector field R used is circular in each $U(K_i)$, since here R is positively transverse to any meridian disk of K_i in $\mathring{V}(K_i)$.*

The proof of Theorem 6.1 works exactly like in the fibered case.

7

A filtered isomorphism

In this chapter we start to investigate the relations between ECK and HFK at the homology level. In light of the proof of the equivalence between ECH and HF given by Colin, Ghiggini and Honda, it is natural to start with the fibered knots.

Let (K, S, ϕ) be an open book decomposition of a three manifold Y, α an adapted contact form and N the mapping torus of (S, ϕ). Let $(\widehat{CF}(S, \phi), \partial^{HF})$ be a chain complex for $\widehat{HF}(Y)$ as defined in Section 4.1. We ask then how the chain map

$$\Phi\colon \widehat{CF}(S, \phi) \longrightarrow PFC_{\leq 2g}(N, \partial N)$$

recalled in Section 4.3, and inducing an isomorphism in homology, behaves with respect to the filtrations induced by K on the chain complexes. The main result of this chapter (see Theorem 7.29) is

Theorem 7.1. *For a suitable choice of the Hamiltonian structure, the chain map Φ respects the knot filtrations.*

The proof of this theorem is carried on in several steps. The main problem to face is that essentially the differentials of $\widehat{CF}(S, \phi)$ and $PFC_{\leq 2g}(N, \partial N)$ are both defined in the complement of a neighborhood of K in Y. The same thing holds also for Φ: this implies that none of the holomorphic curves counted by these maps crosses K, and so we cannot directly apply the methods used to define the knot filtrations in HF and ECH (see for example Proposition 2.17). To solve the problem we modify ϕ by an isotopy into another monodromy $\widetilde{\phi}$ that will allow us to "see" the knot filtrations in the complement of K. We do this in Section 7.1 on the ECH-PFH side and in Section 7.2 on the HF side. In

Section 7.3 we will be finally ready to study the holomorphic curves counted by Φ with respect to the knot filtrations and we will prove the theorem above.

7.1 Adapting the monodromy to the binding

From now on (Y, ξ) will be a contact 3-manifold and $K \subset Y$ an oriented fibered knot which is the binding of the open book decomposition (S, ϕ) of Y compatible with ξ, where S is of fixed genus g. Moreover we will assume all the conventions given in Chapters 1-4. In particular we use the same conventions for the parametrization near the neighborhood $\mathcal{N}(K)$.

7.1.1 Monodromy and contact form near the binding

In this subsection we want to slightly modify the monodromy ϕ and the associated contact form α near ∂S. Before doing this we recall the construction of a family of contact forms in $[a, b] \times T^2$ given in [9, Section 6], where $a < b$ are real numbers.

Parametrize $[a, b] \times S^1 \times \frac{[0,2]}{0\sim 2}$ by (y, ϑ, t) and consider the 1-form

$$\alpha = cyd\vartheta + (C - cF(y))dt \tag{7.1}$$

depending on $c, C \in \mathbb{R}$ and $F : [a, b] \to \mathbb{R}$ smooth. Then

$$d\alpha = cdy \wedge d\vartheta - cF'(y)dy \wedge dt$$

and, taking $C \gg c > 0$, the 3-form

$$\alpha \wedge d\alpha = c(C - cF(y) + cyF'(y))dy \wedge d\vartheta \wedge dt$$

is a positive volume form and so α is contact. By simple calculations we find a basis for the associated contact structure:

$$\xi_\alpha = \ker(\alpha) = \langle \partial_y, (C - cF(y))\partial_\vartheta - cy\partial_t \rangle$$

and for the Reeb vector field

$$R_\alpha = \frac{1}{P_\alpha(y)}(F'(y)\partial_\vartheta + \partial_t)$$

with $P_\alpha(y) = \alpha(F'(y)\partial_\vartheta + \partial_t) = C + c(yF'(y) - F(y))$, which is always positive since $C \gg c > 0$.

Call $f(y) = F'(y)$. All the tori $T_y = \{y\} \times S^1 \times \frac{[0,2]}{0\sim 2}$ are foliated by orbits of R_α and the first return map of R_α is

$$(y, \vartheta) \mapsto (y, \vartheta + f(y)). \tag{7.2}$$

If T_y is linearly foliated by closed orbits, define the *degree* $\deg(T_y) \in \mathbb{N}^* \cup \{\infty\}$ *of the torus* T_y to be the degree of any of the orbits foliating T_y. The differential of the first return map of the flow of R_α on $\xi_{(y,\vartheta)}$ is

$$\begin{pmatrix} 1 & 0 \\ \frac{\deg(T_y) f'(y)}{P_\alpha(y)} & 1 \end{pmatrix} \tag{7.3}$$

(cf. Section 1.3). Since $\deg(T_y), P_\alpha(y) \gneq 0$, the torus T_y is M-B negative if $f'(y) < 0$ and positive if $f'(y) > 0$.

Example 7.2. *The restriction to* $[0,3) \times S^1 \times \frac{[0,2]}{0\sim 2}$ *of the contact form on* $Y \setminus K$ *used in Section 1.4 can be obtained as above by using the function* $f_\delta : [0,3) \to \mathbb{R}$.

Let $\widetilde{S}$ be the surface obtained by removing $[2,3) \times S^1$ from the extended page and then gluing a copy of $[2,4] \times S^1$ along $\{2\} \times S^1$. Now extend $f_\delta|_{[0,2]}$ to a new smooth function $f_{\delta,\varepsilon} : [0,4] \to \mathbb{R}$ depending on a new constant $0 < \varepsilon \ll 2\pi$ and such that:

1. $f_{\delta,\varepsilon}$ coincides with f_δ in $[0,2]$;
2. $f_{\delta,\varepsilon}$ has maximum in $y = 3$ of value ε;
3. $f_{\delta,\varepsilon}(4) = 0$;
4. $df_{\delta,\varepsilon}(y) > 0$ in $[2,3)$ and $df_{\delta,\varepsilon}(y) < 0$ in $(3,4]$;
5. $f_{\delta,\varepsilon} = -y + 4$ in a small neighborhood of $\{y = 4\}$.

See Figure 7.1. Call $\widetilde{\phi}_{\delta,\varepsilon}$ the monodromy on $\widetilde{S}$ obtained by extending ϕ from S to $\widetilde{S}$ via the diffeomorphism given by equation 7.2 using $f_{\delta,\varepsilon}$.

Let $\widetilde{N}$ be the mapping torus of $(\widetilde{S}, \widetilde{\phi})$. Moreover let $\widetilde{\alpha}_{\delta,\varepsilon}$ be a contact form on $\widetilde{N}$ obtained by extending α_δ to $[0,4] \times S^1 \times \frac{[0,2]}{0\sim 2}$ using the equation 7.1 with F primitive of $f_{\delta,\varepsilon}$. When we are not interested in a particular choice of (δ, ε) we will write simply $\widetilde{\phi}$ and $\widetilde{\alpha}$.

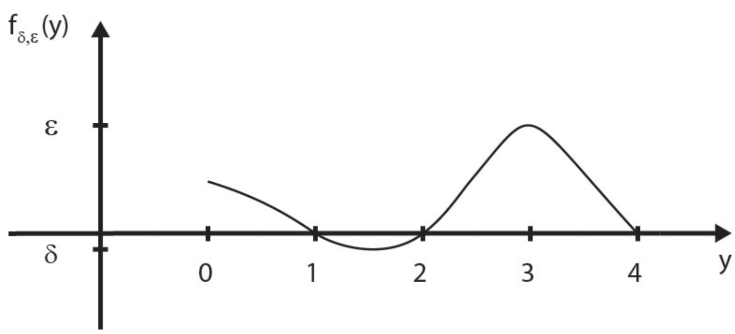

Figure 7.1: The function $f_{\delta,\varepsilon}$ on $[0,4]$.

Observation 7.3. *In the region $[2,4] \times S^1 \times \frac{[0,2]}{\sim}$, the torus T_y is a positive M-B for $y \in [2,3)$ and negative for $y \in (3,4]$. In particular T_4 is a negative M.B. torus foliated by Reeb orbits with slope ∞, exactly like T_1. Moreover the condition 5 above implies that the behavior of the monodromy and the contact form near $\partial \widetilde{N}$ is the same as near ∂N. In fact we can observe that $\widetilde{N}$ is diffeomorphic to N and that we can see $\widetilde{\phi}$ as a slight modification of ϕ by an isotopy which is the identity outside a neighborhood ($\{y \in (1,4]\}$) of the boundary and inside a smallest neighborhood of the boundary (where the monodromy is given by condition 5). Actually we can consider $\widetilde{\phi}$ just to be a modification of ϕ, obtained by composing the latter with a small "finger move" along $\{y = 1,5\}$ in the positive K-direction and another finger move along $\{y = 3\}$ in the opposite direction. It will be useful to keep in mind both interpretations of $\widetilde{N}$.*

Throughout the rest of the chapter we will always assume that $\delta, \varepsilon \ll \frac{2\pi}{2g}$: this will guarantee that, for any $y \in [0,4]$, $\deg(T_y) \gg 2g$ except for the tori T_1, T_2 or T_4, which have degree 1.

If $(\widetilde{S}, \widetilde{\phi})$ is obtained as above we say that it is *adapted to the binding*.

7.1.2 $\widehat{\text{ECH}}$ for open books adapted to the binding

In this subsection we analyze the embedded contact homology of $(\widetilde{N}, \widetilde{\alpha})$. Let $ECH_{\leq 2g}(\widetilde{N}, \widetilde{\alpha})$ be the homology of the chain complex $ECC_{\leq 2g}(\widetilde{N}, \widetilde{\alpha})$ with the ECH boundary (cf. subsection 2.2.1).

Doing the M-B modification on the tori T_1, T_2 or T_4 we get three couples of degree 1 Reeb orbits:

$$\begin{array}{rcl} (e,h) & \subset & T_1; \\ (e_+,h_+) & \subset & T_2; \\ (e_-,h_-) & \subset & T_4. \end{array}$$

These are the only simple Reeb orbits in $\widetilde{N} \cap \{y \in [0,4]\}$ with degree less or equal then $2g$ and their C-Z indices are $\mu(e) = \mu(e_-) = -1$ and $\mu(e_+) = 1$ for the elliptic orbits and $\mu(h) = \mu(h_+) = \mu(h_-) = 0$ for the hyperbolic orbits. Moreover we have the immediate identification:

$$ECH_{\leq 2g}(\widetilde{N}, \widetilde{\alpha}) \equiv ECH^{e_+,h_+,e_-,h_-}_{\leq 2g}(N, \widetilde{\alpha}). \tag{7.4}$$

Observation 7.4. *We remark that the two orbits e_+ and h_+ are not the orbits with the same name in last chapters. On the other hand we decided to call them in the same way because they will play a somewhat analogue role in the definition of the knot filtration.*

By Observation 7.3 we can see $e_- \subset \partial \widetilde{N}$ as the analogue of $e \subset \partial N$, so that, by equation 2.12, we have

$$\widehat{ECH}(\widetilde{N}, \partial \widetilde{N}, \widetilde{\alpha}) \cong \frac{ECH_{\leq 2g}(\widetilde{N}, \widetilde{\alpha})}{e_-\gamma \sim \gamma}.$$

Since embedded contact homology does not change under isotopy of the monodromy, we have

Corollary 7.5. $\widehat{ECH}(\widetilde{N}, \partial\widetilde{N}, \widetilde{\alpha}) \cong \widehat{ECH}(N, \partial N, \alpha)$.

For what follows it is convenient to construct a specific isomorphism between the two homology groups.

Proposition 7.6. *The ECH boundary map on $\{e, h, e_+, h_+, e_-, h_-\}$ acts as follows:*

$$\begin{aligned} \partial e &= \partial e_- = 0 \\ \partial h &= \partial h_- = 0 \\ \partial h_+ &= e_- + e \\ \partial e_+ &= h_- + h \end{aligned} \tag{7.5}$$

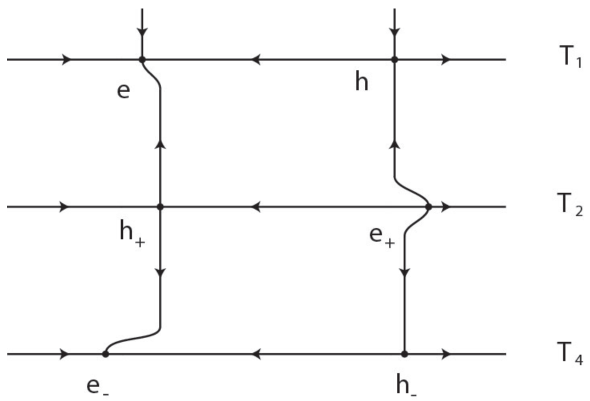

Figure 7.2: Relevant orbits and holomorphic curves in $\widetilde{N} \cap \{y \in [0, 4]\}$ (cf. Figure 2.1).

The proof of Proposition 7.6 is similar to that of Equations 2.6 and we discuss it in the next subsection with some critical details which will be useful later. For the moment we focus on the consequences.

Any equivalence class of orbit sets γ in $\widetilde{N}$ under the relation $e_-\gamma \sim \gamma$ is determined by its element in which e_- has exponent 0. If $\gamma \in \mathcal{O}(\widetilde{N})$, the equivalence class $[\gamma] \in \frac{ECC_{\leq 2g}(\widetilde{N})}{e_-\gamma \sim \gamma}$ will be often denoted simply by γ.

Corollary 7.7. *The ECH differential ∂ on $\widehat{ECH}(\widetilde{N}, \partial\widetilde{N})$ is given by:*

$$\partial(e_+^a h_-^b h_+^c \gamma) = e_+^{a-1} h_-^b (h_- + h) h_+^c \gamma + e_+^a h_-^b h_+^{c-1}(e_- + e)\gamma + e_+^a h_-^b h_+^c \partial\gamma,$$

where the terms are understood to be 0 if they contain factors with negative exponent or hyperbolic orbits with total multiplicity grater then 1.

Proof. Any orbit in $\{y \in ([1,4] \setminus \{1,2,4\})\} \subset \widetilde{N}$ has degree grater then $2g$ and so it does not contribute to the generators of $ECC_{\leq 2g}(\widetilde{N})$. This fact and the Blocking and Trapping lemmas prevent holomorphic curves from crossing T_i along curves with slope ∞, for $i \in \{1,2,4\}$. Moreover applying homological arguments (like in the proof of Lemma 9.5.1 of [9] or in Lemma 7.28 below) on holomorphic curves crossing some T_i along different slope, one can see that these curves would not be contained in $\widetilde{N}$. This implies in particular that our contact form is *nice*, i.e. that any ECH-index 1 M.B. building has at most one connected component of ECH-index grater then 0 (see Definition 4.4.1 and Corollary 9.5.2 in [9]). The result then follows applying Proposition 7.6. $\square$

Observation 7.8. *Note in particular that if* $\gamma \in \mathcal{O}(N)$*, then* $\deg(\partial\gamma) = \deg(\gamma)$.

In the rest of this subsection it will be more convenient to write (equivalence classes of) orbit sets in $\widetilde{N}$ in the form $e_+^a h_-^b \gamma$, with now $\gamma \in \mathcal{O}(N \cup \{h_+\})$ and, again, a, b natural numbers with $b \in \{0,1\}$.

Note now that $\left(ECC_{\leq 2g}^{h_+,e_-}(N,\widetilde{\alpha}), \partial\right)$ is a sub-complex of $\left(ECC_{\leq 2g}(\widetilde{N},\alpha), \partial\right)$.

Lemma 7.9. *Consider the chain map*

$$i : ECC_{\leq 2g}^{h_+,e_-}(N,\widetilde{\alpha}) \longrightarrow ECC_{\leq 2g}(\widetilde{N},\widetilde{\alpha})$$

defined on the generators by

$$i(\gamma) = \gamma.$$

Then i is a homotopy equivalence with homotopy inverse

$$\pi : ECC_{\leq 2g}(\widetilde{N},\widetilde{\alpha}) \longrightarrow ECC_{\leq 2g}^{h_+,e_-}(N,\widetilde{\alpha})$$

defined on the generators by

$$\pi(e_+^a h_-^b \gamma) = \begin{cases} 0 & \textit{if} \quad a > 0 \\ h^b\gamma & \textit{if} \quad a = 0 \end{cases}.$$

Proof. It is evident that $\pi \circ i$ is the identity map on $ECC_{\leq 2g}^{h_+,e_-}(N,\widetilde{\alpha})$. It remains to prove that $i \circ \pi$ is homotopy equivalent to the identity on $ECC_{\leq 2g}(\widetilde{N},\widetilde{\alpha})$, that we will call simply id. Let

$$H : ECC_{\leq 2g}(\widetilde{N},\widetilde{\alpha}) \longrightarrow ECC_{\leq 2g}(\widetilde{N},\widetilde{\alpha})$$

be the map defined on the generators by

$$H(e_+^a h_-^b \gamma) = \begin{cases} 0 & \text{if} \quad b = 0 \\ e_+^{a+1}\gamma & \text{if} \quad b = 1 \end{cases}.$$

We want to prove that H is a homotopy between i and π, i.e.

$$(\partial \circ H + H \circ \partial)|_{\ker\partial} = (id + i \circ \pi)|_{\ker\partial} \tag{7.6}$$

We check relation 7.6 for $a > 0$ and leave to the reader the similar calculation for $a = 0$.

a > 0 and b = 0

$$\begin{aligned}(\partial \circ H + H \circ \partial)(e_+^a \gamma) &= H\big(e_+^{a-1}(h+h_-)\gamma + e_+^a \partial\gamma\big)\\ &= e_+^a \gamma\end{aligned}$$

$$(id + i \circ \pi)(e_+^a \gamma) = e_+^a \gamma$$

a > 0 and b = 1

$$\begin{aligned}(\partial \circ H + H \circ \partial)(e_+^a h_- \gamma) &= \partial(e_+^{a+1}\gamma) + H\big(e_+^{a-1}hh_-\gamma + e_+^a h_- \partial\gamma\big)\\ &= e_+^a(h+h_-)\gamma + e_+^{a+1}\partial\gamma + e_+^a h\gamma + e_+^{a+1}\partial\gamma\\ &= e_+^a h_- \gamma\end{aligned}$$

$$(id + i \circ \pi)(e_+^a h_- \gamma) = e_+^a h_- \gamma$$

□

7.1.3 Holomorphic curves near ∂N

This subsection is devoted to the proof of Proposition 7.6. The key ingredients are results in [60, Section 4.2] and [59, Chapter 3]: we can summarize what we need as follows.

Consider $[a, b] \times T^2$ with $0 < a < b$, parametrized by (y, ϑ, t) as in subsection 7.1.1 and endowed with a contact form

$$\lambda = g(y)d\vartheta + h(y)dt. \tag{7.7}$$

The contact condition is $g'h - gh' > 0$ and the associated Reeb vector field is

$$R_\lambda = \frac{1}{g'h - gh'}(g'\partial_t - h'\partial_\vartheta).$$

Then any torus $\{y_0\} \times T^2$ is foliated by Reeb orbits. Suppose that:

1. g' is a positive function;
2. $g(a), g(b) \neq 0$;
3. $h'(a) = h'(b) = 0$ and $h'(y) \neq 0 \ \forall y \in (a, b)$.

These imply that T_a and T_b are the only tori foliated by Reeb orbits with slope $\frac{g'}{2\pi h'} = \infty$.

Let C_{ϑ_0} be the cylinder $\{\vartheta = \vartheta_0\} \subset [a, b] \times T^2$. For the following two lemmas see [59, Chapter 3].

Lemma 7.10. *There exists a family of finite energy holomorphic cylinders*

$$\{Z_{s,\vartheta}\}_{s\in\mathbb{R},\vartheta\in S^1}$$

that foliate $\mathbb{R}\times[a,b]\times T^2$ and such that, if $\pi_{\mathbb{R}}:\mathbb{R}\times[a,b]\times T^2\longrightarrow[a,b]\times T^2$ is the projection along the $\mathbb{R}$ direction, then $\forall(s_0,\vartheta_0)$

$$\pi_{\mathbb{R}}(Z_{s_0,\vartheta_0})=\mathrm{int}(C_{\vartheta_0}) \tag{7.8}$$

and

$$\forall s_1\in\mathbb{R},\ Z_{s_0+s_1,\vartheta_0} \textit{ is the } s_1\textit{-translate of } Z_{s_0,\vartheta_0}. \tag{7.9}$$

Moreover:

1. *if $h''(a)<0$ and $h''(b)>0$, each $Z_{s,\vartheta}$ is positively asymptotic to T_a and negatively asymptotic to T_b;*
2. *if $h''(a)>0$ and $h''(b)<0$, each $Z_{s,\vartheta}$ is positively asymptotic to T_b and negatively asymptotic to T_a.*

The tori T_a and T_b are M-B and their perturbation behaves nicely with respect to the foliation, in the following sense (see [59, 3.3]). Suppose we are in the case 1 of Lemma above. Then T_a is a positive M-B torus and the corresponding S^1-family of Reeb orbits is perturbed into the couple (e_a,h_a); similarly a perturbation near the negative M-B torus T_b produces a couple of orbits (e_b,h_b). Here $\mu_{CZ}(e_a)=1$, $\mu_{CZ}(h_a)=\mu_{CZ}(h_b)=0$ and $\mu_{CZ}(e_b)=-1$.

Lemma 7.11. *The perturbation near T_a and T_b induces a perturbation of the family $\{Z_{s,\vartheta}\}_{s\in\mathbb{R},\vartheta\in S^1}$ which gives: a Fredholm index 1 holomorphic cylinder from e_a to h_b; another index 1 holomorphic cylinder from h_a to e_b; an infinite family of holomorphic cylinders of index 2 from e_a to e_b.*

The case corresponding to situation 2 of Lemma 7.10 is analogue: T_a is now negative M.B. and T_b is positive and the Conley-Zehnder indices of (e_a,h_a) and (e_b,h_b) are exchanged, as the orientation of the holomorphic cylinders.

A priori there could be other holomorphic curves with image in $[a,b]\times T^2$; under some conditions on the curves this can not happen.

Lemma 7.12 (Lemma 8.4.8, [9])**.** *Suppose we are in the situation 1 (resp. 2) of Lemma 7.10. Let $u:(F,j)\to(\mathbb{R}\times[a,b]\times T^2,J)$ be a holomorphic curve with some ends in $\{b\}\times T^2$ (resp. $\{a\}\times T^2$) and without ends in $(a,b)\times T^2$; then $u(F)$ is equal to $Z_{s,\vartheta}$ for some (s,ϑ).*

Proof. Suppose first that $\pi_{\mathbb{R}}(u(F))\not\subseteq C_\vartheta$ for any ϑ. Then there exists ϑ_0 such that u does not have any end contained in C_{ϑ_0} or crossing it. Let $L:=$

$\pi_{\mathbb{R}}(u(F)) \pitchfork int(C_{\vartheta_0}) \neq \emptyset$. L is a closed 1-dimensional immersed submanifold of C_{ϑ_0} and $\pi_{\mathbb{R}}^{-1}(L)$ is a compact submanifold of $u(F)$ and this implies that

$$\text{for } s \text{ large enough } Z_{s,\vartheta_0} \cap u(F) = \emptyset. \tag{7.10}$$

On the other hand, since $L \neq \emptyset$, $\exists s_0 \in \mathbb{R}$ such that $u(F)$ intersects nontrivially Z_{s_0,ϑ_0} and the intersection is strictly positive by holomorphicity. By property 7.9 it follows that

$$\forall s_1 > 0 \, \#(u \pitchfork Z_{s_0+s_1,\vartheta_0}) = \#(u \pitchfork Z_{s_0,\vartheta_0}) > 0,$$

which contradicts condition 7.10.

Suppose now that $\pi_{\mathbb{R}}(u(F)) \subseteq C_{\vartheta_0}$ for some ϑ_0. The 3-dimensional submanifold $\mathbb{R} \times C_{\vartheta_0}$ of $\mathbb{R} \times [a,b] \times T^2$ is foliated by $\{Z_{s,\vartheta_0}\}_s$ and must contain $u(F)$. If $u(F) \neq Z_{s,\vartheta_0}$ for any s then there exists s_0 such that $u(f)$ intersects transversely and non-trivially Z_{s_0,ϑ_0}. The dimension of this intersection must be 1, which is absurd since $u(F)$ and Z_{s_0,ϑ_0} are both holomorphic. □

Proof of Proposition 7.6. The contact form $\widetilde{\alpha}$ on $[1,4] \times T^2$ satisfies the conditions at the beginning of this subsection. By the Blocking Lemma and the fact that all the orbits in $[1,4] \times T^2 \setminus (T_1 \cup T_2 \cup T_4)$ have degree grater than $2g$, no curve counted by ∂^{ECH} can cross T_i, $i \in \{1,2,4\}$.

Lemma 7.11 corresponding to case 1 of Lemma 7.10 applied to $[2,4] \times T^2$ implies that there exist two Fredholm index 1 families of cylinders, one from h_+ to e_- and one from e_+ to h_-; these cylinders are embedded and the ECH index inequality (see [27]) implies that also their ECH index is 1. For the same reason all the other cylinders of the foliation are of ECH index 2 and are not counted by ∂.

Similarly, applying the lemmas 7.10 (case 2) and 7.11 to $[1,2] \times T^2$ we see that there exist other two ECH-index 1 cylinder, one from h_+ to e and one from e_+ to h, and no other cylinder of the foliation is counted by ∂. By Lemma 7.12, since none of the curves can cross T_4, these four cylinders are the only curves counted by ∂^{ECH} and the result follows. □

7.1.4 The knot filtration inside the open book

In order to drop the $\widetilde{xx}$ in the notations, from now on we will assume that an open book decomposition comes already adapted to its binding, i.e. it is already like $(\widetilde{S}, \widetilde{\phi})$ in last subsections. More precisely:

Definition 7.13. *Let (S,ϕ) be an open book decomposition. If there exists a neighborhood $[0,4] \times \partial S$ of ∂S in S where ϕ can be expressed in terms of a function $f_{\delta,\varepsilon}$ as above, we say that (S,ϕ) is* adapted to the binding. *Moreover the mapping torus of the neighborhood $[0,4] \times \partial S$ will be called* adaptation neighborhood.

Notation. *Let $\partial S \times [0,4]$ be the neighborhood (of the adaptation) of ∂S in S as in last sections. Given $y \in [0,4]$, we will call S_y the surface $S \setminus (y,4]$. Moreover let N_y denote the mapping torus of $(S_y, \phi|_{S_y})$, so that $\partial N_y = T_y$. In particular now $N = N(S,\phi) = N_4$ is the whole mapping torus (containing the adaptation neighborhood too).*

Let (S,ϕ) be an open book decomposition of Y whose monodromy is adapted to the binding K. In this section we explain how to define $\widehat{ECK}(K,Y,\alpha)$ in N. This is obtained essentially by identifying the orbits e_+ and h_+ with their counterpars in the definition of $\widehat{ECK}(K,Y,\alpha)$ that we gave in Section 2.4. By definition of the knot filtration, the Alexander degree will be given by the degree of the orbits as in Definition 2.11.

Consider

$$ECC^{h_+,e_-}(N_1,\partial N,\alpha) := \frac{ECC^{h_+,e_-}(N_1,\alpha)}{e_-\gamma \sim \gamma}.$$

Endow it with the restriction of the differential ∂^{ECH} given by Corollary 7.7: it is easy to see that this is a chain complex whose homology $ECH^{h_+,e_-}(N_1,\partial N,\alpha)$ is isomorphic to $\widehat{ECH}(N,\partial N,\alpha)$.

The isomorphism is obtained by using Lemma 7.9 and Theorem 2.7, identifying e_- with the orbit $\emptyset$: this makes sense in light of the quotient by the equivalence relation $e_-\gamma \sim \gamma$.

Using the second interpretation it is easy to see that, if ∂^{ECK} is the differential on $ECC^{h_+,e_-}(N_1,\partial N,\alpha)$ defined by

$$\partial^{ECK}(h_+^a\gamma) = h^{a-1}e\gamma + h^a\partial^{ECH}\gamma, \tag{7.11}$$

then

$$H\left(ECC^{h_+,e_-}(N_1,\partial N,\alpha),\partial^{ECK}\right) \cong \widehat{ECK}(K,Y,\alpha) \tag{7.12}$$

Similarly to what done for closed manifolds in Section 2.3, we can switch from contact structures and Reeb orbits to stable Hamiltonian structures and periodic orbits. Since the definitions are the same of the case of ECH we avoid the details. If $PFC^{h_+,e_-}(N_1)$ and $PFC^{h_+,e_-}(N_1,\partial N)$ are defined in the same way as their counterparts in ECH and the differential ∂^{PFK} is defined by an expression analogue to Equation 7.11, then the *hat version of the periodic Floer knot homology of the open book* (K,S,ϕ) is:

$$\widehat{PFK}(K,S,\phi) := H\left(PFC^{h_+,e_-}(N_1,\partial N),\partial^{PFK}\right).$$

Obviously this homology is defined only for fibered knots and indeed it should be considered as an invariant of the couple (S,ϕ). In order to simplify the notations we are avoiding to refer to the stable Hamiltonian structure in the formulas.

Observation 7.14. *Note that the quotient by the relation $e_-\gamma \sim \gamma$ is not compatible with the degree defined in Subsection 2.2.1, since $\deg(e_-\gamma) = \deg(\gamma) + 1$. However, since the orbit e_- can be interpreted formally as the empty orbit, the most reasonable definition for the degree of an orbit set γ is*

$$\deg(\gamma) = \min_{\delta \in [\gamma]_\sim} \{\deg(\delta)\}.$$

This is obviously the degree of the (unique) orbit set in $[\gamma]_\sim$ that belongs to $\mathcal{O}(N_3)$ and

$$\deg(\gamma) = \langle \gamma, S' \rangle. \tag{7.13}$$

Note that for every k, $[e_-^k]_\sim = [\emptyset]_\sim$ and, with the last definition of $\deg$, these are the only orbits with degree 0.

7.2 $\widehat{\text{HFK}}$ on a page

In Section 7.1 we modified the monodromy of an open book in order to adapt it to the binding K: this allowed us to "see" the filtration given by K inside the mapping torus of the open book (and then in the codomain of the chain map Φ).

In this section we want to do something similar in spirit for Heegaard Floer homology. The classical definition of the knot filtration for HF should require a stabilization of the Heegaard diagram, as done for example in [25, Section 3.2]: we want to avoid this stabilization in order to get a diagram compatible with K and contained in a page of the open book.

Remember the construction of an Heegaard diagrams adapted to an open book decomposition (K, S, ϕ) of Y given in Section 4.1. Without loss of generality we can assume that near ∂S the curves a_i are such that

$$\begin{aligned} a_i \cap \partial S \times [0,1] &= (\{r(2i-1)\} \cup \{r(2i+1)\}) \times [0,1] \text{ if } i \text{ odd}, \\ a_i \cap \partial S \times [0,1] &= (\{r(2i-2)\} \cup \{r(2i)\}) \times [0,1] \text{ if } i \text{ even}, \end{aligned} \tag{7.14}$$

where r is a real number such that, for a fixed metric for which ∂S has length 2π, $0 < r << \frac{\pi}{2g}$ where g is the genus of S (see figure 7.3).

Note in particular that, for any i, $a_i \cap \partial S \times [0,1]$ is an union of two arcs a_i^- and a_i^+ where $a_i^- = rl \times [0,1]$ and $a_i^+ = r(l+2) \times [0,1]$ for some integer l.

Suppose that (S, ϕ) is adapted to K in the sense of Definition 7.13: in particular $S^1 \times [0,4]$ is a neighborhood of ∂S, $S' = S_3$ and ϕ depends on a $f_{\delta,\varepsilon}$ as in 7.1.1.

Fixed δ and ε, choose r such that $4gr < \varepsilon$ and the diagram $(S, \mathbf{a}, \phi(\mathbf{a}))$ near ∂S is like in figure 7.4 (cf. figure 7.1).

Note in particular that ϕ acts on the arcs $a_i \cap \{y \in [2,4]\}$ as a finger move of length ε in the direction of K: this creates a new bigon in $S \setminus (\mathbf{a} \cup \phi(\mathbf{a}))$ in which we pick a marked point z. If we choose δ rational, the orbit of z under

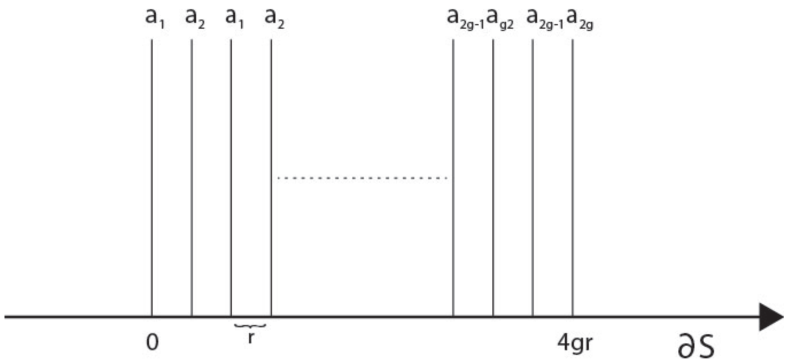

Figure 7.3: The curves a_i near ∂S.

the action of ϕ consists of $\deg(T_3) \gg 2g$ points (see 7.1.1) and z is the only one of them belonging to the bigon, while all the others belong to the region of the diagram where we put the marked point w. In particular we can assume that $w = \phi(z)$.

Definition 7.15. *Let* $\mathbf{x} = (x_1, \dots, x_{2g})$ *be a generator of* $\widehat{CF}(S, \mathbf{a}, \phi(\mathbf{a}))$. *Define the* degree *of* $\mathbf{x}$ *by*

$$\deg(\mathbf{x}) := \langle \mathbf{x}, S' \rangle = \#\{j \mid x_j \in S'\}.$$

Observation 7.16. *Note that last definition can be seen as an analogue for set of chords of the Equation 7.13 for orbit sets.*

Observation 7.17 (Orientations)**.** *We remark that* (K, S, ϕ) *is given as open book decomposition of* Y*; since we are working on* $-Y$*,* K *is now oriented as the boundary of* S *in* $-Y$*. Since the arc connecting* z *to* w *in* $\Sigma \setminus \mathbf{a}$ *is oriented as* ∂S*, the conventions about the role of* z *and* w *in the orientations given in subsection 3.2.1 imply then that* $\widehat{CF}(S, \mathbf{a}, \phi(\mathbf{a}), z, w)$ *is a Heegaard diagram for* $(-Y, -K)$.

Recall the definition of the cobordism W given in Section 4.2. Define moreover $K' = \{y = 3\} = \partial S'$; note that K' is a closed curve on S isotopic to K.

The proof of the following is analogue to that of Lemma 2.17, avoiding the considerations about the signs of the intersections.

Lemma 7.18. *Let* $u : F \to W$ *be a degree* $k \leq 2g$ *holomorphic multisection of* W *with positive end* $\mathbf{x}$ *and negative end* $\mathbf{y}$*. Then for any* $t_0 \in (0, 1)$*:*

$$\deg(\mathbf{x}) - \deg(\mathbf{y}) = \langle u, \mathbb{R} \times \{t_0\} \times K' \rangle.$$

We want to prove that this degree coincides, up to a shift, with the Alexander degree defined using the $Spin^c$-structures.

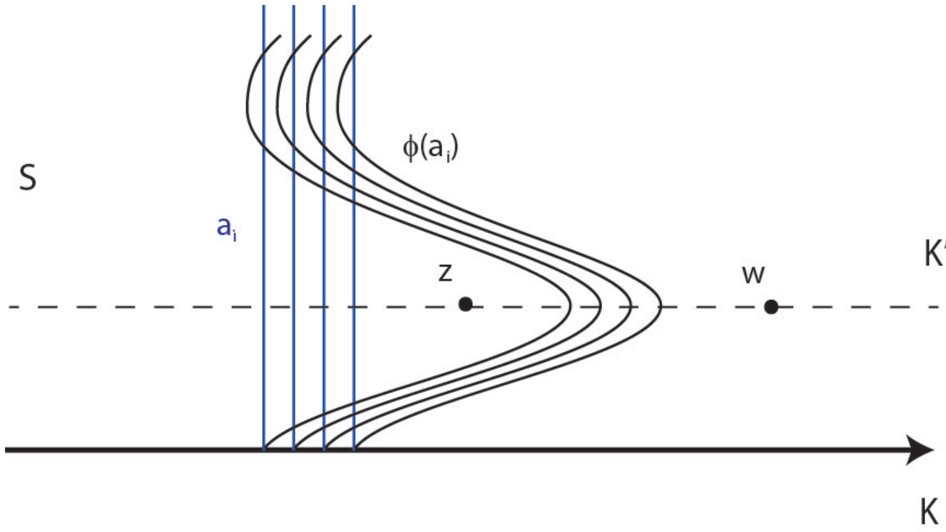

Figure 7.4: The diagram $(S, \mathbf{a}, \phi(\mathbf{a}))$ near ∂S. Note the new bigon in the diagram containing the marked point z.

7.2.1 The knot filtration on the page

Let $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ be the Heegaard diagram for Y compatible with (S, ϕ) built in Subsection 4.1. Recall in particular that $\Sigma = S_0 \sqcup_{\partial S_0} S_{\frac{1}{2}}$ and $(S, \mathbf{a}, \phi(\mathbf{a}))$ can be identified with $(S_0, \boldsymbol{\alpha} \cap S_0, \boldsymbol{\beta} \cap S_0)$. We will place the marked points later.

Let U_1 be a small open disk in the bigon created by the finger move along K' and such that it non trivially intersects K'. Define $U_2 = \phi(U_1)$ and stabilize S_0 (resp. Σ) by removing U_1 and U_2 and gluing a handle $[0, 1] \times S^1$ as showed in figure 7.5; let $\underline{S}_0$ (resp. $\underline{\Sigma}$) be the resulting surface. See [25, Section 3] for an analogous stabilization.

$K' \setminus (U_1 \cup U_2)$ has two connected components, one intersecting the curves $\mathbf{a}$ and not $\phi(\mathbf{a})$ and the other one intersecting $\phi(\mathbf{a})$ and not $\mathbf{a}$. Define:

- $\beta_0 = \{\frac{1}{2}\} \times S^1$ contained in the handle $[0, 1] \times S^1$;
- α_0 obtained by closing the component of $K' \setminus (U_1 \cup U_2)$ intersecting $\phi(\mathbf{a})$ with an arc in the handle crossing β_0 only once;
- λ' obtained by closing the component of $K' \setminus (U_1 \cup U_2)$ intersecting $\mathbf{a}$ with an arc in the handle intersecting β_0 only once and disjoint from α_0. Then modify λ' by a finger move twisting twice along β_0 and call λ the resulting curve.

Note that α_0 does not intersect the other $\boldsymbol{\alpha}$-curves and that λ does not intersect the $\boldsymbol{\beta}$-curves. Place the marked points z and w near β_0 like in the picture. Then $(\underline{\Sigma}, \boldsymbol{\beta} \cup \beta_0, \boldsymbol{\alpha} \cup \alpha_0, w, z)$ is a Heegaard diagram for $(-Y, -K)$ with λ and β_0 respectively longitude and meridian of a close copy of K in Y.

We remark that this stabilization is necessary to define a Heegaard diagram

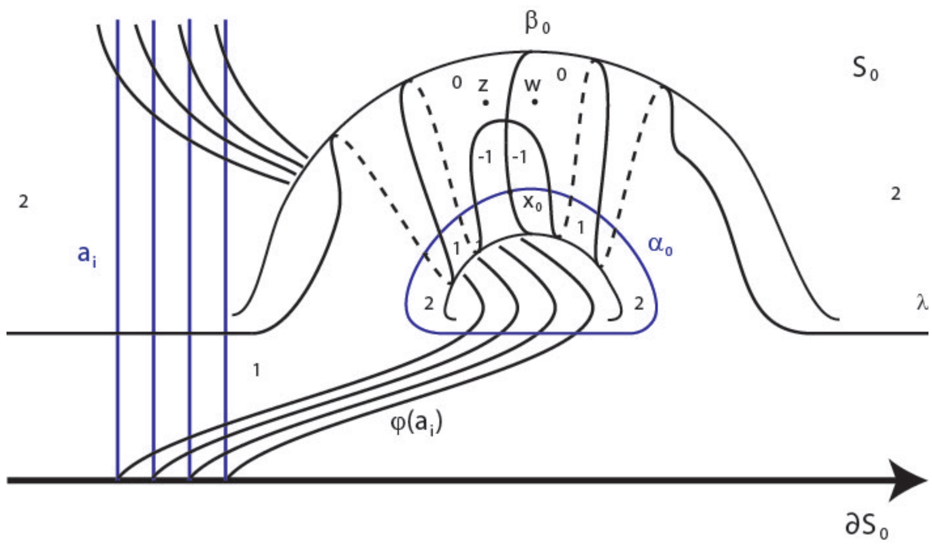

Figure 7.5: The stabilized Heegaard diagram with the multiplicities of $\mathcal{P}$.

for the three-manifold $-Y_0(-K)$ obtained by 0-surgery on $-Y$ along $-K$: this will allow us to associate to any generator $\mathbf{x}$ of $\widehat{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ a Spin^c-structure $\underline{\mathfrak{s}}_{\mathbf{x}} \in \mathrm{Spin}^c(-Y_0(-K))$ (see Section 3.2).

Let $CF(\underline{\Sigma}, \boldsymbol{\beta} \cup \beta_0, \boldsymbol{\alpha} \cup \alpha_0, w)$ be the free $\mathbb{Z}_2$-module generated by $(2g+1)$-tuples of intersection points as usual and let $\widehat{HF}(\underline{\Sigma}, \boldsymbol{\beta} \cup \beta_0, \boldsymbol{\alpha} \cup \alpha_0, w)$ be the associated Heegaard Floer homology. By the invariance of HF under stabilizations of Heegaard diagrams (Theorem 10.1, [42]), we have

$$\widehat{HF}(\underline{\Sigma}, \boldsymbol{\beta} \cup \beta_0, \boldsymbol{\alpha} \cup \alpha_0, w) \cong \widehat{HF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, w).$$

The isomorphism is induced by the isomorphism between the chain complexes defined on the generators by

$$\begin{array}{rcl} \widehat{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, w) & \longrightarrow & \widehat{CF}(\underline{\Sigma}, \boldsymbol{\beta} \cup \beta_0, \boldsymbol{\alpha} \cup \alpha_0, w) \\ \mathbf{x} = (x_1, \ldots, x_{2g}) & \longmapsto & \underline{\mathbf{x}} := (x_0, x_1, \ldots, x_{2g}), \end{array}$$

where x_0 is the unique point in $\beta_0 \cap \alpha_0$. Indeed, since x_0 is the only intersection point between $\{\alpha_i\}_i$ and μ, this must be a component of all the generators of $\widehat{CF}(\underline{\Sigma}, \boldsymbol{\beta} \cup \beta_0, \boldsymbol{\alpha} \cup \alpha_0, w)$, and in particular, in any generator, no other point in μ or α_0 can exist.

Now, if v is one of the two points in $\lambda \cap \alpha_0$ nearest to x_0, to each $\underline{\mathbf{x}}$ we can univocally associate the $(2g+1)$-tuple $\mathbf{x}' = (v, x_1, \ldots, x_{2g})$ of intersection points in the diagram $(\underline{\Sigma}, \boldsymbol{\beta} \cup \lambda, \boldsymbol{\alpha} \cup \alpha_0, w)$ for $-Y_0(-K)$

Let $\mathcal{P}$ be the periodic domain on $(\underline{\Sigma}, \boldsymbol{\beta} \cup \lambda, \boldsymbol{\alpha} \cup \alpha_0, w)$ with boundary $\alpha_0 \cup \lambda$ with multiplicity 2 in all the regions of $\underline{S}_0 \setminus \{y \in [3,4]\}$, 1 in those of $\{y \in$

$[3,4]\}\cup S_{\frac{1}{2}}$ and as in figure 7.5 in the handle. $\mathcal{P}$ is a periodic domain representing the homology class of $\widehat{S}$, obtained by capping off S along K (actually K') in $-Y_0(-K)$.

In 3.2.1 we recalled that to this data we can associate a filtration on the Floer complex, whose degree is given by the integer $\frac{1}{2}\langle c_1(\mathfrak{s}_w(\underline{\mathbf{x}})), [\widehat{S}]\rangle$, where $\mathfrak{s}_w(\underline{\mathbf{x}}) = \mathfrak{s}_w(\mathbf{x}')$ is the Spin^c structure in $-Y_0(-K)$ determined by $\mathbf{x}'$. This quantity can be calculated in terms of $\mathcal{P}$ by using equation 3.3:

$$\langle c_1(\mathfrak{s}_w(\mathbf{x}')), [\widehat{S}]\rangle = \chi(\mathcal{P}) + 2\big(n_u(\mathcal{P}) + \sum_{i=1}^{2g} n_{x_i}(\mathcal{P})\big).$$

Recalling that we defined $S' = S \setminus \{y \in [3,4]\}$, in our case we have:

- $n_u(\mathcal{P}) = \frac{1}{4}(0+1+0-1) = 0$.
- $\bar{n}_{x_i}(\mathcal{P}) = \begin{cases} 1 & \text{if } x_i \in \underline{\Sigma} \setminus S' \\ 2 & \text{if } x_i \in S' \end{cases}$
- $\chi(\mathcal{P})$ is calculated by dividing $\underline{\Sigma}$ into pieces corresponding to regions with different multiplicity, we compute (see for example [45, Section 3]) $\chi(\mathcal{P}) = \sum_{i=-1}^{2} i\chi(R_i) = -6g$.

Substituting in the formula above, we have:

$$\langle c_1(\mathfrak{s}_w(\underline{\mathbf{x}})), [\widehat{S}]\rangle = 2\big(-3g + \#\{i \mid x_i \in \Sigma \setminus S'\} + 2\#\{i \mid x_i \in S'\}\big).$$

where we consider S' as contained in Σ. The following is then straightforward:

Lemma 7.19. *Let $\mathbf{x} = (x_1, \ldots, x_{2g})$ be a generator of $\widehat{CF}(S, \mathbf{a}, \phi(\mathbf{a}))$ with (S, ϕ) adapted to K and $\underline{\mathbf{x}} = (x_0, x_1, \ldots, x_{2g})$. Then*

$$\frac{1}{2}\langle c_1(\mathfrak{s}_w(\underline{\mathbf{x}})), [\widehat{S}]\rangle = \deg(\mathbf{x}) - g.$$

7.2.2 The homology

Consider the (non-stabilized) diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, w)$, with z and w placed as in Figure 7.4. Note that the placement of w is different from that in Section 4.1, but since they are in the same connected component of $\Sigma \setminus (\boldsymbol{\beta} \cup \boldsymbol{\alpha})$ the two choices give the same constraints on the holomorphic curves counted by the HF-differential.

Define $\widehat{CFK}_{*,i}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, w)$ to be the subcomplex of $CF_*(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, w)$ generated by $2g$-tuples $\mathbf{x}$ of intersection points with $\deg(\mathbf{x}) = i$. Note that this is not the standard symmetrized Alexander grading of knot Floer homology: our convention differs from the last just by a shift of $-g$ on the degrees. The computation of last subsection implies that $\widehat{CFK}_{*,i}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, w)$ is isomorphic (as $\mathbb{Z}_2$-module) to the standard knot Floer complex as defined in 3.2.

Notation. *Given* $y_0 \in [0,4]$*, we will denote by* A_{y_0} *the annulus* $([0,1] \times \{y_0\} \times S^1) \subset [0,1] \times \Sigma$*. For example* $[0,1] \times K' = A_3$*.*

Recall that $W = \mathbb{R} \times [0,1] \times \Sigma$ and for any $P \in \Sigma$, $\chi_P = [0,1] \times \{P\}$ is the Reeb chord in $[0,1] \times \Sigma$ passing through P. Define $\sigma_P = \mathbb{R} \times \chi_P \subset W$. If $u : \dot{F} \to W$ is a holomorphic curve, call $n_z(u)$ the algebraic intersection number $\langle u, \sigma_z \rangle$. Since u and σ_z are holomorphic, every intersection point between them has positive sign.

Define moreover

$$G_3 := \mathbb{R} \times [0,1] \times K' \subset \mathbb{R} \times [0,1] \times S_{\frac{1}{2}}.$$

Σ is the non-stabilized Heegaard surface, so that G_3 is connected.

Proposition 7.20. *If* $u : (\dot{F}, j) \to (W, J)$ *is a holomorphic curve counted by* ∂^{HF}*, which tends to* $\mathbf{x}$ *for* $s \to +\infty$ *and to* $\mathbf{y}$ *for* $s \to -\infty$ *then*

$$\deg(\mathbf{x}) - \deg(\mathbf{y}) = n_z(u). \tag{7.15}$$

Proof. This result is the analogue of Equation 3.5 and can be recovered using the fact that the filtration given by deg is just a translation of the Alexander filtration, for which the result holds. However it will be convenient for us to proceed reasoning as follows.

Since the set of the branched points of u is finite, up to slightly pushing K' in the y direction, by Theorem 1.10 we can suppose that u is always transverse to $\partial_t|_{G_3}$ and that it has no branched points in G_3. By Lemma 7.18, we have

$$\deg(\mathbf{x}) - \deg(\mathbf{y}) = \langle u, \mathbb{R} \times \{t_0\} \times K' \rangle \tag{7.16}$$

up to choosing $t_0 \in (0,1)$ such that all the intersections are transverse. Let us prove that

$$\langle u, \mathbb{R} \times \{t_0\} \times K' \rangle = n_z(u). \tag{7.17}$$

Since in G_3 the image of u is always transverse to ∂_t, by holomorphicity it is also transverse to ∂_s. Then u intersects G_3 transversely along a finite set of curves $\mathfrak{C}(u) = \{\mathfrak{c}_1, \ldots, \mathfrak{c}_h\}$ with

$$\partial \mathfrak{c}_i \subset (\mathbb{R} \times \{0\} \times \phi(\mathbf{a})) \sqcup (\mathbb{R} \times \{1\} \times \mathbf{a})$$

for any i. Then $\mathfrak{c}_i$ can be of one of the following two kinds (see Figure 7.6):

1. $\partial \mathfrak{c}_i \subset \mathbb{R} \times \{0\} \times (\phi(\mathbf{a}) \cap K')$ or $\partial \mathfrak{c}_i \subset \mathbb{R} \times \{1\} \times (\mathbf{a} \cap K')$;
2. $\mathfrak{c}_i$ goes from $\mathbb{R} \times \{0\} \times (\phi(\mathbf{a}) \cap K')$ to $\mathbb{R} \times \{1\} \times (\mathbf{a} \cap K')$.

Consider the homology

$$\overline{H}_*(G_3) := H_*(G_3, \mathbb{R} \times \{0,1\} \times (K' \setminus P)),$$

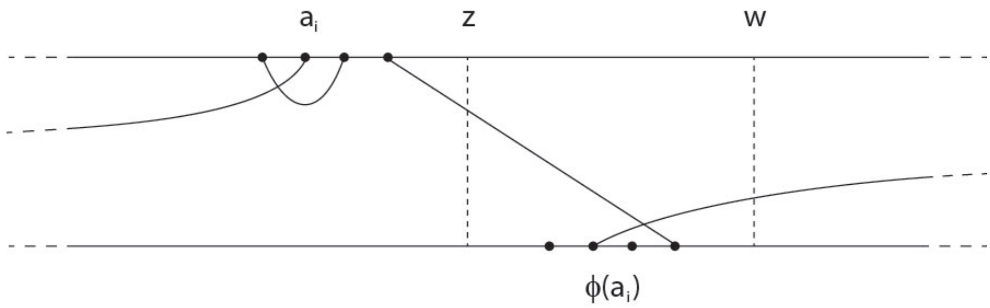

Figure 7.6: Examples of curves $\mathfrak{c}_i \in \mathfrak{C}(u)$ projected on $[0,1] \times K'$. Note the curve crossing $[0,1] \times \{w\}$ can not exists if $n_w(u) = 0$.

where P is a point in K' inside the region of w.

Fix the basis (μ, λ) of $\overline{H}_1(G_3) \cong \mathbb{Z}^2$ with generators given, in order, by a copy of a Reeb chord and of K'. If $[\mathfrak{c}_i] = (m_i, l_i)$, then for the two cases above we have respectively

1. $(m_i, l_i) = (0, 0)$;
2. $(m_i, l_i) = (1, 0)$.

Indeed, since u is a positive multisection and $n_w(u) = 0$, then $\forall i\ m_i \geq 0$ and $l_i = 0$. Moreover in case 1 $\mathfrak{c}_i$ is homotopic, relatively to its boundary, to a curve contained either in $\mathbb{R} \times \{0\} \times K'$ or in $\mathbb{R} \times \{1\} \times K'$ so that $m_i = 0$. Finally in case 2 $\mathfrak{c}_i$ is simple (since u has no branched points in G_3) and connected and so $m_i = 1$.

Note now that the curves of type 1 do not give contributions to either of sides of Equation 7.17. Suppose that $\mathfrak{c}_i$ is of type 2. Since u is always transverse to $\partial_t|_{G_3}$, the placement of z implies $\#\{\mathfrak{c}_i \cap \sigma_z\} = 1$ and the positivity of intersections between holomorphic curves gives

$$n_z(u) = \sum_{\{i | \mathfrak{c}_i \text{ of type } 2\}} 1.$$

On the other hand, by the position of the curves of the diagram, we can parametrize $\mathfrak{c}_i$ in a way that $\dot{\mathfrak{c}}_i = (c_s, c_t, 0, c_\vartheta)$ has c_ϑ always strictly negative, where the coordinates are expressed in terms of the positive basis $(\partial_s, \partial_t, \partial_y, \partial_\vartheta)$ for TW. If $P \in \mathfrak{c}_i \cap (\mathbb{R} \times \{t_0\} \times K')$ then $(\partial_s, \partial_\vartheta, \dot{\mathfrak{c}}_i, J(\dot{\mathfrak{c}}_i))$ is a positive basis of $T_P W$ if and only if $c_t > 0$ and so

$$\langle u, \mathbb{R} \times \{1\} \times K' \rangle = \sum_{\{i | \mathfrak{c}_i \text{ of type } 2\}} \left(\sum_{\{P \in \mathfrak{c}_i \cap (\mathbb{R} \times \{t_0\} \times K')\}} \operatorname{sign}(c_t) \right).$$

But the fact that $\mathfrak{c}_i$ has a homology class (m_i, l_i) with $m_i = 1$ implies that

$$\sum_{\{P \in \mathfrak{c}_i \cap (\mathbb{R} \times \{t_0\} \times K')\}} \operatorname{sign}(c_t) = 1$$

for any i and the result follows.

□

Since the signs of the intersections in equation 7.17 are all positive, ∂^{HF} respects the filtration given by $\deg$. Let ∂^{HFK} be the part of ∂^{HF} that strictly preserves the (filtration) degree, i.e. ∂^{HFK} counts the holomorphic curves u of ∂^{HF} such that $n_z(u) = n_w(u) = 0$. We can see ∂^{HFK} as a family of differentials

$$\partial_i^{HFK} : \widehat{CFK}_{*,i}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, w) \longrightarrow \widehat{CFK}_{*,i}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, w) \tag{7.18}$$

for $i \in \{-g, \dots, g\}$.

Corollary 7.21. $H\big(\widehat{CFK}_{*,i}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, w), \partial_i^{HFK}\big) \cong \widehat{HFK}_{*,i}(-K, -Y)$.

Proof. For any i, Lemma 7.19 implies that the map $\underline{\mathbf{x}} \mapsto \mathbf{x}$ induces a bijection between $\widehat{CFK}_{*,i}(\underline{\Sigma}, \boldsymbol{\beta} \cup \beta_0, \boldsymbol{\alpha} \cup \alpha_0, z, w)$ and $\widehat{CFK}_{*,i}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, w)$. The constraints imposed by $z, w \in \underline{\Sigma}$ on the curves counted by ∂^{HFK} imply that the differential counts holomorphic curves $\underline{u}$ which are a disjoint union of a trivial strip on x_0 (of index 0) and a $2g$-multisection u of index 1 not intersecting the regions containing z or w. So u can be seen as a curve counted by ∂_i^{HFK}. Then there is a bijection $\underline{u} \mapsto u$ between the curves counted by the differentials, which is compatible with the bijection on the complexes (in the sense that if $\underline{u}$ flows from $\underline{\mathbf{x}}$ to $\underline{\mathbf{y}}$ then u flows from $\mathbf{x}$ to $\mathbf{y}$). □

In order to define the knot Floer complex on a page (and not on the entire Σ) it is sufficient to take into account only $2g$-tuples in S_0 and quotient the groups by relation 4.2 as done in Section 4.1. We will call $\widehat{CFK}_{*,i}(S, \boldsymbol{a}, \phi(\boldsymbol{a}), z)$ the resulting groups. The proof that the homology does not change under the quotient is the same of that for HF in [10, Section 4.9]. Then we have

$$\begin{array}{rcl} \widehat{HFK}_{*,i}(S, \boldsymbol{a}, \phi(\boldsymbol{a}), z, w) & := & H_*(\widehat{CFK}_i(S, \boldsymbol{a}, \phi(\boldsymbol{a}), z), \partial_i^{HFK}) \\ & \cong & \widehat{HFK}_{*,i}(-K, -Y) \end{array} \tag{7.19}$$

Notation. *Similarly to the notation used for $\widehat{ECK}$ and $\widehat{PFK}$, we will define*

$$\widehat{CFK}_{*,\leq i}(S, \boldsymbol{a}, \phi(\boldsymbol{a}), z) = \bigoplus_{j \leq i} \widehat{CFK}_{*,i}(S, \boldsymbol{a}, \phi(\boldsymbol{a}), z)$$

7.3 Φ and the degree filtrations

In this section we prove that Φ respects the knot filtrations described. Let us begin with some definition and notation.

Definition 7.22. *We say that ϑ_0* is far from the curves of the diagram $(S, \mathbf{a}, \phi(\mathbf{a}))$ *(or simply far from the curves) if $([0,4] \times \{\vartheta_0\}) \cap \phi(\mathbf{a}) = \emptyset$. In this case we say also that, for any y_0, $P = (y_0, \vartheta_0)$* is far from the curves.

Observation 7.23. *Note that, if $\phi = \phi_{\delta,\varepsilon}$ and r is like in the definition of the curves* $\mathbf{a}$ *of the diagram given in Section 4.1, then ϑ_0 is far from the curves of the diagram if and only if $\vartheta_0 \in (4gr + \varepsilon, 2\pi - \delta)$.*

From now on we consider the M-B perturbations of the tori T_y, $y = 1, 2, 4$, such that the orbits h, e_+, h_- cross S_0 far from the curves. By this we mean that, if

$$\begin{array}{rcl} h \cap S_0 & = & (\vartheta_h, 1), \\ e_+ \cap S_0 & = & (\vartheta_{e_+}, 2), \\ h_- \cap S_0 & = & (\vartheta_{h_-}, 4), \end{array} \tag{7.20}$$

then ϑ_h, ϑ_{e_+} and ϑ_{h_-} are far from the curves.

Let us introduce some other notations that we will use in this and the next sections.

Notations.
- *Given $\vartheta_0 \in [0, 2\pi]$ and $j = 1, 2, 4$, let $\delta^j_{\vartheta_0}$ denote the simple orbit $\{\vartheta = \vartheta_0\}$ contained in the M.B. torus T_j (e.g. $h = \delta^1_{\vartheta_h}$).*
- *Given X, $u : F \to \mathbb{R} \times X$ and $s_0 \in \mathbb{R}$, we denote $u_{\leq s_0}$ the restriction of u to $u^{-1}(\{s \leq s_0\})$. Similarly defined are the functions $u_{\geq s_0}$, $u_{=s_0}$, etc.*
- *Sometimes, given a manifold X and a coordinate x on some subset of X, we will denote by $X_{\{x \in [x_0, x_1]\}}$ the subset of points of X with coordinate x included in $[x_0, x_1]$. Similarly defined are $X_{\{x = x_0\}}$, $X_{\{x \in (x_0, x_1)\}}$ etc. We will use a similar notations for subsets of X defined by conditions on more then one coordinate like $X_{\{x \in [x_0, x_1], z = z_0\}}$, etc. (for example $W_{\{t = t_0, y = 3\}} = \mathbb{R} \times \{t_0\} \times K' \subset W$ and $N_{\{y = y_0\}} = T_{y_0} \subset N$).*
- *Given $y_0 \in [0, 4]$, in analogy with the submanifold G_3 of W of last section, we call $G^+_{y_0}$ the 3-dimensional proper submanifold $W_{+\{y = y_0\}}$ of W_+.*

7.3.1 Properties of the Φ-curves near ∂S

In this subsection we describe some properties about the curves counted by Φ in $N_{\{y \in [1,4]\}}$. We will refer to these curves simply as Φ-*curves*.

Fix a basis $([\mu], [\lambda])$ for $H_1(N_{\{y \in [0,4]\}})$ where μ is a meridian for K and λ is a longitude contained in S_0. For any y_0, consider the chain of maps

$$G^+_{y_0} \hookrightarrow W_{+\{y \in [0,4]\}} \hookrightarrow W_{\{y \in [0,4]\}} \twoheadrightarrow N_{\{y \in [0,4]\}},$$

where the first two maps are the natural inclusions and the last one is the projection along the coordinate s. The chain above induces then the following one in homology:

$$H_1(G^+_{y_0}) \to H_1(W_{+\{y \in [0,4]\}}) \to H_1(W_{\{y \in [0,4]\}}) \to H_1(N_{\{y \in [0,4]\}}).$$

These maps are all isomorphisms: we will keep to call $([\mu], [\lambda])$ the pre-images of the generators of $H_1(N_{\{y \in [0,4]\}})$ in each of the groups above.

Let $u : (\dot{F}, j) \to (W_+, J_+)$ be a degree k multisection and let $y_0 \in [0, 4]$ be such that u is always transverse to $\partial_t|_{G^+_{y_0}}$ and does not have any branched point or limit in $G^+_{y_0}$ (the set of the allowed y_0 is dense in $[0, 4]$ by Theorem 1.10). By holomorphicity, u is transverse to $G^+_{y_0}$ and $Im(u) \cap G^+_{y_0}$ consists in a finite (possibly empty) collection of properly embedded curves

$$\mathfrak{C}_{y_0}(u) = \{\mathfrak{c}_1, \dots, \mathfrak{c}_h\}$$

of the following two kinds:

– a curve with boundary in

$$G^+_{y_0} \cap \big((2, +\infty) \times \big((\phi(\mathbf{a}) \times \{0\}) \cup (\mathbf{a} \times \{1\})\big)\big) \subset \partial G^+_{y_0}$$

called *b-component*;

– a close curve, called *c-component*.

Consider the homology

$$\overline{H}_*(G^+_{y_0}) := H_*(G^+_{y_0}, \{s > 2\} \times \{0, 1\} \times (K' \setminus P)),$$

with P a point far from the curves. To any (b- or c-) component $\mathfrak{c}_i$ we can associate its homology class $[\mathfrak{c}_i] = (m_i, l_i) \in \overline{H}_1(G^+_{y_0})$ with respect to the basis $([\mu], [\lambda])$.

Recall that in section 7.1 we saw that $N_{\{y\in[2,4]\}}$ and $N_{\{y\in[1,2]\}}$ are foliated by two families of holomorphic cylinders $Z_{s,\vartheta}$. In $N_{\{y\in[2,4]\}}$ the cylinders are such that, $\forall (s_0, \vartheta_0)$:

1. $\pi_{\mathbb{R}}(Z_{s_0,\vartheta_0}) = C_{((2,4),\vartheta_0)}$;
2. $\forall s_1 \in \mathbb{R}$ $Z_{s_0+s_1,\vartheta_0}$ is the s_1-translate of Z_{s_0,ϑ_0};
3. $\lim_{s\to+\infty} Z_{s_0,\vartheta_0} = \delta^2_{\vartheta_0}$;
4. $\lim_{s\to-\infty} Z_{s_0,\vartheta_0} = \delta^4_{\vartheta_0}$.

Define

$$Z^+_{s,\vartheta} := Z_{s,\vartheta} \cap W_+.$$

The proof of the following Lemma uses similar arguments of that in Lemma 6.6.5 in [10].

Lemma 7.24. *Let u be a Φ-curve. Then, for any $y_0 \in [1, 4]$, u intersects $G^+_{y_0}$ in a set of curves, each with homology class of the form $(m, 0)$ with respect to the basis $([\mu], [\lambda])$.*

Proof. We prove the result for $y_0 \in [2, 4]$ using the holomorphic foliation in $N_{\{y\in[2,4]\}}$. The proof in $y_0 \in [1, 2]$ is analogue.

Note first that, since u can not cross T_4, the result is true for $y_0 = 4$. Suppose there exists $y_0 \in (2, 4]$ for which the statement is false. Then there exists $\mathfrak{c} \in \mathfrak{C}_{y_0}(u)$ with homology class $(m, l) \in \overline{H}_1(G^+_{y_0})$ in the coordinates $([\mu], [\lambda])$ with

$l \neq 0$. Then for any ϑ_0 such that $P = (y_0, \vartheta_0)$ is far from the curves of the diagram, there exists $s_0 \in \mathbb{R}$ such that u intersects $Z^+_{s_0,\vartheta_0}$ and

$$\langle u, Z^+_{s_0,\vartheta_0} \rangle > 0$$

by the positivity of the intersection.

Since we are far from the curves, for any $s < s_0$ $Z^+_{s_0,\vartheta_0}$ is homotopic to Z^+_{s,ϑ_0} through an homotopy whose image $\mathcal{Z}$ does not intersect the Lagrangian $L_{\mathbf{a}}$, so that $\mathcal{Z}$ does not cross the boundary of u or the chords at the positive ends of u. Since moreover u can not have negative limits in $N_{\{y \in (2,4)\}}$ we have

$$\langle u, Z^+_{s,\vartheta_0} \rangle = \langle u, Z^+_{s_0,\vartheta_0} \rangle > 0$$

for any $s < s_0$. Since each Z^+_{s,ϑ_0} has negative limit in T_4, this forces u to intersect all the G^+_y for $y \in [y_0, 4)$. The argument applied to all the ϑ far from the curves implies that the result is false also for T_4, contradiction.

□

Corollary 7.25. *Let u be a Φ-curve. Then, for any $y_0 \in [0,4]$, u intersects $G^+_{y_0}$ in a set of curves whose homology class is either $(1,0)$ or $(0,0)$.*

Proof. By the last lemma any $\mathfrak{c} \in \mathfrak{C}_{y_0}(u)$ has homology class $[\mathfrak{c}] = (m, 0)$. Moreover since $\mathfrak{c}$ is connected and always transverse to ∂_t, then either $m = 0$ or $m = 1$. □

Observation 7.26. *Note that $[\mathfrak{c}] = (0,0)$ only if $\mathfrak{c}$ is a curve with either $\partial\mathfrak{c} \subset \mathbb{R} \times \{0\} \times (\phi(\mathbf{a}) \cap K')$ or $\partial\mathfrak{c} \subset \mathbb{R} \times \{1\} \times (\mathbf{a} \cap K')$ (cf. the curves of type 1 in the proof of Proposition 7.20).*

Corollary 7.27. *Let u be a Φ-curve. Any negative end of u in T_2 is in h_+.*

Proof. Let u' be the restriction of u to a neighborhood of the puncture associated to a negative end in T_2. If $\mathrm{Im}(u')$ crosses a torus $T_{2\pm\varepsilon}$ along a closed curve, then last lemma implies that it must have slope ∞ and Trapping Lemma should force the puncture to be positive. The only possibility is that $\mathrm{Im}(u')$ contains a flow trajectory of the Morse function of T_2, which can have only h_+ as negative end.

□

Let us continue to study how Φ-curves can approach the orbits on T_i for $i \in \{1,2,4\}$. Now we will use arguments similar to that in the proof of Lemma 9.5.1 in [9].

Fix an orbit $\delta = \delta^i_{\vartheta_0} \subset T_i$ (we are using the notation introduced at the beginning of this section). Suppose that u contains δ at its negative limit. Given a thin tubular neighborhood $\mathcal{N}(\delta)$ of δ in N, look at

$$\mathrm{Im}(u) \cap \{s = \bar{s}\} \subset W_{+\{s=\bar{s}\}}, \tag{7.21}$$

where $\bar{s} \ll 2$ is fixed. If $\bar{s}$ is small enough, this is a braid around δ contained in $\mathcal{N}(\delta)$ with connected components, say, $\{b_1, \ldots, b_k\}$, each of which is associated to an end in δ. Fix the longitude $\mathfrak{l} \subset \partial\mathcal{N}(\delta)$ for δ, obtained by taking one of the two curves of the intersection $\partial\mathcal{N}(\delta) \cap N_{\{y\in[0,4],\vartheta=\vartheta_0\}}$, oriented as a positive meridian for K. If $\mathfrak{m} = \partial\mathcal{N}(\delta) \cap S_0$ oriented in some way, $([\mathfrak{m}], [\mathfrak{l}])$ is a basis for $H_1(\partial\mathcal{N}(\delta))$. Suppose $[b_j] = (f_j, g_j) \in H_1(\partial\mathcal{N}(\delta))$ with respect to this basis.

Lemma 7.28. *Let u be a Φ-curve having some negative end tending to $\delta = \delta^i_{\vartheta_0}$ and let $\mathcal{N}(\delta)$ and let $([\mathfrak{m}], [\mathfrak{l}])$ be defined as above. If $\{b_1, \ldots, b_k\}$ is the bride given by Equation 7.21 then for any j:*

$$[b_j] = (0, g_j).$$

Proof. To prove that $f_j = 0$ we argue as follows. If $0 < \varepsilon << 1$, consider the thickened torus $T_{i,\varepsilon} = S^1 \times \{y \in [i-\varepsilon, i+\varepsilon]\} \times S^1 \subset N$ and suppose that $\mathcal{N}(\delta)$ is thin enough to be contained in $\mathrm{int}(T_{i,\varepsilon})$. Let $([\mu_\pm], [\lambda_\pm])$ be a basis of $H_1(G^+_{i\pm\varepsilon})$ like the basis $([\mu], [\lambda])$ defined in the last subsection. Call $\mathcal{H}$ the first homology group of $\big(\mathbb{R} \times (T_{i,\varepsilon} \setminus \mathcal{N}(\delta))\big) \cap W_+$ (this can be seen as the product of B_+ with the annulus $S_{\{y\in[i-\varepsilon,i+\varepsilon]\}}$ with a small disk near $\delta \cap S$ removed). Then in $\mathcal{H}$ we have the relations

$$\begin{aligned} [\mathfrak{m}] &= \pm([\lambda_+] - [\lambda_-]) && (7.22)\\ [\mathfrak{l}] &= [\mu]. && (7.23) \end{aligned}$$

where the sign in the first depends on the choice of the orientation of $\mathfrak{m}$. Let A_j be the connected component of $u \cap \big(\mathbb{R} \times (T_{i,\varepsilon} \setminus \mathcal{N}(\gamma))\big) \cap W_+$ such that $b_j \subset A_j$. Then obviously

$$[\partial A_j] = 0 \in \mathcal{H}.$$

So if $f_j \neq 0$ for some j and ε is small enough, u should cross both $G^+_{i+\varepsilon}$ and $G^+_{i-\varepsilon}$ along curves with non trivial homology along the components λ_+ or λ_-, which contradicts Lemma 7.24. □

7.3.2 Φ is filtered

We are finally ready to prove that the chain map

$$\Phi : \widehat{CF}(S, \mathbf{a}, \phi(\mathbf{a})) \to PFC_{2g}(N, \partial N)$$

of Subsection 4.3 is filtered with respect to the degrees on the knot chain complexes.

Theorem 7.29. *For any $i \in \{0, \ldots, 2g\}$:*

$$\Phi\big(\widehat{CFK}_{\leq i}(S, \mathbf{a}, \phi(\mathbf{a}), z)\big) \subset PFC^{h_+,e_-}_{\leq i}(N_1, \partial N).$$

Observation 7.30. *There is here a slight disambiguation in the use of the degree, due to the fact that we changed definition in this chapter (see Observation 7.14). Indeed in the definition of the chain complex $PFC_{2g}(N, \partial N)$, the number "$2g$" refers to the total intersection of the orbit sets with S, while in the theorem the "$\leq i$" refers to the intersection with S'. In the rest of this section, unless stated otherwise, we will always refer to the degree* deg *as to the second interpretation (which gives the knot filtration in PFH as defined in Subsection 7.1.4). This degree is then different from the degree of the multisections of W_+ counted by Φ (which is always $2g$).*

The first problem to face is the good definition of Φ in the theorem. In fact the image of Φ is contained in $PFC_{\leq 2g}(N, \partial N)$ but not a priori in the subcomplex $PFC^{h_+,e_-}_{\leq 2g}(N_1, \partial N)$. We have then to prove that there are no Φ-curves with some negative end that is asymptotic to either $\{e_+\}$ or $\{h_-\}$.

So far we made assumptions only on the orbits h_-, e_+ and h_+, requiting that they must be far from the curves of the diagram. Now we need to make an assumption also on the position of h_+, imposing that it must be *near the curves of the diagram*: by this we mean that $\vartheta_{h_+} \in (0, 4gr)$, where r is the value used in Section 4.1 in the definition of the diagram.

Proposition 7.31. *Let $u : \dot{F} \to \mathbb{R} \times N$ be a Φ-curve negatively asymptotic to some $\gamma \in \mathcal{O}_{\leq 2g}(N)$. Under the hypothesis made on the orbits and the Heegaard diagram in $N_{\{y \in [0,4]\}}$, we have*

$$\gamma \in \mathcal{O}_{\leq 2g}(N_1 \cup \{h_+, e_-\}).$$

Proof. By Corollary 7.27 u can not have e_+ as negative end. Let us prove now that u can not even have h_- as negative end. The proof uses again Wendl's holomorphic foliations of [59] and is similar to that of Lemma 7.24.

Let u' be an irreducible component of u which limits to h_- and study the intersections of u' with the holomorphic submanifolds $Z^+_{s,\vartheta}$ of W_+ defined in the previous section.

Note first that $\pi_{\mathbb{R}}(u' \cap \{y \in [2, 4]\})$ can not contain the entire cylinder $C_{([2,4],\vartheta_{h_-})}$. Indeed, since ϑ_{h_-} is far from the curves, the only possibility is that u' contains a flow trajectory of the Morse function associated to T_2 flowing from some chords in the direction of ϑ_{h_-}. This is not possible because of the fact that e_+ is far from the curves and h_+ is the minimum of the Morse function of T_2 and it is near the curves of the diagram. In fact the proof of Corollary 7.27 shows that u' should contain a gradient line of T_2 flowing from the curves of the diagram in the direction of ϑ_{h_+}; but this is not possible since h_+ is near the curves and so the flow line can not glue with $C_{([2,4],\vartheta_{h_-})} \cap T_2$ along $\delta^2_{\vartheta_{h_-}}$ (which is far from the curves).

Since ϑ_{h_-} is far from the curves of the diagram, there exists then ϑ_0 close to ϑ_{h_-} (and far from the curves) such that:

1. u' does not have any end in $C_{([2,4],\theta_0)}$;
2. $Q := \pi_{\mathbb{R}}(Im(u') \cap \{s \leq 2\}) \pitchfork C_{((2,4),\vartheta_0)} \neq \emptyset$.

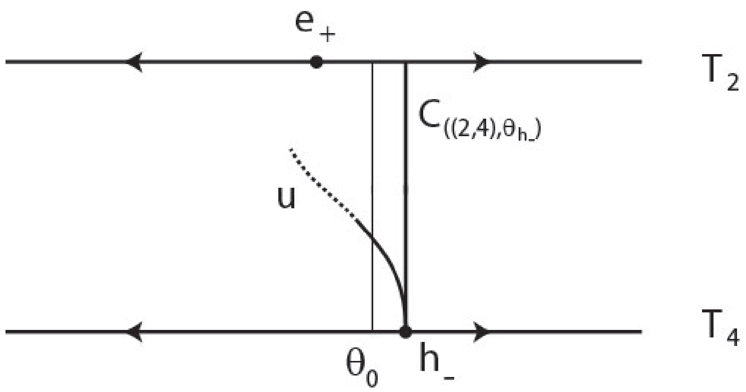

Figure 7.7: The projection of Q to N. The image of u should approach $C_{([2,4],\vartheta_{h\_})}$ for s tending to $-\infty$ but also to some ϑ near the curves of the diagram for s tending to 2.

By 2 there exists s_0 such that

$$\langle u', Z^+_{s_0,\theta_0} \rangle > 0$$

(since u' and $Z^+_{s_0,\theta_0}$ are both holomorphic), while 1 implies that $\pi_{\mathbb{R}}^{-1}(Q)$ is compact in $u(\dot{F})$ and

$$Im(u') \cap Z^+_{s_1,\theta_0} = \emptyset$$

for $s_1 \ll s_0$.

On the other hand, for any $s' < s_0$, Z^+_{s',θ_0} is homotopic to $Z^+_{s_0,\theta_0}$ through a homotopy whose image $\mathcal{Z}$ is the union of the submanifolds Z^+_{s,θ_0}, $s \in [s', s_0]$. Since θ_0 is far from the curves and $\mathcal{Z}$ does not intersect any end of u'

$$\langle u', Z^+_{s',\theta_0} \rangle = \langle u', Z^+_{s_0,\theta_0} \rangle > 0$$

and for $s' = s_1$ we get a contradiction. □

Before giving the final part of the proof of Theorem 7.29 we need a last lemma, whose proof is very similar to that of Lemma 2.17 (avoiding the considerations about the positivity of intersections).

Lemma 7.32. *Let $u : \dot{F} \to W_+$ be a degree $k \leq 2g$ holomorphic multisection of W_+ with positive end $\mathbf{x}$ and negative end γ. Suppose that u is always transverse to $\partial_t|_{G_3^+}$ and does not have any branched point in G_3^+. Then*

$$\deg(\mathbf{x}) - \deg(\gamma) = \langle u, \mathbb{R} \times \{t_0\} \times K' \rangle$$

for any $t_0 \in (0, 1)$ such that the intersections above are transverse.

Proof. of Theorem 7.29

The proof is very similar to that of Proposition 7.20. By the last Lemma we need just to prove that

$$\langle u, \mathbb{R} \times \{t_0\} \times K' \rangle \geq 0.$$

Since u is always transverse to ∂_t, by holomorphicity it is also transverse to ∂_s. Then u intersects G_3^+ transversely along a finite set of simple curves $\mathfrak{C}_3(u) = \{\mathfrak{c}_1, \ldots, \mathfrak{c}_h\}$ of the following three kinds

1. $\partial \mathfrak{c}_i \subset \mathbb{R} \times \{0\} \times (\phi(\mathbf{a}) \cap K')$ or $\partial \mathfrak{c}_i \subset \mathbb{R} \times \{1\} \times (\mathbf{a} \cap K')$;
2. $\mathfrak{c}_i$ goes from $\mathbb{R} \times \{0\} \times (\phi(\mathbf{a}) \cap K')$ to $\mathbb{R} \times \{1\} \times (\mathbf{a} \cap K')$;
3. $\mathfrak{c}_i$ is a closed curve.

By Observation 7.26 $\mathfrak{c}_i$ is homologically trivial in $H_1(G_3^+)$ if it is of type 1 and a meridian for K if it is of type 2 or 3. In any case we can parametrize $\mathfrak{c}_i$ in a way that $\dot{\mathfrak{c}}_i = (c_s, c_t, 0, c_\vartheta)$ has c_ϑ always strictly negative, where the coordinates are expressed in terms of the positive basis $(\partial_s, \partial_t, \partial_y, \partial_\vartheta)$ for TW: since $f_{\delta,\epsilon}$ is positive on $\{y = 3\}$, this orientation makes curves of type 2 and 3 homologically equivalent to positively oriented meridians. Then

$$\begin{aligned}
\langle u, \mathbb{R} \times \{t_0\} \times K' \rangle &= \sum_i \sum_{\{P \in \mathfrak{c}_i \cap (\mathbb{R} \times \{t_0\} \times K')\}} \operatorname{sign}(c_t) \\
&= \sum_{\{i | \mathfrak{c}_i \text{ of type 2 or 3}\}} \sum_{\{P \in \mathfrak{c}_i \cap (\mathbb{R} \times \{t_0\} \times K')\}} \operatorname{sign}(c_t) \\
&= \sum_{\{i | \mathfrak{c}_i \text{ of type 2 or 3}\}} 1 \geq 0
\end{aligned}$$

and the result follows.

□

We end the chapter by observing that Theorem 7.29 is not only interesting by itself, but it can be seen as a first step in the proof of Conjecture 2.27 in the case of fibered knots. Indeed, let $\Psi : \widehat{ECC}_*(Y, \alpha) \to \widehat{CF}_*(-Y)$ be the chain map that induces in homology the inverse isomorphism of Φ. Let H (G) be chain homotopies between $\Psi \circ \Phi$ ($\Phi \circ \Psi$) and the identity map of $\widehat{CF}(-Y)$ ($\widehat{ECC}(Y, \alpha)$). All these maps are defined (in [10] and [11]) by counting certain holomorphic curves in symplectic four manifolds with boundary. By standard results about spectral sequences, if one can prove that all the maps above are filtered with respect of the knot filtrations on $\widehat{CF}_*(-Y)$ and $\widehat{ECC}_*(Y, \alpha)$, then conjecture 0.2 for fibered knots is true. Finally, in light of Proposition 6.10, it should be possible to generalize the result to any knot.

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PÔLE DE RECHERCHE ET D'ENSEIGNEMENT SUPÉRIEUR

UNIVERSITÉ DE NANTES

Thèse de Doctorat

Gilberto SPANO

Invariants de nœuds en homologie de contact plongée

Knot invariants in embedded contact homology

Résumé
Soit (Y,α) une 3-variété de contact et $\widehat{HF}(Y)$, $\widehat{ECH}(Y,\alpha)$ respectivement les homologies de Heegaard Floer et de contact plongée associées. Dans une serie d'articles, Colin, Ghiggini et Honda prouvent qu'il existe un morphisme de chaînes Φ qui induit un isomorphisme $\Phi_* : \widehat{HF}(Y) \to \widehat{ECH}(Y,\alpha)$ en homologie. Étant donné un nœud K dans Y, une version chapeau $\widehat{ECK}(K,Y,\alpha)$ de l'homologie de contact plongée pour les nœuds est définie dans [13] et un isomorphisme avec l'homologie de Heegaard Floer $\widehat{HFK}(K,Y)$ est conjecturé. Ces deux homologies peuvent être définies comme la première page de suites spectrales déterminées par des filtrations induites par K sur des complexes de chaînes pour $\widehat{ECH}(Y,\alpha)$ et $\widehat{HF}(Y)$.
Le but de cette thèse est de fournir des indices sur la véracité de cette conjecture. On définie une version complète ECK de l'homologie $\widehat{ECK}$ et on généralise les définitions de ECK et $\widehat{ECK}$ aux entrelacs.
On calcule ensuite les caractéristiques d'Euler de ces homologies pour les nœuds et entrelacs dans les trois-sphères d'homologie (munies d'une forme de contact convenable) et on prouve que, dans S^3, l'homologie ECK est une catégorification du polynôme d'Alexander à multivariables. Ce fait, associé à un résultat bien connu analogue en HFK, implique que la conjecture est vraie au niveau de caractéristiques d'Euler en S^3.
Finalement, nous montrons que, à homotopies de chaînes près, le morphisme Φ préserve les filtrations du nœud. Ceci peut être considéré comme la première étape d'une preuve de la conjecture pour les nœuds fibrés.

Mots clés
Théorie des nœuds, homologie de Heegaard Floer, homologie de contact plongée, polynôme d'Alexander.

Abstract
Given a contact 3-manifold (Y,α), let $\widehat{HF}(Y)$ and $\widehat{ECH}(Y,\alpha)$ be the associated Heegaard Floer and, respectively, embedded contact homologies. In a series of papers Colin, Ghiggini and Honda proved that there exists a chain map Φ that induces an isomorphism $\Phi_* : \widehat{HF}(Y) \to \widehat{ECH}(Y,\alpha)$ in homology. Given a knot K in Y, in [13] a hat embedded contact knot homology $\widehat{ECK}(K,Y,\alpha)$ is defined and an isomorphism with the hat Heegaard Floer knot homology $\widehat{HFK}(K,Y)$ is conjectured. These two homologies can be defined as first pages of spectral sequences arising from filtrations induced by K on chain complexes for $\widehat{ECH}(Y,\alpha)$ and $\widehat{HF}(Y)$.
The aim of this thesis is to provide some evidences about the veracity of this conjecture. We define a full ECK homology and we generalize the definitions of $\widehat{ECK}$ and ECK to any link.
We compute then the Euler characteristics of these homologies for knots and links in homology three-spheres (endowed with a suitable contact form) and we prove that in S^3 the ECK homology is a categorification of the multivariable Alexander polynomial. This fact, together with a well known analogous result in HFK, implies that the conjecture is true at the level of Euler characteristics in S^3.
Finally we show that, up to chain homotopies, the chain map Φ preserves the knot filtrations. This can be considered as a first step of a proof of the conjecture for fibered knots.

Key Words
Knot theory, Heegaard Floer homology, embedded contact homology, Alexander polynomial.