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Sur les propriétés spectrales des opérateurs générés par un système d'équations différentielles

JURY

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Contents

Introduction

Version Française

On considère l'opérateur différentiel non-autoadjoint $L_m(Q)$,dans l'espace $L_2^m[0,1]$ engendré par la forme différentielle

$$-\mathbf{y}''(x) + Q(x)\,\mathbf{y}(x) \tag{0.1}$$

et les conditions aux limites

$$U_i(\mathbf{y}) = \alpha_i \mathbf{y}^{(k_i)}(0) + \alpha_{i,0}\mathbf{y}(0) + \beta_i \mathbf{y}^{(k_i)}(1) + \beta_{i,0}\mathbf{y}(1) = 0, \ \ i = 1,2 \tag{0.2}$$

qui est dans le cas scalaire ($m = 1$)

$$U_i(y) = \alpha_i y^{(k_i)}(0) + \alpha_{i,0}y(0) + \beta_i y^{(k_i)}(1) + \beta_{i,0}y(1) = 0, \ i = 1,2 \tag{0.3}$$

fortement regulière, où $0 \leq k_2 \leq k_1 \leq 1$, et $\alpha_i, \alpha_{i,0}, \beta_i, \beta_{i,0}$ sont des nombres complexes, tels que pour chaque indice i, l'un au moins des nombres α_i, β_i est non-nul. Ici, $\mathbf{y}(x) = (y_1(x), y_2(x), ..., y_m(x))^T$ et $L_2^m[0,1]$ est l'espace des fonctions à valeurs vectorielles $\mathbf{f}(x) = (f_1(x), f_2(x), ..., f_m(x))$, telles que $f_k \in L_2[0,1]$, pour $k = 1,2,...,m$ et $Q(x) = (b_{i,j}(x))$ est une matrice $m \times m$ dont les composantes $b_{i,j}$ sont complexes et de carré integrable. On définit la norme $\|.\|$ et le produit scalaire $(.,.)$ dans l'espace $L_2^m[0,1]$ par

$$(f,g) = \int_0^1 \langle f(x), g(x)\rangle\, dx, \ \ \|f\| = \left(\int_0^1 |f(x)|^2\, dx\right)^{\frac{1}{2}}$$

où $|.|$ et $\langle .,. \rangle$ sont respectivement la norme et le produit scalaire dans $\mathbb{C}^m$.

Les operateurs différentiels non-autoadjoints apparait dans plusieurs les problèmes de diffusion inélastique et les problèmes de la physique mathematique utilisant la transformée de Fourier. Les premiers travaux concernant ces operateurs ont été initiés, au debut du 20^{eme} siècle, dans [7]-[11] et [54]-[56]. Dans le cas scalaire ($m = 1$), les conditions aux limites qui sont fortement regulières sont les plus couramment utilisées. Si les conditions aux limites sont fortement regulières, alors les fonctions propres et les fonctions associées de l'opérateur généré, dans l'espace $L_2\left[0,1\right]$,à partir de l'équation différentielle ordinaire forment une base de Riesz. Ce résultat a été montré dans [21], [29] et [43]. Dans le cas où les conditions aux limites sont regulières mais pas fortement regulières, en général, les fonctions propres et les fonctions associées ne forment pas une base usuelle.

A. A. Shkalikov a montré que les fonctions propres et les fonctions associées des operateurs, régis par une équation différentelle ordinaire, dont les coefficients de la matrice sont intégrables et avec des conditions aux limites regulières forment une base de Riesz avec les parenthèses devraient être contenues que les fonctons associées aux vecteurs propres qui sont scindés, (voir [48]-[53]).

L. M. Luzhina a generalisé ce résultat pour des problèmes aux limites avec coefficients dépendant du paramètre spectral (voir [32,33]).

Dans [58], Veliev a considéré l'operateur differentiel $T_t(Q)$, définit dans l'espace $L_2^m\left[0,1\right]$, par l'équation différentielle (0.1) et les conditions quasi-périodiques :

$$\mathbf{y}^{'}\left(1\right)=e^{it}\mathbf{y}^{'}\left(0\right),\ \mathbf{y}\left(1\right)=e^{it}\mathbf{y}\left(0\right),$$

pour $t\in(0,2\pi)$ et $t\neq\pi$. Il a montré que la différence des valeurs propres $\lambda_{k,j}$ de $T_t(Q)$ et les valeurs propres de l'operateur $T_t(C)$ est d'ordre $O\left(\frac{\ln k}{k}\right)$, où

$$C=\int_0^1 Q\left(x\right)dx. \tag{0.4}$$

Notons que, afin d'obtenir des formules asymptotiques d'ordre $O(\frac{1}{k})$ pour les valeurs-propres $\lambda_{k,j}$ de l'operateur differentiel défini par (0.1), en utilisant un développement asymptotique classique pour les solutions de l'équation matricielle $-Y^{''}+Q\left(x\right)Y=\lambda Y$, il est nécessaire que Q soit différentiable (voir [12], [35], [36], [44]). La méthode proposée dans [59] nous donne la possibilité d'obtenir les formules asymptotiques d'ordre $O(k^{-1}\ln|k|)$ pour les valeurs propres $\lambda_{k,j}$ et les fonctions propres normalisées $\Psi_{k,j}(x)$ de $T_t(Q)$ lorsqu'il n'existe pas de condition sur la regularité des coefficients $b_{i,j}$ de Q. En suite, dans [60]-[64], Veliev a étudié le spectre et la propriété fondamentale des fonctions propres et des fonctions associées, des operateurs differentiels engendrés, dans $L_2^m\left[0,1\right]$, par la forme différentielle d'ordre arbitraire et par les conditions aux limites $t-$périodiques, périodiques et anti-périodiques, en utilisant la méthode développée dans [59]. Puis, Veliev a appliqué ces résultats aux équations différentielles avec coefficients matriciels périodiques.

Dans [61], Veliev a considéré un operateur differentiel $L(P_2,P_3,...,P_n)\equiv L$, engendré dans l'espace $L_2^m(-\infty,\infty)$, par la forme differentielle

$$l(y)=y^{(n)}(x)+P_2\left(x\right)y^{(n-2)}(x)+P_3\left(x\right)y^{(n-3)}(x)+...+P_n(x)y, \tag{0.5}$$

et un autre operateur différentiel $L_t(P_2, P_3..., P_n) \equiv L_t$, engendré dans $L_2^m(0,1)$, par la même forme différentielle et les conditons aux limites

$$U_{\nu,t}(y) \equiv y^{(\nu)}(1) - e^{it}y^{(\nu)}(0) = 0,\ \nu = 0,1,...,(n-1), \tag{0.6}$$

où $n \geq 2$ et $P_\nu = (p_{\nu,i,j})$ est une matrice $m \times m$ à coefficients complexes intégrables $p_{\nu,i,j}$, telle que $P_\nu(x+1) = P_\nu(x)$, pour $\nu = 2,3,...n$. Veliev a ainsi montré que les valeurs propres $\mu_1, \mu_2, ..., \mu_m$ de la matrice

$$C_2 = \int_0^1 P_2(x)\,dx$$

sont simples et que $y = (y_1, y_2, ..., y_m)$ est une fonction à valeurs vectorielles.

Il est bien connu que le spectre $\sigma(L)$ de L est la réunion des spectres $\sigma(L_t)$ de L_t, pour $t \in [0, 2\pi)$(voir [22], [23], [39]-[41], [46]). Dans le but de déterminer le spectre de L_t, une formule asymptotique est produite pour les valeurs propres et les fonctions propres qui est uniforme par rapport à t dans $Q_\varepsilon(n)$, où

$$Q_\varepsilon(2\mu) = \{t \in Q : |t - \pi k| > \varepsilon, \forall k \in \mathbb{Z}\},\ Q_\varepsilon(2\mu+1) = Q,\ \varepsilon \in (0, \frac{\pi}{4}),\ \mu = 1,2,...,$$

et Q est un sous-ensemble compact de $\mathbb{C}$ contenant un voisinage de l'intervalle $[-\frac{\pi}{2}, 2\pi - \frac{\pi}{2}]$. En utilisant ces formules, il a été montré dans [60] que les fonctions propres et les fonctions associées de L_t pour $t \in \mathbb{C}(n)$ forment une base de Riesz dans l'espace $L_2^m(0,1)$, où $\mathbb{C}(2\mu) = \mathbb{C} \backslash \{\pi k : k \in \mathbb{Z}\}$, $\mathbb{C}(2\mu+1) = \mathbb{C}$.

Dans [64], Veliev a étudié le cas où l'operateur $L(P_2, P_3, ..., P_n)$ est engendré dans l'espace $L_2^m[0,1]$, par la forme différentielle (0.5) et les conditions aux limites periodiques

$$y^{(\nu)}(1) = y^{(\nu)}(0),\ \nu = 0,1,...,(n-1),$$

où n est un entier pair et $P_\nu(x) = (p_{\nu,i,j}(x))$ est une matrice $m \times m$ ayant les coefficients $p_{\nu,i,j}(x)$ à valeurs complexes intégrables pour $v = 2,3...n$. Il a déterminé, dans un premier lieu, les formules asymptotiques pour les valeurs propres et les fonctions propres de L. Ensuite, il a trouvé les conditions nécessaires et suffisantes sur le coefficient $P_2(x)$ pour lesquelles les fonctions propres et les fonctions associées de l'opérateur L forment une base de Riesz dans l'espace $L_2^m[0,1]$. Des résultats similaries ont été obtenus, dans ce même article, pour l'opérateur $A(P_2, P_3, ..., P_n)$ généré par (0.5) et les conditions aux limites anti-périodiques

$$y^{(\nu)}(1) = -y^{(\nu)}(0),\ \nu = 0,1,...,(n-1).$$

A ce stade, notons que la propriété de la base de Riesz pour l'operateur différentiel avec conditions aux limites périodiques et anti-périodiques, a été étudiée dans les articles: [15], [17-20], [24], [28], [34], [37], [38], [42], [53] et [58].

Les travaux de cette thèse portent sur l'étude des propriétés spectrales de l'operateur $L_m(Q)$ généré dans $L_m^2[0,1]$ par l'équation de Sturm-Liouville, dont le potentiel est une matrice $m \times m$ et avec des conditions aux limites qui sont fortement regulières dans le cas scalaire ($m = 1$). Ils concernent aussi bien l'approximation du problème de valeurs propres (0.1)-(0.2) en utilisant la méthode de différences finies.

La compréhension et le développement des méthodes de résolution numérique des problèmes aux limites pour les équations différentielles ordinaires ont augmenté de façon significative au début des années 80. Bien que des développements théoriques et pratiques importantes ont eu lieu sur un certain nombre de fronts, ils n'ont pas été décrit d'une façon compréhensible auparavant. Les études les plus remarquables dans ce domaine sont données par [5] et [27]. Il est clair qu'aucun spécialiste des mathématiques appliquées moderne, physicien ou ingénieur ne peut se passer de l'utilisation des méthodes numériques.

Plusieurs méthode sur l'estimation des valeurs propres pour l'opérateur de Sturm-Liouville, dans le cas scalaire ont été présentées, voir par exemple [1-4], [6], [13], [14], [16], [26], [45], [65]. Cependant, á notre connaisance l'estimation numérique de valeurs propres pour un opérateur différentiel généré par un système d'équations différentielles est étudiée pour la première fois dans cette thèse. Pour cela, la méthode des différences finies a été utilisée pour l'approximation de système. Cette dernière est facile à implémenter comparée à d'autres méthodes numériques.

On considère le problème de valeurs propres régi par l'équation de Sturm-Liouville

$$-y^{''} + q(x)y = \lambda y, \quad 0 \leq x \leq \pi, \tag{0.7}$$

avec conditions aux limites

$$y(0) = y(\pi) = 0. \tag{0.8}$$

une discrétization de ce problème par la méthodee des différences finies sur la partition uniforme de $[0, \pi]$:

$$G = \{x_j; x_j = jh, j = 0, 1, 2, ..., n+1, h = \pi/(n+1)\},$$

permet d'écrire (0.7)-(0.8) sous forme du problème de valeurs propres matriciel suivant

$$(-A + D)\underset{\sim}{u} = \lambda^{(n)}\underset{\sim}{u}, \tag{0.9}$$

(où $D \equiv 0$ si et seulement si $q \equiv 0$). Il est bien connu que les valeurs propres $\lambda_1^{(n)}, \lambda_2^{(n)}, ..., \lambda_n^{(n)}$ du système (0.9) permettent d'avoir des valeurs propres approchées de celles du problème (0.7)-(0.8). Il peuvent, en particulier, donner une approximation de la premiére valeur propre λ_1 et des premierès valeurs $\lambda_2, \lambda_{3,...,}\lambda_m$(avec $m \ll n$). Par exemple, dans le cas où $q \equiv 0$, en utilisant la formule des différences finies centrées pour l'approximation de $-y''$ sur G, on obtient alors les valeurs propres algébraiques (i.e. les valeurs propres de $-A$):

$$4\sin^2(kh/2)/h^2, k = 1, 2, ..., n,$$

et l'erreur correspondante

$$\varepsilon_k^{(n)} = k^2 - 4\sin^2(kh/2)/h^2, k = 1, 2, ..., n,$$

qui satisfait

$$\varepsilon_k^{(n)} = O(k^4h^2).$$

Ce qui illustre clairement la croissance rapide de $\varepsilon_k^{(n)}$ en function de k.

Dans les travaux [2], [4], [6], [45], [65], le problème de valeurs propres de Sturm-Liouville avec conditions aux limites de type Dirichlet a été considéré.

Il a été démontré dans [45], que les valeurs propres approchées du problème (0.7)-(0.8), pour un q général, peuvent être corigées pour obtenir des approximations sensiblement améliorées.

En effet, les valeurs propres de

$$-y'' = \mu y, y(0) = y(\pi) = 0,$$

sont connues ($\mu_k = k^2$), et les valeurs propres algébriques définie par

$$-A\underset{\sim}{u} = \mu^{(n)}\underset{\sim}{u},$$

peuvent être évaluées analytiquement, dans ce cas l'erreur

$$k^2 - \mu_k^{(n)},$$

peut être utilisée pour estimer le comportement asymptotique de $\lambda_k - \lambda_k^{(n)}$ et ainsi générer les approximation corrigées des valeurs propres approchées corrigées

$$\widetilde{\lambda}_k^{(n)} = \lambda_k^{(n)} + k^2 - \mu_k^{(n)}.$$

Il a été prouvé dans [45] que si $q \in C^2[0, \pi]$, alors il existe $\alpha > 0$, indépendant de n, tel que

$$\widetilde{\lambda}_k^{(n)} + \varepsilon_k^{(n)} = \lambda_k^{(n)} + O(kh^2), 1 \leq k \leq \alpha n, \alpha < 1. \tag{0.11}$$

Le même problème de valeurs propres (0.7) avec conditions aux limites générales:

$$\alpha_1 y^{'}(0) - \alpha_2 y(0) = 0,$$
$$\beta_1 y^{'}(\pi) + \beta_2 y(\pi) = 0,$$

a été considéré dans [1], puis Anderssen et Hoog ont montré une relation analogue à celle dans (0.11), tout en utilisant des résultats de [26].

D'autre part, l'étude numérique du problème de valeurs propres périodique et anti periodique a été fait dans [3], [13], [14], [16].

Les travaux de cette thèse sont présentés sous forme de quatre chapitres.

Dans le premier chapitre, nous présentons quelques définitions préliminaires, puis nous donnons quelques résultats très utiles pour aborder le deuxième et le troisème chapitre.

Le deuxième chapitre porte sur l'étude de l'opérateur $L_m(Q)$ défini par (0.1)-(0.2) dans l'espace des fonctions vectorielles $L_2^m[0,1]$. Nous commençons par montrer que si les conditions aux limites (0.3) sont régulierès, alors les conditions aux limites (0.2) sont aussi régulières. Ensuite, nous considérons l'opérateur $L_m(Q)$, comme une perturbation de l' opérateur $L_m(C)$ par $Q - C$, puis nous obtenons une formulation asymptotique pour les valeurs propes et les fonctions propes de $L_m(Q)$ en terms des valeurs propres et des fonctions propres de $L_m(C)$, lorsque C est défini par (0.4). Finalement, en utilisant ces formules asymptotiques et le théorème de Bari (voir [25]), nous montrons que les valeurs propes de la matrice C sont simples, ainsi les fonctions propres et les fonctions associées de l'operateur $L_m(Q)$ forment une base de Riesz. Les résultats du chapitre 2 ont été publiés dans Mathematical Notes (voir [47]).

Le troisème chapitre concerne l'étude numérique du problème par la méthode des différence finies, ceci dans le but d'estimer la plus petite valeur propre de $L_m(Q)$. Pour cela, nous commençons par considérer le problème avec conditions de Dirichlet, ensuite nous considérons le cas des conditions aux limites générales. Ainsi, en utilisant une approximation par différences finies de (0.1) et les conditions aux limites, nous l'écrivons sous forme matricielle, puis nous déterminons les valeurs propres avec des erreurs de l'ordre $O(h^{3/2})$ et $O(h^{1/2})$ respectivement pour chaque cas.

Dans le dernier chapitre, nous donnons quelques résultats numerique qui illustrent les résultats mentionnés dans les deux chapitres précédents.

English Version

We consider the non-self-adjoint differential operator $L_m(Q)$, in the space $L_2^m[0,1]$ generated by the differential expression

$$-\mathbf{y}''(x)+Q(x)\mathbf{y}(x) \tag{0.1}$$

and the boundary conditions

$$U_i(\mathbf{y})=\alpha_i\mathbf{y}^{(k_i)}(0)+\alpha_{i,0}\mathbf{y}(0)+\beta_i\mathbf{y}^{(k_i)}(1)+\beta_{i,0}\mathbf{y}(1)=0,\ \ i=1,2 \tag{0.2}$$

whose scalar case (the case $m=1$)

$$U_i(y)=\alpha_i y^{(k_i)}(0)+\alpha_{i,0}y(0)+\beta_i y^{(k_i)}(1)+\beta_{i,0}y(1)=0,\ \ i=1,2 \tag{0.3}$$

are strongly regular, where $0\leq k_2\leq k_1\leq 1$, $\alpha_i,\alpha_{i,0},\beta_i,\beta_{i,0}$ are complex numbers and for each value of the index i at least one of the numbers α_i,β_i is nonzero. Here, $\mathbf{y}(x)=(y_1(x),y_2(x),...,y_m(x))^T$ and $L_2^m[0,1]$ is the set of vector-functions $\mathbf{f}(x)=(f_1(x),f_2(x),...,f_m(x))$ with $f_k\in L_2[0,1]$ for $k=1,2,...,m$ and $Q(x)=(b_{i,j}(x))$ is a $m\times m$ matrix with the complex-valued square integrable entries $b_{i,j}$. The norm $\|.\|$ and inner product $(.,.)$ in $L_2^m[0,1]$ are defined by

$$\|f\|=\left(\int_0^1|f(x)|^2dx\right)^{\frac{1}{2}},\ (f,g)=\int_0^1\langle f(x),g(x)\rangle dx,$$

where $|.|$ and $\langle .,.\rangle$ are respectively the norm and the inner product in $\mathbb{C}^m$.

Non-self-adjoint differential operators arise in the theory of open resonators, in problems of inelastic scattering and in problems of mathematical physics, when the Fourier method is used. The early works concerned with these operators were investigated in [7]-[11] and [54]-[56], at the beginning of the 20th century. In the scalar case ($m=1$), the strongly regular boundary conditions are the ones which are studied more commonly. If the boundary conditions are strongly regular, then the root functions (eigenfunctions and associated functions) of the operators generated in the space $L_2[0,1]$ by the ordinary differential expression form a Riesz basis. This result was proved independently in [21], [29] and [43]. In the case when an operator is regular but not strongly regular, the root functions generally do not form even usual basis.

A. A. Shkalikov proved that the root functions of the operators generated by an ordinary differential expression with summable matrix coefficients and regular boundary conditions form a Riesz basis with parenthesis and in the parenthesis,only the functions corresponding to splitting eigenvalues should be included. (see [48]-[53]).

L. M. Luzhina generalized this result for the boundary value problems when the coefficients depend on the spectral parameter. (see [32],[33]).

In the paper of Veliev, (see [59]), the differential operator $T_t(Q)$ generated in the space $L_2^m[0,1]$ by the differential expression (0.1) and the quasiperiodic conditions

$$\mathbf{y}'(1) = e^{it}\mathbf{y}'(0)\,,\ \mathbf{y}(1) = e^{it}\mathbf{y}(0)\,,$$

for $t \in (0, 2\pi)$ and $t \neq \pi$ was considered . It was proved that the eigenvalues $\lambda_{k,j}$ of $T_t(Q)$ lie in the $O\left(\frac{\ln k}{k}\right)$ neighborhoods of the eigenvalues of the operator $T_t(C)$, where

$$C = \int_0^1 Q(x)\,dx. \tag{0.4}$$

Note that, to obtain the asymptotic formulas of order $O(\frac{1}{k})$ for the eigenvalues $\lambda_{k,j}$ of the differential operators generated by (0.1), using the classical asymptotic expansions for the solutions of the matrix equation $-Y'' + Q(x)Y = \lambda Y$, it is required that Q be differentiable (see [12], [35], [36], [44]). The suggested method in [59] gives the possibility of obtaining the asymptotic formulas of order $O(k^{-1}\ln|k|)$ for the eigenvalues $\lambda_{k,j}$ and the normalized eigenfunctions $\Psi_{k,j}(x)$ of $T_t(Q)$ when there is not any condition about smoothness of the entries $b_{i,j}$ of Q. Then, in papers [60]-[64], using the method of [59], the spectrum and basis property of the root functions of differential operators generated in $L_2^m[0,1]$ by the differential expression of arbitrary order and by the $t-$periodic, periodic, antiperiodic boundary conditions were considered and these investigations were applied to the differential operators with periodic matrix coefficients.

In [61], the following investigations were done: Let $L(P_2, P_3, ..., P_n) \equiv L$ be the differential operator generated in the space $L_2^m(-\infty, \infty)$ by the differential expression

$$l(y) = y^{(n)}(x) + P_2(x)\,y^{(n-2)}(x) + P_3(x)\,y^{(n-3)}(x) + ... + P_n(x)y, \tag{0.5}$$

and $L_t(P_2, P_3..., P_n) \equiv L_t$ be the differential operator generated in $L_2^m(0,1)$ by the same differential expression and the boundary conditions

$$U_{\nu,t}(y) \equiv y^{(\nu)}(1) - e^{it}y^{(\nu)}(0) = 0,\ \nu = 0, 1, ..., (n-1), \tag{0.6}$$

where $n \geq 2$, $P_\nu = (p_{\nu,i,j})$ is a $m \times m$ matrix with the complex-valued summable entries $p_{\nu,i,j}$, and $P_\nu(x+1) = P_\nu(x)$ for $\nu = 2, 3, ...n$. The eigenvalues $\mu_1, \mu_2, ..., \mu_m$ of the matrix

$$C_2 = \int_0^1 P_2(x)\,dx$$

are simple and $y = (y_1, y_2, ..., y_m)$ is a vector valued function.

It is well-known that the spectrum $\sigma(L)$ of L is the union of the spectra $\sigma(L_t)$ of L_t for $t \in [0, 2\pi)$. (see [22], [23], [39]-[41], [46]). Therefore the investigation of the boundary condition (0.6) depends on this fact. First, an asymptotic formula was derived for the eigenvalues and eigenfunctions of L_t which is uniform with respect to t in $Q_\varepsilon(n)$, where

$$Q_\varepsilon(2\mu) = \{t \in Q : |t - \pi k| > \varepsilon, \forall k \in \mathbb{Z}\},\ Q_\varepsilon(2\mu+1) = Q,\ \varepsilon \in (0, \frac{\pi}{4}),\ \mu = 1, 2, ...,$$

and Q is a compact subset of $\mathbb{C}$ containing a neighborhood of the interval $[-\frac{\pi}{2}, 2\pi - \frac{\pi}{2}]$. Using these formulas, it was proved that the root functions of L_t for $t \in \mathbb{C}(n)$ form a Riesz basis in $L_2^m(0,1)$, where $\mathbb{C}(2\mu) = \mathbb{C} \backslash \{\pi k : k \in \mathbb{Z}\}$, $\mathbb{C}(2\mu + 1) = \mathbb{C}$.

In [64], the operator $L(P_2, P_3, ..., P_n)$ generated in $L_2^m[0,1]$ by the differential expression (0.5) and the periodic boundary conditions

$$y^{(\nu)}(1) = y^{(\nu)}(0),\ \nu = 0, 1, ..., (n-1),$$

where n is an even integer, $P_\nu(x) = (p_{\nu,i,j}(x))$ is a $m \times m$ matrix with the complex-valued summable entries $p_{\nu,i,j}(x)$ for $\nu = 2, 3, ...n$ was investigated. First, asymptotic formulas were obtained for the eigenvalues and eigenfunctions of L. Then, necessary and sufficient conditions were found on the coefficient $P_2(x)$ for which the root functions of the operator L form a Riesz basis in $L_2^m[0,1]$. The similar results were obtained for the operator $A(P_2, P_3, ..., P_n)$ generated by (0.5) and the antiperiodic boundary conditions

$$y^{(\nu)}(1) = -y^{(\nu)}(0),\ \nu = 0, 1, ..., (n-1).$$

Note that the Riesz basis property of the differential operator with periodic and antiperiodic boundary conditions was investigated in the papers; [15], [17-20], [24], [28], [34], [37], [38], [42], [53] and [58].

In this thesis, we are interested in the investigation of spectral properties of non-self-adjoint operator $L_m(Q)$ generated in $L_2^m[0,1]$ by the differential expression (0.1) and the boundary conditions (0.2). First we obtain asymptotic formulas for the eigenvalues and eigenfunctions of $L_m(Q)$ and then find a condition on the potential for which the root functions of the operator form a Riesz basis. Besides the spectral properties of the operator $L_m(Q)$ with strongly regular boundary conditions, we also study the eigenvalue problem for (0.1)-(0.2) by the numerical methods.

The knowledge and understanding of methods for the numerical solution of boundary value problems for ordinary differential equations has increased significantly at the beginning of 1980s. Although important theoretical and practical developments have taken place on a number of fronts, they have not previously been comprehensively described in any text. The most remarkable studies in this area are given by [5] and [27]. It is clear that no modern applied mathematician, physical scientist or engineer can be properly trained without some understanding of numerical methods.

There are a lot of methods and a lot of papers about estimation of the eigenvalues for Sturm-Liouville operator in scalar case, for instance see [1-4], [6], [13], [14], [16], [26], [45], [65] and references on them. However, to my best knowledge, the numerical estimation of the eigenvalues for the differential operator generated by a system of differential equation is investigated in this thesis for the first time. Comparing with numerical methods, it seems finite difference methods are more uniform and admit a more unified theory since the problem can be expressed by matrix form after applying finite difference approximations.

Let us consider the Sturm-Liouville eigenvalue problem

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{0.7}$$

with boundary conditions

$$y(0) = y(\pi) = 0. \tag{0.8}$$

If finite difference approximations were used on a grid

$$G = \{x_j; x_j = jh, j = 0, 1, 2, ..., n+1, h = \pi/(n+1)\},$$

then, the problem (0.7)-(0.8) replaces by an algebraic eigenvalue problem of order n such that

$$(-A + D)\underset{\sim}{u} = \lambda^{(n)}\underset{\sim}{u}, \tag{0.9}$$

(viz. where $D \equiv 0$ if and only if $q \equiv 0$). It is well known that algebraic eigenvalues $\lambda_1^{(n)}, \lambda_2^{(n)}, ..., \lambda_n^{(n)}$ of (0.9) only yield satisfactory approximations for the fundamental eigenvalues of (0.7)-(0.8), i.e., λ_1 and the first few harmonics $\lambda_2, \lambda_{3,...,}\lambda_m (m \ll n)$. For example, if $q \equiv 0$ and a central difference formula is used to approximate $-y''$ on G, then the corresponding algebraic eigenvalues (i.e. the algebraic eigenvalues of $-A$) are given by

$$4\sin^2(kh/2)/h^2, k = 1, 2, ..., n,$$

while the corresponding error is

$$\varepsilon_k^{(n)} = k^2 - 4\sin^2(kh/2)/h^2, k = 1, 2, ..., n, \tag{0.10}$$

which satisfies

$$\varepsilon_k^{(n)} = O(k^4h^2).$$

This clearly illustrates the rapid growth of $\varepsilon_k^{(n)}$ as a function of k.

In the papers [2], [4], [6], [45], [65], Sturm-Liouville eigenvalue problem with Dirichlet boundary conditions was considered.

In [45] , it was shown how approximate algebraic eigenvalues $\lambda_k^{(n)}$ derived for (0.7)-(0.8), for general q can be corrected to yield substantially improved approximations. The eigenvalues of

$$-y^{''} = \mu y, y(0) = y(\pi) = 0,$$

are known($\mu_k = k^2$), and the algebraic eigenvalues defined by

$$-A\underset{\sim}{u} = \mu^{(n)}\underset{\sim}{u},$$

can be evaluated analytically, the error

$$k^2 - \mu_k^{(n)},$$

can be used to estimate the asymptotic behaviour of $\lambda_k - \lambda_k^{(n)}$, and thereby generate the corrected eigenvalue approximations

$$\widetilde{\lambda}_k^{(n)} = \lambda_k^{(n)} + k^2 - \mu_k^{(n)}.$$

It was proved in [45] that when $q \in C^2[0, \pi]$, there exists an α, independent of n, such that

$$\widetilde{\lambda}_k^{(n)} + \varepsilon_k^{(n)} = \lambda_k^{(n)} + O(kh^2), 1 \le k \le \alpha n, \alpha < 1. \tag{0.11}$$

In [1], using the results in [26], the same eigenvalue problem (0.7) was considered with the general boundary conditions

$$\alpha_1 y^{'}(0) - \alpha_2 y(0) = 0,$$
$$\beta_1 y^{'}(\pi) + \beta_2 y(\pi) = 0,$$

and the same result was proved as in (0.11).

Besides, in the papers [3], [13], [14], [16], periodic and antiperiodic eigenvalue problem were investigated by numerical method.

The thesis consists of four chapters. The first chapter presents preliminary definitions and formulations of some results to be used in Chapter 2 and Chapter 3.

In Chapter 2, we investigate the operator $L_m(Q)$ defined by (0.1)-(0.2) in the space of vector functions. First we prove that if the boundary conditions (0.3) are regular, then the boundary conditions (0.2) are also regular. Then, we consider the operator $L_m(Q)$ as a perturbation of $L_m(C)$ by $Q - C$ and obtain asymptotic formulas for the eigenvalues and eigenfunctions of $L_m(Q)$ in term of the eigenvalues and eigenfunctions of $L_m(C),$ where C is defined in (0.4). Finally, using the obtained asymptotic formulas and the theorem of Bari (see, [25]), we prove that if the eigenvalues of the matrix C are simple, then the root functions of the operator $L_m(Q)$ form a Riesz basis. These results are published in Mathematical Notes, see [47].

In Chapter 3, we investigate the numerical estimation of small eigenvalues of $L_m(Q)$ by finite difference method. First, we consider Dirichlet boundary conditions and then we consider general separated boundary conditions. Applying finite difference approximations to (0.1) and boundary conditions, we express the problem in matrix form. Then we find the errors in each case as $O(h^{3/2})$ and $O(h^{1/2})$ respectively.

In Chapter 4, we give some examples which summarize the notions mentioned in Chapter 2 and Chapter 3.

1

Preliminary Facts

1.1 Strongly Regular Boundary Conditions in Scalar Case

Consider the $n-th$ order linear ordinary differential expression

$$l(y) = y^{(n)} + p_1(x)y^{(n-1)} + ... + p_n(x)y, \tag{1.1}$$

given on the interval $[0,1]$. The functions $p_1(x), p_2(x), ..., p_n(x)$ are called the coefficients of the differential expression. The coefficients $p_s(x)$ will be assumed Lebesgue integrable and complex valued functions on $[0,1]$ where $s = 1, 2, ..., n$.

Let $B(y)$ be a linear form in the variable $y_a, y_a^{'}, ..., y_a^{(n-1)}, y_b, y_b^{'}, ..., y_b^{(n-1)}$ at the boundary points a and b of the interval $[a,b]$, that is,

$$B(y) = \alpha_0 y_a + \alpha_1 y_a' + ... + \alpha_{n-1} y_a^{(n-1)} + \beta_0 y_b + \beta_1 y_b' + ... + \beta_{n-1} y_b^{(n-1)}, \tag{1.2}$$

where $y_a^{(k)} = y^{(k)}(a)$ and $y_b^{(k)} = y^{(k)}(b)$,for $k = 0, 1, 2, ..., n-1$. If $B_1(y)$, $B_2(y), ..., B_n(y)$ are independent linear forms then the conditions

$$B_v(y) = 0; v = 1, 2, ..., n, \tag{1.3}$$

are called homogeneous boundary conditions.

Definition 1.1.1 *Let $D(L)$ be subspace of $L_2[0,1]$ defined by*

$$D(L) = \{y \in L_2[0,1] : \exists y^{(n-1)} \in AC[0,1], l(y) \in L_2[0,1], B_v(y) = 0, v = 1, 2, ..., n\}$$

where $AC[0,1]$ is the set of absolutely continuous functions on $[0,1]$. We say that operator L is generated by the differential expression $l(y)$ and the boundary conditions (1.3) if $Ly = l(y)$ for $y \in D(L)$.

The problem of determining a function $y \in D(L)$ which satisfies the conditions

$$\begin{aligned} l(y) &= 0, \\ B_v(y) &= 0,\ v = 1, 2, ..., n, \end{aligned} \tag{1.4}$$

is called the homogeneous boundary value problem.

A number λ is called an eigenvalue of an operator L if there exists a function $y \neq 0$ in the domain of definition of the operator L such that $Ly = \lambda y$. The function y is called the eigenfunction of the operator L for the eigenvalue λ.

It can be shown in an elementary fashion that there exists a set of linearly independent solutions which are entire in the parameter λ. Let the elements of this set be $y_1, y_2, ..., y_n$.

The eigenvalues of the operator L are determined by the zeros of the characteristic determinant $\Delta(\lambda)$, which has the form

$$\Delta(\lambda) = \begin{vmatrix} B_1(y_1) & ... & B_1(y_n) \\ . & ... & . \\ B_n(y_1) & ... & B_n(y_n) \end{vmatrix}.$$

If $\Delta(\lambda)$ vanishes identically, then any number λ is an eigenvalue of the operator L.

An eigenvalue λ may be multiple zero of $\Delta(\lambda)$. In this case we have the following definition:

Definition 1.1.2 *An eigenvalue λ_0 of the boundary value problem (1.4) is said to have multiplicity p if λ_0 is root of multiplicity p of the function $\Delta(\lambda)$. An eigenvalue λ_0 of (1.4) is called simple if λ_0 is a simple zero of the characteristic determinant $\Delta(\lambda)$.*

There is also one more notion called associated function. Denote by

$$\varphi_{n,0}(x) \equiv \varphi_n(x)$$

the eigenfunction of the operator L corresponding to the eigenvalue λ_n. The function $\varphi_{n,p}(x)$ of the operator L, for $p = 1, 2, ..., m_p$, is said to be associated function of order p corresponding to the same eigenvalue λ_n and the eigenfunction $\varphi_{n,0}(x)$ if all the functions $\varphi_{n,p}(x)$ satisfy the following equations

$$\begin{aligned} (L - \lambda_n)\varphi_n(x) &= 0, \\ (L - \lambda_n)\varphi_{n,p}(x) &= \varphi_{n,p-1}(x),\ p = 1, 2, ..., m_p, \end{aligned}$$

where m_p is called the length of the system of associated functions.

The set of all eigenfunctions and associated functions is called root functions.

To define adjoint operator L^*, first we need to present the definition of adjoint differential expression and adjoint boundary conditions. To do this, first we shall define the Lagrange's formula:

Assume that the coefficients $p_k(x), k = 0, 1, 2, ..., n$ of the differential expression

$$l(y) = p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{(n-1)}y}{dx^{n-1}} + ... + p_n(x)y,$$

have continuous derivatives up to the order $(n-k)$ inclusive on the interval $[a, b]$. Further let y and z be two arbitrary functions in $\mathbb{C}^{(n)}$. By k partial integrations we get

$$\int_a^b p_{n-k}\overline{z}y^{(k)}dx = [p_{n-k}\overline{z}y^{(k-1)} - (p_{n-k}\overline{z})'y^{(k-2)} + ... \\ + (-1)^{k-1}(p_{n-k}\overline{z})^{(k-1)}y]_{x=a}^{x=b} + (-1)^k \int_a^b y(p_{n-k}\overline{z})^{(k)}dx. \tag{1.5}$$

Here $\overline{z}$ denotes the complex number conjugate to z, and $\overline{z} = \overline{z(x)}$ denotes the function whose values are conjugate to those of $z(x)$. If we put $k = n, n-1, ..., 0$ in (1.5) and add the resulting equations we obtain the formula

$$\int_a^b l(y)\overline{z}dx = P(\eta, \zeta) + \int y\overline{l^*(z)}dz, \tag{1.6}$$

where

$$l^*(z) = (-1)^n(\overline{p}_0 z)^{(n)} + (-1)^{n-1}(\overline{p}_1 z)^{(n-1)} + (-1)^{n-2}(\overline{p}_2 z)^{(n-2)} + ... + \overline{p}_n z, \tag{1.7}$$

and $P(\eta, \zeta)$ is a certain bilinear form in the variables

$$\eta = y_a, y_a', ..., y_a^{(n-1)}, y_b, y_b', ..., y_b^{(n-1)},$$
$$\zeta = z_a, z_a', ..., z_a^{(n-1)}, z_b, z_b', ..., z_b^{(n-1)}.$$

The differential expression $l^*(z)$ defined by the formula (1.7) is called the adjoint differential expression of $l(y)$, and (1.6) is called Lagrange's formula. A differential expression $l(y)$ is said to be self adjoint if $l = l^*$.

Now, let $B_1, B_2, ..., B_m$ be linearly independent forms in the variables

$$y_a, y_a', ..., y_a^{(n-1)}, y_b, y_b', ..., y_b^{(n-1)}$$

if $m < 2n$, we shall supplement them with other forms $B_{m+1}, ..., B_{2n}$ to obtain a linearly independent system of $2n$ forms $B_1, B_2, ..., B_{2n}$. Since these forms are linearly independent, the variables $y_a, y'_a, ..., y_a^{(n-1)}, y_b, y'_b, ..., y_b^{(n-1)}$ can be expressed as linear combinations of the forms $B_1, B_2, ..., B_{2n}$.

We substitute these expressions in the bilinear form $P(\eta, \zeta)$ which occured in Lagrange's formula.(see (1.6)). Then $P(\eta, \zeta)$ becomes a linear, homogeneous form in the variables $B_1, B_2, ..., B_{2n}$, and its $2n$ coefficients are themselves linear, homogeneous forms, which we denote by $V_{2n}, V_{2n-1}, ..., V_1$, in the variables $z_a, z'_a, ..., z_a^{(n-1)}, z_b, z'_b, ..., z_b^{(n-1)}$. Lagrange's formula now takes the form

$$\int_a^b l(y)\overline{z}dx = B_1V_{2n} + B_2V_{2n-1} + ... + B_{2n}V_1 + \int_a^b y\overline{l^*(z)}dx.$$

The forms $V_1, V_2, ..., V_{2n}$ are linearly independent. Therefore, the boundary conditions

$$V_1 = 0, V_2 = 0, ..., V_{2n-m} = 0, \tag{1.8}$$

(and all boundary conditions equivalent to them) are said to be adjoint to the original boundary conditions

$$B_1 = 0, B_2 = 0, ..., B_m = 0. \tag{1.9}$$

Boundary conditions are self adjoint if they are equivalent to their adjoint boundary conditions.

Definition 1.1.3 *Let L be the operator generated by the expression $l(y)$ and the boundary conditions (1.9). The operator generated by $l^*(y)$ and the boundary conditions (1.8) will be denoted by L^* and called the adjoint operator to L.*

Now let us define the process for boundary conditions, called normalization; before the definition of regular boundary conditions.

We wish to investigate the different systems $B_v(y)$, $v = 1, 2, ..., n$, of linear forms which are defined by a given differential operator. If $y_0^{(k)}$ or $y_1^{(k)}$ appear explicitly in the form $B(y)$ but $y_0^{(v)}$ and $y_1^{(v)}$ do not, for any $v > k$, then we say that the form $B(y)$ has order k. We consider the forms $B_v(y)$ of order $(n-1)$, if there are any. By replacing them, if necessary, by equivalent linear combinations, we can arrange that the maximum number of forms of order $(n-1)$ is ≤ 2. The remaining forms have orders $\leq (n-2)$; we apply the same process to the forms of order $(n-2)$ and reduce their number to a minimum; and so on.

The operations described are called the normalization of the boundary conditions, and the boundary conditions are said to be normalized. From the way in which they are constructed it follows that the normalized boundary conditions must have the form

$$B_v(y) =: B_{v0(y)} + B_{v1(y)} = 0,$$

where

$$B_{v0(y)} = \alpha_v y_0^{(k_v)} + \sum_{j=0}^{k_v-1} \alpha_{vj} y_0^{(j)},$$

$$B_{v_1}(y) = \beta_v y_1^{(k_v)} + \sum_{j=0}^{k_v-1} \beta_{vj} y_1^{(j)},$$

$$n-1 \geq k_1 \geq k_2 \geq ... \geq k_n \geq 0,\ k_{v+2} < k_v,$$

and for each value of the suffix v at least one of the numbers α_v, β_v is non-zero.

Now we are ready to define regular boundary conditions:

Definition 1.1.4 *Suppose n is even. The normalized boundary conditions are said to be regular if the numbers θ_{-1} and θ_1 defined by the identity*

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s =$$

$$\begin{vmatrix} \alpha_1\omega_1^{k_1} & ... & \alpha_1\omega_{\mu-1}^{k_1} & (\alpha_1 + s\beta_1)\omega_\mu^{k_1} & (\alpha_1 + \frac{1}{s}\beta_1)\omega_{\mu+1}^{k_1} & \beta_1\omega_{\mu+2}^{k_1} & ... & \beta_1\omega_n^{k_1} \\ \alpha_2\omega_1^{k_2} & ... & \alpha_2\omega_{\mu-1}^{k_2} & (\alpha_2 + s\beta_2)\omega_\mu^{k_2} & (\alpha_2 + \frac{1}{s}\beta_2)\omega_{\mu+1}^{k_2} & \beta_2\omega_{\mu+2}^{k_2} & ... & \beta_2\omega_n^{k_2} \\ . & ... & . & . & . & . & ... & . \\ \alpha_n\omega_1^{k_n} & ... & \alpha_n\omega_{\mu-1}^{k_n} & (\alpha_n + s\beta_n)\omega_\mu^{k_n} & (\alpha_n + \frac{1}{s}\beta_n)\omega_{\mu+1}^{k_n} & \beta_n\omega_{\mu+2}^{k_n} & ... & \beta_n\omega_n^{k_n} \end{vmatrix}$$

are different from zero. Here $\omega_1, \omega_2, ..., \omega_n$ are different $n-$ th roots of -1 arranged in an order in each case to suit later requirements.

Remark 1.1.1 *Note that, in the case $n = 2$, the determinant in the Definition 1.1.4 has the form*

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s = \det \begin{bmatrix} (\alpha_1 + s\beta_1)\omega_1^{k_1} & (\alpha_1 + \frac{1}{s}\beta_1)\omega_2^{k_1} \\ (\alpha_2 + s\beta_2)\omega_1^{k_2} & (\alpha_2 + \frac{1}{s}\beta_2)\omega_2^{k_2} \end{bmatrix},$$

where $\omega_1 = i$ and $\omega_2 = -i$.

Definition 1.1.5 *The boundary conditions are said to be strongly regular if*

$$\theta_0^2 - 4\theta_1\theta_{-1} \neq 0.$$

Theorem 1.1.1 *A differential operator of the $n-th$ order which is generated by an expression and by regular boundary conditions has precisely denumerably many eigenvalues, whose behaviour at infinity is specified for even $n = 2\mu$, and $\theta_0^2 - 4\theta_1\theta_{-1} \neq 0$ by the following formulae:*

$$\lambda_k' = (-1)^\mu (2k\pi)^{2\mu} \left\{ 1 \mp \frac{\mu \ln_0 \xi'}{k\pi i} + O(\frac{1}{k^2}) \right\},$$

$$\lambda_k'' = (-1)^\mu (2k\pi)^{2\mu} \left\{ 1 \mp \frac{\mu \ln_0 \xi''}{k\pi i} + O(\frac{1}{k^2}) \right\},$$

where ξ' and ξ'' are the roots of the equation

$$\theta_1\xi^2 + \theta_0\xi + \theta_{-1} = 0$$

where the upper or lower sign is to be taken according as $n = 4v$ or $n = 4v + 2$.

For even n, $n = 2\mu$, and $\theta_0^2 - 4\theta_1\theta_{-1} = 0$, the following sequences are obtained:

$$\begin{aligned}
\lambda'_k &= (-1)^\mu (2k\pi)^{2\mu} \left\{ 1 \mp \frac{\mu \ln_0 \xi}{k\pi i} + O(\frac{1}{k^{3/2}}) \right\}, \\
\lambda''_k &= (-1)^\mu (2k\pi)^{2\mu} \left\{ 1 \mp \frac{\mu \ln_0 \xi}{k\pi i} + O(\frac{1}{k^{3/2}}) \right\}, \\
k &= N, N+1, ...,
\end{aligned}$$

where ξ is the double root, occuring in this case. The signs are to be chosen in the same way.

In the first case, all eigenvalues of sufficiently large modulus are simple; but in the second case all eigenvalues of sufficiently large modulus can be either simple or double.

Remark 1.1.2 *It follows from Theorem 1.1.1 that the boundary conditions are strongly regular if and only if $\xi' \neq \xi''$, that is, the eigenvalues are far from each other. (see [49]).*

1.2 On Sturm-Liouville Operators

In this section, we consider Sturm-Liouville operators generated by the most general boundary conditions which have the form

$$\begin{aligned}
B_1 &=: a_1 y'_0 + b_1 y'_1 + a_0 y_0 + b_0 y_1 = 0, \\
B_2 &=: c_1 y'_0 + d_1 y'_1 + c_0 y_0 + d_0 y_1 = 0.
\end{aligned} \tag{1.10}$$

Proposition 1.2.1 *In the following three cases the boundary conditions (1.10) are strongly regular:*

(a) $a_1 d_1 - b_1 c_1 \neq 0$,

(b) $a_1 d_1 - b_1 c_1 = 0, |a_1| + |b_1| > 0, b_1 c_0 + a_1 d_0 \neq 0, a_1 \neq \pm b_1$ *and* $c_0 \neq \pm d_0$,

(c) $a_1 = b_1 = c_1 = d_1 = 0, a_0 d_0 - b_0 c_0 \neq 0$.

Proof. (a) $a_1 d_1 - b_1 c_1 \neq 0$

By solving (1.10) for y'_0 and y'_1, with the condition $a_1 d_1 - b_1 c_1 \neq 0$, we have the boundary conditions in the form

$$\begin{aligned}
y'_0 + a_{11} y_0 + a_{12} y_1 = 0, \\
y'_1 + a_{21} y_0 + a_{22} y_1 = 0.
\end{aligned} \tag{1.11}$$

Here, $\alpha_1 = 1, \alpha_2 = 0, \beta_1 = 0$ and $\beta_2 = 1$, so, by Remark 1.1.1 we have,

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s = \begin{vmatrix} i & -i \\ si & -\frac{1}{s}i \end{vmatrix} = \frac{1}{s} - s.$$

It is clear that the boundary conditions are regular since $\theta_1 = -1$ and $\theta_{-1} = 1$. Moreover, since $\theta_0^2 - 4\theta_1\theta_{-1} = 0 - 4\cdot(-1)\cdot 1 = 4 \neq 0$, the boundary conditions are strongly regular.

(b) $a_1d_1 - b_1c_1 = 0, |a_1| + |b_1| > 0$

In this case we can transform the conditions (1.10) to

$$\begin{aligned} a_1y^{'}(0) + b_1y^{'}(1) + a_0y(0) + b_0y(1) = 0, \\ c_0y(0) + d_0y(1) = 0. \end{aligned} \tag{1.12}$$

Hence,

$$\begin{aligned} \frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s &= \begin{vmatrix} (a_1 + sb_1)i & -(a_1 + \frac{1}{s}b_1)i \\ (c_0 + sd_0) & c_0 + \frac{1}{s}d_0 \end{vmatrix} \\ &= i(b_1c_0 + a_1d_0)\left(s + \frac{1}{s}\right) + 2(a_1c_0 + b_1d_0)i. \end{aligned}$$

which implies that

$$\theta_0 = 2(a_1c_0 + b_1d_0)i, \ \ \theta_1 = \theta_{-1} = i(b_1c_0 + a_1d_0)$$

We see that the boundary conditions are regular if $b_1c_0 + a_1d_0 \neq 0$. Moreover, the conditions are strongly regular if

$$\theta_0^2 - 4\theta_1\theta_{-1} = -4(a_1c_0 + b_1d_0)^2 + 4(b_1c_0 + a_1d_0)^2 = (a_1^2 - b_1^2)(c_0^2 - d_0^2) \neq 0,$$

that is, the conditions $a_1 \neq \pm b_1$ and $c_0 \neq \pm d_0$ hold.

(c) $a_1 = b_1 = c_1 = d_1 = 0$

In this case the boundary conditions are in the form

$$\begin{aligned} a_0y_0 + b_0y_1 = 0, \\ c_0y_0 + d_0y_1 = 0. \end{aligned} \tag{1.13}$$

Hence,

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s = \begin{vmatrix} a_0 + sb_0 & a_0 + \frac{1}{s}b_0 \\ c_0 + sd_0 & c_0 + \frac{1}{s}d_0 \end{vmatrix} = (a_0d_0 - b_0c_0)(s - \frac{1}{s})$$

Therefore the boundary conditions are regular if $a_0d_0 - b_0c_0 \neq 0$. Since

$$\theta_0^2 - 4\theta_1\theta_{-1} = 4(a_0d_0 - b_0c_0)^2 \neq 0,$$

the boundary conditions are also strongly regular. ■

Besides the regular boundary conditions can be classified and investigated in following forms (see, [31]). To do this, let

$$A = \begin{bmatrix} a_1 & b_1 & a_0 & b_0 \\ c_1 & d_1 & c_0 & d_0 \end{bmatrix},$$

be the coefficient matrix associated with B_1, B_2. For integers i, j with $1 \leq i \leq j \leq 4$, let $A(ij)$ denote the 2×2 submatrix of A obtained by retaining the $i-$th and $j-$th columns and let

$$A_{ij} = \det A(ij).$$

In [30], it was proved that A_{ij} satisfies the following fundamental quadratic equation

$$A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23} = 0.$$

In terms of the A_{ij}, the self adjoint of L is characterized by the following theorem:

Theorem 1.2.1 *$L = L^*$ iff there exists a complex number $\gamma \neq 0$ such that*

$$\begin{aligned} \overline{A_{12}} &= \gamma A_{12}, & \overline{A_{23}} &= \gamma A_{23}, \\ \overline{A_{24}} &= \gamma A_{24}, & \overline{A_{13}} &= \gamma A_{13}, \\ \overline{A_{14}} &= \gamma A_{14}, & \overline{A_{34}} &= \gamma A_{34}. \end{aligned}$$

Moreover, the characteristic determinant is given by

$$\begin{aligned} \Delta(\rho) = &-[A_{12}\rho^2 - i(A_{14} + A_{23})\rho + A_{34}]e^{i\rho} \\ &+ [A_{12}\rho^2 + i(A_{14} + A_{23})\rho + A_{34}]e^{-i\rho} + 2i[A_{13} + A_{24}]\rho. \end{aligned}$$

Now we present the theorems which are useful to calculate the eigenvalues:

Theorem 1.2.2 *The point $\lambda_0 = \rho_0^2 \neq 0$ is an eigenvalue of L iff the point $\rho_0 \neq 0$ is a zero of Δ, in which case the algebraic multiplicity of λ_0 is equal to the order of ρ_0 as a zero of Δ. Moreover, $\lambda_0 = 0$ is an eigenvalue of L iff*

$$A_{34} - (A_{14} + A_{23}) - (A_{13} + A_{24}) = 0.$$

Note that $\Delta(-\rho) = -\Delta(\rho)$, so ρ_0 is a zero of Δ iff -ρ_0 is a zero of Δ. Also, when $A_{12} = 0$ and $A_{14} + A_{23} \neq 0$, then associated with Δ, the quadratic polynomial is defined as

$$Q(z) = i(A_{14} + A_{23})z^2 + 2i(A_{13} + A_{24})z + i(A_{14} + A_{23}),$$

and Q has two distinct roots iff $A_{14} + A_{23} \neq \mp(A_{13} + A_{24})$.

Theorem 1.2.3 *Let $\xi_0 \neq 0$ and $\eta_0 \neq 0$ be constants with $\xi_0 \neq \eta_0$, let*

$$f(\rho) = [e^{i\rho} - \xi_0][e^{i\rho} - \eta_0],$$

and let

$$h(\rho) = [e^{i\rho} - \xi_0]^2.$$

Then

(a) the zeros of f are given by the two sequences

$$\mu'_k = (Arg\xi_0 + 2k\pi) - i\ln|\xi_0|\,, k = 0, \pm1, \pm2, ...,$$
$$\mu''_k = (Arg\eta_0 + 2k\pi) - i\ln|\eta_0|\,, k = 0, \pm1, \pm2, ...,$$

where $\mu'_k \neq \mu''_l$ for all k, l and each μ''_k is a zero of order 1 of f.

(b) the zeros of h are given by the sequence

$$\mu_k = (Arg\xi_0 + 2k\pi) - i\ln|\xi_0|\,, k = 0, \pm1, \pm2, ...,$$

where each μ_k is a zero of order 2 of h.

In addition, if

$$g(\rho) = \frac{1}{\rho}[A_2e^{2i\rho} + A_1e^{i\rho} + A_0] + \frac{1}{\rho^2}[B_2e^{2i\rho} + B_1e^{i\rho} + B_0],$$

where the A_i, B_i are constants and if $F = f + g$,then

(c) the zeros of $F$are given by two sequences

$$\rho'_k = \mu'_k + \varepsilon'_k, |\varepsilon'_k| \leq \frac{\gamma}{|k|}, k = \pm k_0, \pm(k_0 + 1),$$
$$\rho''_k = \mu''_k + \varepsilon''_k, |\varepsilon''_k| \leq \frac{\gamma}{|k|}, k = \pm k_0, \pm(k_0 + 1),$$

plus a finite number of additional zeros, where $\gamma > 0$ is a constant and k_0 is a positive integer, $\rho'_k \neq \rho'_l$ for all k, l and each ρ'_k and each ρ''_k is a zero of order 1 of F.

Now, let us present the cases related to strongly regular boundary conditions in terms of the notation used in [31].

Case 1: The differential operator L belongs to Case 1 provided

$$A_{12} \neq 0 \text{ and } A_{13} = A_{14} = A_{23} = A_{24} = A_{34} = 0.$$

In this case the characteristic determinant is given by

$$\Delta(\rho) = -A_{12}\rho^2e^{i\rho}[e^{i\rho} - 1][e^{i\rho} + 1],$$

and the nonzero zeros of Δ are clearly determined by the function

$$f(\rho) = [e^{i\rho} - 1][e^{i\rho} + 1].$$

Applying Theorem 1.2.3(a) with $\xi_0 = 1$ and $\eta_0 = -1$, we see that the nonzero zeros of Δ are precisely

$$\rho_k = k\pi, k = \pm 1, \pm 2, ...$$

From Theorem 1.2.2 it follows that the nonzero eigenvalues of L are

$$\lambda_k = (k\pi)^2, k = 1, 2, ...$$

Finally, in Case 1 there is only one possible normalized form for the coefficient matrix A, namely

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

which corresponds to Neumann boundary conditions and L is always self adjoint.

Case 2: The principal strategy for studying Case 2 is to treat it as a perturbation of Case 1. For the differential operator L to belong to Case 2, it must satisfy the conditions

$$A_{12} \neq 0 \text{ and } A_{13}, A_{14}, A_{23}, A_{24}, A_{34} \text{ are not all zero.}$$

The characteristic determinant for this case is given by

$$\begin{aligned}\Delta(\rho) &= -A_{12}\rho^2 e^{-i\rho}[e^{i\rho} - 1][e^{i\rho} + 1] \\ &\quad + A_{12}\rho^2 e^{-i\rho}\frac{1}{A_{12}\rho}\left[i(A_{14} + A_{23})e^{2i\rho} + 2i(A_{13} + A_{24})e^{i\rho} + i(A_{14} + A_{23})\right] \\ &\quad - A_{12}\rho^2 e^{-i\rho}\frac{A_{34}}{A_{12}\rho^2}[e^{2i\rho} - 1],\end{aligned}$$

and the nonzero zeros of Δ are clearly determined by the function

$$F(\rho) = f(\rho) + g(\rho),$$

where

$$f(\rho) = [e^{i\rho} - 1][e^{i\rho} + 1],$$

and

$$g(\rho) = -\frac{1}{A_{12}\rho}\left[i(A_{14} + A_{23})e^{2i\rho} + 2i(A_{13} + A_{24})e^{i\rho} + i(A_{14} + A_{23})\right] + \frac{A_{34}}{A_{12}\rho^2}[e^{2i\rho} - 1].$$

Using Theorem 1.2.3(c), with $\xi_0 = 1$ and $\eta_0 = -1$, the nonzero zeros of Δ are given by a sequence

$$\rho_k = k\pi + \varepsilon_k \text{ with } |\varepsilon_k| \leq \frac{\gamma}{|k|}, k = \pm k_0, \pm(k_0 + 1), ...,$$

plus a finite number of additional zeros, where $\gamma > 0$ is a constant and k_0 is a positive integer and where each ρ_k is a zero of order 1 of Δ. It follows that the eigenvalues of L are given by the sequence

$$\lambda_k = \rho_k^2, k = k_0, k_0+1, ...,$$

plus a finite number of additional eigenvalues. The coefficient matrix A can have only one possible normalized form in Case 2, namely,

$$A = \begin{bmatrix} 1 & 0 & a_0 & b_0 \\ 0 & 1 & c_0 & d_0 \end{bmatrix},$$

with a_0, b_0, c_0, d_0 not all zero. For the normalized form, L is self adjoint iff $-\overline{b}_0 = c_0, \overline{d}_0 = d_0$ and $\overline{a}_0 = a_0$.

Case 3: It is possible to explicitly calculate all the spectral quantities, although some of the calculations are quite complicated. To belong to Case 3, the differential operator L must satisfy the conditions

$$A_{12} = 0, A_{14} + A_{23} \neq 0, A_{14} + A_{23} \neq \mp(A_{13} + A_{24}), A_{34} = 0.$$

The characteristic determinant for this case is given by

$$\Delta(\rho) = i(A_{14} + A_{23})\rho e^{-i\rho}[e^{i\rho} - \xi_0][e^{i\rho} - \eta_0],$$

where ξ_0, η_0 are the roots of the quadratic polynomial Q. Clearly $\xi_0\eta_0 = 1$, while $\xi_0 \neq \eta_0$ by the third condition, i.e. $A_{34} = 0$ and hence, $\xi_0 \neq \pm 1$ and $\eta_0 \neq \pm 1$. Also, the nonzero zeros of Δ are clearly determined by the function

$$f(\rho) = [e^{i\rho} - \xi_0][e^{i\rho} - \eta_0],$$

whose zeros are all unequal to 0 because $\xi_0 \neq 1$ and $\eta_0 \neq 1$.

Applying Theorem 1.2.3(a), we see that the nonzero zeros of Δ are given precisely by the two sequences

$$\begin{aligned} \rho_k &= (Arg\xi_0 + 2k\pi) - i\ln|\xi_0|, \quad k = 0, \pm 1, \pm 2, ..., \qquad (*) \\ \zeta_k &= (-Arg\xi_0 + 2k\pi) + i\ln|\xi_0|, \quad k = 0, \pm 1, \pm 2, ..., \end{aligned}$$

where $\rho_k \neq \zeta_l$ for all k, l and where each ρ_k and each ζ_k is a zero of order 1 of Δ. Observe that $\zeta_k = -\rho_{-k}$ for $k = 0, \pm 1, \pm 2, ...$ and that

$$\Delta^{(1)}(\rho_k) = (A_{14} + A_{23})(\eta_0 - \xi_0)\rho_k, \quad k = 0, \pm 1, \pm 2, ...$$

We conclude that the nonzero eigenvalues of L are given by the sequence

$$\lambda_k = \rho_k^2, \quad k = 0, \pm 1, \pm 2, ...$$

Finally in Case 3 the coefficient matrix A can have three possible normalized forms, viz

$$A = \begin{bmatrix} 1 & b_1 & 0 & 0 \\ 0 & 0 & 1 & d_0 \end{bmatrix},$$

with $d_0 \neq -b_1, b_1 \neq \pm 1$ and $d_0 \neq \pm 1$ or

$$A = \begin{bmatrix} 1 & b_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with $b_1 \neq \pm 1$, or

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_0 \end{bmatrix},$$

with $d_0 \neq \pm 1$. If A has the normalized form (first one), then L is self adjoint iff $\overline{b}_1 d_0 = 1$.

Case 4: It is treated as a perturbation of Case 3, the technique being similar to the way Case 2 was treated as a perturbation of Case 1. The differential operator L belongs to Case 4 provided it satisfies the conditions

$$A_{12} = 0, A_{14} + A_{23} \neq 0, A_{14} + A_{23} \neq \mp(A_{13} + A_{24}), A_{34} \neq 0.$$

In this case,

$$\Delta(\rho) = i(A_{14} + A_{23})\rho e^{-i\rho} \times \\ \left\{ [e^{i\rho} - \xi_0][e^{i\rho} - \eta_0] - \frac{A_{34}}{i(A_{14} + A_{23})\rho} \left[e^{2i\rho} - 1\right] \right\},$$

for the characteristic determinant, where ξ_0, η_0 are the roots of the quadratic polynomial Q with $\xi_0 \eta_0 = 1$ and $\xi_0 \neq \eta_0$. Clearly, the nonzero zeros of Δ are determined by the function

$$F(\rho) = f(\rho) + g(\rho),$$

where

$$f(\rho) = [e^{i\rho} - \xi_0][e^{i\rho} - \eta_0],$$

and

$$g(\rho) = -\frac{A_{34}}{i(A_{14} + A_{23})\rho} \left[e^{2i\rho} - 1\right].$$

It follows from Theorem 1.2.3(c) that the nonzero zeros of Δ are given by two sequences

$$\rho_k = (Arg\xi_0 + 2k\pi) - i \ln |\xi_0| + \epsilon_k \text{ with } |\epsilon_k| \leq \frac{\gamma}{|k|}, \tag{**}$$

$$\zeta_k = (-Arg\xi_0 + 2k\pi) + i \ln |\xi_0| - \varepsilon_k \text{ with } |\varepsilon_k| \leq \frac{\gamma}{|k|},$$

$k = \pm k_0, \pm(k_0+1), ...$, plus a finite number of additional zeros, where $\gamma > 0$ is a constant and k_0 is a positive integer, where $\rho_k \neq \zeta_l$ for all k, l and where each ρ_k and each ζ_l is a zero of order 1 of Δ. We can assume without loss of generality that $\zeta_k = -\rho_{-k}$ for $k = \pm k_0, \pm(k_0+1), ...$Thus, the eigenvalues of L are given by the sequence

$$\lambda_k = \rho_k^2, k = \pm k_0, \pm(k_0+1), ...,$$

plus a finite number of additional eigenvalues.

The coefficient matrix A can have three possible normalized forms in Case 4:

$$A = \begin{bmatrix} 1 & b_1 & 0 & b_0 \\ 0 & 0 & 1 & d_0 \end{bmatrix},$$

with $d_0 \neq -b_1$,$b_1 \neq \pm 1, d_0 \neq \pm 1$, and $b_0 \neq 0$; or

$$A = \begin{bmatrix} 1 & b_1 & a_0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with $b_1 \neq \pm 1$ and $a_0 \neq 0$; or

$$A = \begin{bmatrix} 0 & 1 & 0 & b_0 \\ 0 & 0 & 1 & d_0 \end{bmatrix},$$

with $d_0 \neq \pm 1$ and $b_0 \neq 0$. When A is in the normalized form L is selfadjoint iff $\overline{b_1} d_0 = 1$ and $Arg b_0 = Arg(\pm b_1)$.

Case 5: It is simple to treat because all the spectral quantities are easily computed. To belong to Case 5, the differential operator L must satisfy the conditions

$$A_{12} = 0, A_{14} + A_{23} = 0, A_{34} \neq 0, A_{13} + A_{24} = 0, A_{13} = A_{24}.$$

We can easily see that the characteristic determinant for this case is given by

$$\Delta(\rho) = -A_{34} e^{-i\rho}[e^{i\rho} - 1][e^{i\rho} + 1],$$

and obviously the function

$$f(\rho) = [e^{i\rho} - 1][e^{i\rho} + 1]$$

determines the nonzero zeros of Δ. It follows from Theorem 1.2.3(a) with $\xi_0 = 1$ and $\eta_0 = -1$ that the nonzero zeros of Δ are

$$\rho_k = k\pi, k = \pm 1, \pm 2, ...,$$

with each ρ_k being a zero of order 1 of Δ. Therefore, the nonzero eigenvalues of L are

$$\lambda_k = (k\pi)^2, k = 1, 2, ...$$

We note that there is only one normalized form for the coefficient matrix A, namely,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which corresponds to Dirichlet boundary conditions. It should also be noted that L is always self-adjoint in this case.

Theorem 1.2.4 *In cases 1-5, the boundary conditions are strongly regular.*

Proof. One can easily observe that, Case 1 and Case 2 are related to the case which is mentioned in Proposition 1.2.1(a), since the condition $a_1d_1 - b_1c_1 \neq 0$ holds. Case 5 is familiar with the case written in Proposition 1.2.1(c), when the conditions $a_1 = b_1 = c_1 = d_1 = 0$ and $a_0d_0 - b_0c_0 \neq 0$ hold. In Case 3 and Case 4, one can easily see from the formulas (*) and (**), the conditions $\xi_0\eta_0 = 1$ and $\xi_0 \neq \eta_0$ hold, that is, the eigenvalues are far from each other. Therefore, from Remark 1.1.1 we say that Case 3 and Case 4 are also strongly regular. ■

1.3 Linear operators in the space of vector-functions

Let $\mathbb{C}^m$ denote an $m-$dimensional complex vector space; i.e. $\mathbb{C}^m$ consists of all vectors $\mathbf{y} = (y_1, y_2, ..., y_m)$ where each y_r is a complex number. Functions $\mathbf{y} = \mathbf{y}(x)$, of the real independent variable x, whose values are not numbers but vectors in $\mathbb{C}^m$ are called vector functions. A vector function is therefore simply a system of m complex-valued functions

$$\mathbf{y}(x) = (y_1(x), y_2(x), ..., y_m(x)),$$

and each of the scalar functions $y_r(x)$ is called a component of the vector function $\mathbf{y}(x)$.

The function $\mathbf{y}(\mathbf{x})$ is said to be continuous at the point x_0 if all its components are continuous at x_0.Similarly, a function $\mathbf{y}(\mathbf{x})$ is said to be differentiable if each of its components is differentiable, and by definiton

$$\mathbf{y}'(x) = (y_1'(x), y_2'(x), ..., y_m'(x)).$$

Derivatives of higher order are defined in a similar way. It may be easily seen that:

$$(\mathbf{y} + \mathbf{z})' = \mathbf{y}' + \mathbf{z}', (\lambda\mathbf{y})' = \lambda'\mathbf{y} + \lambda\mathbf{y}',$$
$$(\mathbf{y}, \mathbf{z})' = (\mathbf{y}', \mathbf{z}) + (\mathbf{y}, \mathbf{z}'),$$

where

$$(\mathbf{y}, \mathbf{z}) = \sum_{k=1}^{m} \mathbf{y}_k(x)\overline{\mathbf{z}_k(x)}.$$

In addition to vector functions we shall also be concerned with operator functions. The values of operator functions are linear operators in $\mathbb{C}^m$. Such operators can be represented by means of square matrices $A(x) = [a_{jn}(x)]$ of order m, whose elements are scalar functions.

Essentially, operator-functions are also vector functions, since the aggregate of all linear operators is a vector space of dimension m^2. Consequently an operator-function $A(x)$ will be said to be continuous at the point x_0 if all its functions $a_{jk}(x)$ are continuous at x_0, and to be differentiable at x_0 if all the $a_{jk}(x)$ are differentiable at x_0. So, by definition, $A'(x)$ is the matrix whose elements are $a_{jk}^{'}(x)$. We see that the following rules hold:

$$(A + B)' = A' + B', (\lambda A)' = \lambda' A + \lambda A',$$
$$(AB)' = A'B + AB', (Ay)' = A'y + Ay'.$$

Definition 1.3.1 *Let $P_0(x), P_1(x), ..., P_n(x)$ be operator functions which are continuous in $[a, b]$ and suppose $detP_0(x) \neq 0$ in $[a, b]$. An expression of the form*

$$l(\mathbf{y}) = P_0(x)\mathbf{y}^{(n)} + P_1(x)\mathbf{y}^{(n-1)} + ... + P_n(x)\mathbf{y},$$

is called a linear differential expression in the space of vector functions.

We remark that, essentially, $l(\mathbf{y})$ is a system of m differential expressions of the $n-$th order which depend on m scalar functions $y_1(x), y_2(x), ..., y_m(x)$.

We denote by $\mathbf{y}_a, \mathbf{y}_a^{'}, ..., \mathbf{y}_a^{(n-1)}; \mathbf{y}_b, \mathbf{y}_b^{'}, ..., \mathbf{y}_b^{(n-1)}$ the value of the vector function and its first $(n - 1)$ derivatives at the points a and b respectively, so that $\mathbf{y}_a, ..., \mathbf{y}_b^{(n-1)}$ are vectors in the space $\mathbb{C}^m$. We put

$$U(y) = A_0\mathbf{y}_a + A_1\mathbf{y}_a^{'} + ... + A_{n-1}\mathbf{y}_0^{(n-1)} + B_0\mathbf{y}_b + B_1\mathbf{y}_b^{'} + ... + B_{n-1}\mathbf{y}_b^{(n-1)}, \quad (1.14)$$

where $A_0, ..., A_{n-1}, B_0, ..., B_{n-1}$ are fixed linear operators in the space $\mathbb{C}^m$.

Definition 1.3.2 *If several such forms (1.14) are given, $U_1(\mathbf{y}), U_2(\mathbf{y}), ..., U_q(\mathbf{y})$, then equations of the form*

$$U_1(\mathbf{y}) = 0, U_2(\mathbf{y}) = 0, ..., U_q(\mathbf{y}) = 0 \quad (1.15)$$

are called boundary conditions.

Definition 1.3.3 *Let $\mathbf{D}(L)$ be subspace of $L_2^m[0, 1]$ defined by*

$$\mathbf{D}(L) = \{\mathbf{y} \in L_2^m[0, 1] : \exists \mathbf{y}^{(n-1)} \in AC[0, 1], l(y) \in L_2[0, 1], U_v(\mathbf{y}) = 0, v = 1, 2, ..., m\}$$

where $AC[0, 1]$ is the set of absolutely continuous functions on $[0, 1]$. We say that operator L is generated by the differential expression $l(\mathbf{y})$ and the boundary conditions (1.15) if $L\mathbf{y} = l(\mathbf{y})$ for $y \in \mathbf{D}(L)$.

We will assume in the definiton of $\mathbf{D}(L)$ that the forms

$$U_v(\mathbf{y}) = A_{v,0}\mathbf{y}_a + ... + A_{v,n-1}\mathbf{y}_a^{(n-1)} + B_{v,0}\mathbf{y}_b + ... + B_{v,n-1}\mathbf{y}_b^{(n-1)}, v = 1, 2, ...q,$$

are linearly independent; this implies that the rank of the matrix formed from all the elements of the matrices $[A_{vj}], [B_{vj}], viz.$

$$\begin{bmatrix} A_{10}, & ..., & A_{1,n-1}, & B_{10,} & ..., & B_{1,n-1} \\ A_{20}, & ..., & A_{2,n-1,} & B_{20}, & ..., & B_{2,n-1} \\ . & ..., & . & . & . & . \\ A_{q0}, & ..., & A_{q,n-1}, & B_{q0}, & ..., & B_{q,n-1} \end{bmatrix},$$

is equal to mq; for each form $U_v(\mathbf{y})$ has m components. From now on we shall mainly be concerned with the case $q = n$.

Definition 1.3.4 *The problem of determining a vector-function* $\mathbf{y}$ *which shall satisfy the conditions*

$$l(\mathbf{y}) = 0, \tag{1.16}$$
$$U_v(\mathbf{y}) = 0, \ v = 1, 2, ..., n, \tag{1.17}$$

is called the homogeneous boundary value problem.

We consider the $n^2 \times n^2$ matrix,

$$U = \begin{bmatrix} U_1(Y_1) & U_1(Y_2) & ... & U_1(Y_n) \\ U_2(Y_1) & U_2(Y_2) & ... & U_2(Y_n) \\ . & . & ... & . \\ U_n(Y_1) & U_n(Y_2) & ... & U_n(Y_n) \end{bmatrix},$$

where $Y_1, Y_2, ..., Y_n$ are the solutions of the homogeneous equation

$$l(Y) = P_0(x)Y^{(n)} + P_1(x)Y^{(n-1)} + ... + P_nY = 0.$$

If these solutions are linearly independent, it can be seen that any solution of the equation $l(\mathbf{y}) = 0$ has the form

$$y = Y_1c_1 + Y_2c_2 + ... + Y_nc_n,$$

where $c_1, c_2, ..., c_n$ are arbitrary constant vectors in $\mathbb{C}^m$.

Therefore, a homogeneous boundary-value problem (1.16), (1.17) has a non trivial solution if and only if the determinant of the matrix U vanishes.

Now let us give the definition of eigenvalue of a differential operator in the space of vector functions.

A number λ is called an eigenvalue of an operator L if there exists a function $\mathbf{y} \neq 0$ in the domain of definition of the operator L such that $L\mathbf{y} = \lambda\mathbf{y}$. In particular, the eigenvalues are the zeros of the characteristic determinant

$$\Delta(\lambda) = \begin{vmatrix} U_1(Y_1) & . & . & . & U_1(Y_n) \\ . & . & . & . & . \\ U_n(Y_1) & . & . & . & U_n(Y_n) \end{vmatrix},$$

where $Y_1, Y_2, ..., Y_n$ are linearly independent solutions of the operator equation $l(Y) - \lambda Y = 0$.

In fact, according to the definitions given until now, it is not difficult to see that the ordinary eigenvalue problem in the space of vector functions is equivalent to a certain generalized eigenvalue problem for scalar functions.

Now let us state the definitons of adjoint differential expression, adjoint boundary conditions and finally adjoint operator.

We now further require that, for $k = 0, 1, 2, ..., n$, the coefficient matrices $P_k(x)$ shall each be continuously differentiable $(n-k)$ times. We denote the scalar product of the vectors $\mathbf{y}$,$\mathbf{z} \in \mathbb{C}^m$ by $(\mathbf{y}, \mathbf{z})$. Integrating by parts, we obtain

$$\int_a^b (l(\mathbf{y}), \mathbf{z})dx = P(\eta, \zeta) + \int_a^b (\mathbf{y}, l^*(\mathbf{z}))dx, \tag{1.18}$$

where $P(\eta, \zeta)$ is a bilinear form in

$$\eta = (\mathbf{y}_a, \mathbf{y}_a^{'}, ..., \mathbf{y}_a^{(n-1)}; \mathbf{y}_b, \mathbf{y}_b^{'}, ..., \mathbf{y}_b^{(n-1)}),$$

and

$$\zeta = (\mathbf{z}_a, \mathbf{z}_a^{'}, ..., \mathbf{z}_a^{(n-1)}; \mathbf{z}_b, \mathbf{z}_b^{'}, ..., \mathbf{z}_b^{(n-1)}),$$

and where

$$l^*(\mathbf{z}) = (-1)^n (P_0^* \mathbf{z})^{(n)} + (-1)^{n-1} (P_1^* \mathbf{z})^{(n-1)} + ... + P_n^* \mathbf{z}.$$

The differential expression $l^*(\mathbf{z})$ is said to be adjoint to $l(\mathbf{z})$. Formula (1.18) is called Lagrange's formula. A differential expression $l(\mathbf{y})$ is said to be self-adjoint if $l^*(\mathbf{y}) = l(\mathbf{y})$.

We supplement any given set of linearly independent forms $U_{1,...}, U_n$, to form a complete system of linearly independent forms $U_1, U_2, ..., U_{2n}$. We can tansform the formula (1.18) to

$$\int_a^b (l(\mathbf{y}), \mathbf{z})dx = (U_1, V_{2n}) + (U_2, V_{2n-1}) + ... + (U_{2n}, V_1) + \int_a^b (\mathbf{y}, l^*(\mathbf{z}))dx, \tag{1.19}$$

where $V_1, V_2, ..., V_{2n}$ are linearly independent forms in the variables

$$\mathbf{z}_a, \mathbf{z}_a^{'}, ..., \mathbf{z}_a^{(n-1)}; \mathbf{z}_b, \mathbf{z}_b^{'}, ..., \mathbf{z}_b^{(n-1)}$$

The boundary conditions

$$V_v = 0, v = 1, 2, ..., n, \tag{1.20}$$

(or any conditions equivalent to them) are said to be adjoint to the boundary conditions

$$U_v = 0, v = 1, 2, ..., n. \tag{1.21}$$

Definition 1.3.5 *The operator generated by the differential expression $l^*(\mathbf{y})$ and the boundary conditions (1.20) is said to be adjoint to the operator L generated by the differential expression $l(\mathbf{y})$ and the boundary conditions (1.21). It will be denoted by L^*.*

It follows from the formula (1.19) that for the operators L and L^*, the equation

$$\int_a^b (L\mathbf{y}, \mathbf{z})dx = \int_a^b (\mathbf{y}, L^*\mathbf{z})dx,$$

holds. An operator L is self adjoint if $L^* = L$. In other words, an operator L is self adjoint if it is generated by a self-adjoint differential expression and self-adjoint boundary conditions.

Now we will give the definiton of normalization for the boundary conditions in vectoral case.

A given differential operator is characterised by the boundary conditions $U_v(\mathbf{y}) = 0, v = 1, 2, ...n$. The number k is called the order of a form $U(\mathbf{y})$ if $U(\mathbf{y})$ contains at least one of the vectors $\mathbf{y}_0^{(k)}$ and $\mathbf{y}_1^{(k)}$ but does not contain the vectors $\mathbf{y}_0^{(v)}$ or $\mathbf{y}_1^{(v)}$ for $v > k$. We consider the forms $U(\mathbf{y})$ of order $(n-1)$, if there are any; they have the form

$$U_v(\mathbf{y}) = A_{v,n-1}\mathbf{y}_0^{(n-1)} + B_{v,n-1}\mathbf{y}_1^{(n-1)} + ...$$

The rectangular matrix $[A_{v,n-1}, B_{v,n-1}]$ has m rows and $2m$ columns. The maximum number of linearly independent rows of $2m$ elements is, however, $2m$; if, then, we replace the rows in the forms of order $(n-1)$ by linear combinations of these rows (if this process is necessary), we can arrange that not more than two forms of order $(n-1)$ occur.

Continuing in the same way with the remaining forms, we can, after a finite number of such steps, reduce the boundary conditions to the form

$$U_v(\mathbf{y}) = U_{v0}(\mathbf{y}) + U_{v1}(\mathbf{y}) = 0$$

where

$$U_{v0}(\mathbf{y}) = A_v\mathbf{y}_0^{(k_v)} + \sum_{j=0}^{k_v-1} A_{vj}\mathbf{y}_0^{(j)}, \tag{1.22}$$

$$U_{v1}(\mathbf{y}) = B_v\mathbf{y}_1^{(k_v)} + \sum_{j=0}^{k_v-1} B_{vj}\mathbf{y}_1^{(j)},$$

$$n-1 \geq k_1 \geq k_2 \geq ... \geq k_n \geq 0, k_{v+2} > k_v,$$

and where, for each v, $v = 1, 2, ..., n$, at least one of the matrices A_v, B_v is different from the zero-matrix.

The operations just described are referred as the normalization of the boundary conditions, and the finally boundary conditions of the form (1.22) are called normalized boundary conditions.

The asymptotic formulae are going to be derived for a particular class of boundary conditions which we shall call regular. The definiton of regular boundary conditions depends on whether n is even or odd. In our problem, n is even. We consider a fixed domain S_k, and number $\omega_1, \omega_2, ..., \omega_n$ so that, for $\rho \in S$,

$$\Re(\rho\omega_1) \leq \Re(\rho\omega_2) \leq ... \leq \Re(\rho\omega_n).$$

Definition 1.3.6 *Suppose n is even; $n = 2\mu$. The normalized boundary conditions (1.22) are said to be regular if both the numbers θ_{-m} and θ_m defined by the equation*

$$\theta_{-m}s^{-m} + \theta_{-m+1}s^{-m+1} + ... + \theta_m s^m =$$

$$\begin{vmatrix} A_1\omega_1^{k_1} & ... & A_1\omega_{\mu-1}^{k_1} & (A_1+sB_1)\omega_\mu^{k_1} & (A_1+\frac{1}{s}B_1)\omega_{\mu+1}^{k_1} & B_1\omega_{\mu+2}^{k_1} & ... & B_1\omega_n^{k_1} \\ A_2\omega_1^{k_2} & ... & A_2\omega_{\mu-1}^{k_2} & (A_2+sB_2)\omega_\mu^{k_2} & (A_2+\frac{1}{s}B_2)\omega_{\mu+1}^{k_2} & B_2\omega_{\mu+2}^{k_2} & ... & B_2\omega_n^{k_2} \\ . & ... & . & . & . & . & ... & . \\ A_n\omega_1^{k_n} & ... & A_n\omega_{\mu-1}^{k_n} & (A_n+sB_n)\omega_\mu^{k_n} & (A_n+\frac{1}{s}B_n)\omega_{\mu+1}^{k_n} & B_n\omega_{\mu+2}^{k_n} & ... & B_n\omega_n^{k_n} \end{vmatrix}$$

don't vanish.

We can see at once regularity does not depend on the particular domain S_k selected. In questions concerning the asymptotic behaviour of the eigenvalues, the equation

$$\theta_{-m}s^{-m} + \theta_{-m+1}s^{-m+1} + ... + \theta_m s^m = 0, \tag{1.23}$$

for even n, plays an important part. If the boundary conditions are regular, this equation has order $2m$, and all the roots are different from zero. In the following theorem we assume that the coefficients of the differential expression considered are continuous matrix functions in the interval $[0, 1]$.

Theorem 1.3.1 *Let L be a differential operator of $n-th$ order, defined in the interval $[0,1]$, whose differential expression contains no derivative of the $(n-1)-th$ order, and whose boundary conditions are regular. If n is even, then to each simple root ξ of the equation (1.23) for the domain S_0 corresponds a sequence λ_k of eigenvalues of the operator L, and*

$$\lambda_k = (2k\pi i)^n \left[1 \mp \frac{n \ln_0 \xi}{2k\pi i} + O\left(\frac{1}{k^2}\right)\right]$$
$$k = N, (N+1), ..$$

where the upper or lower signs hold according as $n = 4v$ or $n = 4v+2$. To each multiple zero ξ of equation (1.23), with multiplicity r, correspond r sequences of eigenvalues $\lambda_{k,j}$ of the operator L, and

$$\lambda_{k,j} = (2k\pi i)^n \left[1 \mp \frac{n \ln_0 \xi}{2k\pi i} + O\left(\frac{1}{k^{1+1/r}}\right)\right]$$
$$j = 1, 2, ..., r; k = N, (N+1), ..$$

and again the sign is $-$ or $+$ according as $n = 4v$ or $n = 4v+2$.

(see [44]).

1.4 On Riesz Bases

In this section, we give the basic definitons for Riesz basis. As it is mentioned in introduction, we need these definitions to determine if root functions form a Riesz basis or not. These descriptions are clearly given in [25].

A sequence $\{\phi_j\}_1^\infty$ of vectors of a Banach space Ω is called a basis of this space if every vector $x \in \Omega$ can be expanded in a unique way in a series

$$x = \sum_{j=1}^{\infty} c_j \phi_j, \tag{1.24}$$

which converges in the norm of the space Ω. In this expansion the coefficients c_j are obviously linear functionals of the element $x \in \Omega$:

$$c_j = \varphi_j(x), \quad j = 1, 2, ... \tag{1.25}$$

Moreover by a well known theorem of Banach, these linear functionals are continuous $(\varphi_j \in \Omega^*; j = 1, 2, ...)$ and there exists a constant C_ϕ associated with them such that

$$|\phi_j|^{-1} \leq |\varphi_j| \leq C_\phi |\phi_j|^{-1}. \tag{1.26}$$

We shall apply these generel results to a basis $\{\phi_j\}$ of a Hilbert space $\Omega = H$. In this case the relations (1.25) can be written in the form

$$c_j = (x, \varphi_j) \quad (\varphi_j \in H; j = 1, 2, ...) \tag{1.27}$$

Setting $x = \phi_k(k = 1, 2, ...)$, we obtain

$$(\phi_k, \varphi_j) = \delta_{jk} \quad (j, k = 1, 2, ...)$$

Let us recall that two sequences $\{\varsigma_j\}$ and $\{\nu_j\}$ with elements from H are said to be biorthogonal, if

$$(f_j, g_k) = \delta_{jk} \quad (j, k = 1, 2, ...)$$

For a given sequence $\{f_j\}_1^\infty \in H$ a biorthogonal sequence $\{g_j\}_1^\infty \in H$ exists if and only if each element f_j $(j = 1, 2, ...)$ lies outside the closed linear hull Υ_j of all the other elements f_k $(k \neq j)$. If this condition is fullfilled then the biorthogonal sequence $\{g_j\}_1^\infty$ will be uniquely determined if and only if the system $\{f_j\}_1^\infty$ is complete in H. In this case the orthogonal complement $\varrho_j^\perp = H \ominus \Upsilon_j(j = 1, 2, ...)$ is one dimensional and the element g_j is determined by the conditions $g_j \in \varrho_j^\perp, (g_j, f_j) = 1$ $(j = 1, 2, ...)$.

Thus for every basis $\{\phi_j\}_{j=1}^\infty$ the biorthogonal sequence $\{\varphi_j\}_1^\infty$ is defined uniquely.

From the equalities (1.24) and (1.27) it follows that any vector f which is orthogonal to all the vectors $\varphi_j(j = 1, 2, ...)$ equals zero. Consequently the sequence biorthogonal to a basis is always complete in H.

Theorem 1.4.1 *The sequence $\{\varphi_j\}_1^\infty$, biorthogonal to a basis $\{\phi_j\}_1^\infty$ of a Hilbert space H, is also a basis of H.*

We shall say that a sequence $\{\phi_j\}$ of vectors from H is almost normalized if

$$\inf_n |\phi_n| > 0 \text{ and } \sup_n |\phi_n| < \infty$$

If the basis $\{\phi_j\}_1^\infty$ of the space H is almost normalized, then the biorthogonal basis $\{\varphi_j\}_1^\infty$ is almost normalized.

Let $\{\phi_j\}$ be an arbitrary orthonormal basis of the space H, and A some bounded linear invertible operator. Then for any vector $f \in H$ one has

$$A^{-1}f = \sum_{j=1}^\infty (A^{-1}f, \phi_j)\phi_j = \sum_{j=1}^\infty (f, A^{*-1}\phi_j)\phi_j$$

and consequently

$$f = \sum_{j=1}^\infty (f, f_j)\varphi_j$$

where

$$\varphi_j = A\phi_j, f_j = A^{*-1}\phi_j \quad j = 1, 2, ...$$

Obviously

$$(\varphi_j, f_j) = \delta_{jk} \quad j, k = 1, 2, ...$$

Therefore if

$$f = \sum_{j=1}^{\infty} c_j \varphi_j$$

then

$$c_j = (f, f_j) \quad j = 1, 2, ...$$

i.e. the expansion is unique.

Thus every bounded invertible operator transforms any orthonormal basis into some other basis of the space H. A basis $\{\varphi_j\}$ of the space H which is obtained from an orthonormal basis by means of such a transformation is called a basis equivalent to an orthonormal basis (in the terminology of N. K. Bari, a Riesz basis).

Theorem 1.4.2 *The followings are equivalent:*

(i) $\{\varphi_j\}$ *is a Riesz basis.*

(ii) The sequence $\{\varphi_j\}$ *becomes an orthonormal basis of* H *following the appropriate replacement of the inner product* (f, g) *by some new one* $(f, g)_1$ *,*

$$c_1(f, f) \le (f, f)_1 \le c_2(f, f)$$

.

(iii) The sequence $\{\varphi_j\}$ *is complete in* H *and there exist positive constants* $a_1, a_2, ...$ *such that for any positive integer* n *and any complex numbers* $\gamma_1, ..., \gamma_n$ *one has*

$$a_2 \sum_{j=1}^{n} |\gamma_j|^2 \le \left| \sum_{j=1}^{n} |\gamma_j \varphi_j| \right|^2 \le a_1 \sum_{j=1}^{n} |\gamma_j|^2 .$$

(iv) The sequence $\{\varphi_j\}$ *is complete in* H *and its Gram matrix* $\|(\varphi_j, \varphi_k)\|_1^{\infty}$ *generates a bounded invertible operator in* l_2.

(v) $\{\varphi_j\}$ *is complete in* H*,there exist a complete biorthogonal sequence* $\{x_j\}$ *and for any* $f \in H$*one has*

$$\sum |(f, \varphi_j)|^2 < \infty,$$

and

$$\sum |(f, x_j)|^2 < \infty.$$

Moreover, let us give the definiton of basis of subspaces:

A sequence $\{H_k\}_1^\infty$ of nonzero subspaces $H_k \subset H$ is said to be a basis (of subspaces) of the space H, if any vector $x \in H$ can be expanded in a unique way in a series of the form

$$x = \sum_{k=1}^{\infty} x_k,$$

where $x_k \in H_k(k = 1, 2, ...)$. If the subspaces $H_k, (k = 1, 2, ...,)$ are one-dimensional, then they form a basis of the space H if and only if unit vectors $\phi_k \in H_k(k = 1, 2, ...)$ form a vector basis of H.

Now it will be useful to present a simple result which establishes connections between bases of subspaces and vector bases:

If the sequence of subspaces $\{H_k\}_1^\infty$ is a basis of the space H equivalent to an orthogonal one, then any sequence $\{\phi_j\}_1^\infty$, obtained as the union of orthonormal bases of all the subspaces $H_k, (k = 1, 2, ...)$, is a basis of the space H equivalent to orthonormal one.

1.5 On the Finite Difference Methods and Numerical Solutions

To solve differential equations numerically we can replace the derivatives in the equation with finite difference approximations on a discretized domain. This results in a number of algebraic equations that can be solved one at a time (explicit methods) or simultaneously (implicit methods) to obtain values of the dependent function y_i corresponding to values of the independent function x_i in the discretized domain.

A finite difference is a technique by which derivatives of functions are approximated by differences in the values of the function between a given value of the independent variable say x_0, and a small increment $(x_0 + h)$. For example, from the definiton of the derivative,

$$df/dx = \lim_{h \to 0}(f(x+h) - f(x))/h,$$

we can approximate the value of df/dx by using the finite difference approximation

$$(f(x+h) - f(x))/h$$

with a small value of h.

The error, i.e., the difference between the numerical derivative $\Delta f/\Delta x$ and the actual value,varies linearly with the increment h in the independent variable. It is very common to indicate this dependency by saying that "the error is of order h", or error $=O(h)$. The magnitude of the error can be estimated by using Taylor series expansions of the function $f(x+h)$.

The Taylor series expansion of the function $f(x)$ about the point $x = x_0$ is given by the formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

where $f^{(n)}(x_0) = (d^n f/dx^n)\mid_{x=x_0}$, and $f^{(0)}(x_0) = f(x_0)$.

If we let $x = x_0 + h$, then $x - x_0 = h$, then the series can be written as

$$f(x_0 + h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}h^n = f(x_0) + \frac{f^{'}(x_0)}{1!}h + \frac{f^{''}(x_0)}{2!}h^2 + O(h^3),$$

where the expansion $O(h^3)$ represents the remaining terms of the series and indicates that the leading term is of order h^3. Because h is a small quantity, we can write $1 > h$, and $h > h^2 > h^3 > h^4 > ...$ Therefore, the ramaining of the series represented by $O(h^3)$ provides the order of the error incurred in neglecting this part of the series expansion when calculating $f(x_0 + h)$.

From the Taylor series expansion shown above we can obtain an expression for the derivative $f^{'}(x_0)$ as

$$f^{'}(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} + \frac{f''(x_0)}{2!}h + O(h^2) = \frac{f(x_0 + h) - f(x_0)}{h} + O(h)$$

In practical applications of finite differences, we will replace the first-order derivative df/dx at $x = x_0$, with the expression $(f(x_0 + h) - f(x_0))/h$, selecting an appropriate value for h, and indicating that the error introduced in the calculation is of order h, i.e. error$=O(h)$.

The approximation

$$df/dx = (f(x_0 + h) - f(x_0))/h$$

is called a forward difference formula because the derivative is based on the value $x = x_0$ and it involves the function $f(x)$ evaluated at $x = x_0 + h$, i.e., at a point located forward from x_0 by an increment h.

If we include the values of $f(x)$ at $x = x_0 - h$, and $x = x_0$, the approximation is written as

$$df/dx = (f(x_0) - f(x_0 - h))/h$$

and is called a backward difference formula. The order of the error is still $O(h)$.

A centered difference formula for df/dx will include the point $(x_0 - h, f(x_0 - h))$ and $(x_0 + h, f(x_0 + h))$. To find the expression for the formula as well as the order of the error we use the Taylor series expansion of $f(x)$ once more. First we write the equation corresponding to a forward expansion

$$f(x_0 + h) = f(x_0) + f'(x_0)h + 1/2f''(x_0)h^2 + 1/6f^{(3)}(x_0)h^3 + O(h^4)$$

Next, we write the equation for a backward expansion

$$f(x_0 - h) = f(x_0) - f'(x_0)h + 1/2f''(x_0)h^2 - 1/6f^{(3)}(x_0)h^3 + O(h^4)$$

Subtracting these two equations results in

$$f(x_0 + h) - f(x_0 - h) = 2f'(x_0)h + 1/3f^{(3)}(x_0)h^3 + O(h^5).$$

Notice that the even terms in h, vanish. Therefore, the order of the remaining terms in this last expression is $O(h^5)$. Solving for $f'(x_0)$ from the last result produces the following centered difference formula for the first derivative

$$\frac{df}{dx}\,|_{x=x_0}= \frac{f(x_0+h) - f(x_0-h)}{2h} + \frac{1}{3}f^{(3)}(x)h^2 + O(h^4),$$

or

$$\frac{df}{dx} = \frac{f(x_0+h) - f(x_0-h)}{2h} + O(h^2)$$

This result indicates that the centered difference formula has an error of order $O(h^2)$, while the forward and backward difference formulas had an error of the order $O(h)$. Since $h^2 < h$, the error introduced in using the centered difference formula to approximate a first derivative will be smaller than if the forward or backward difference formulas are used.

To obtain a centered finite difference formula for the second derivative, we'll start by using the equations for the forward and backward Taylor series expansions from the previous section but including terms up to $O(h^5)$, i.e.,

$$f(x_0+h) = f(x_0)+f'(x_0)h+1/2f''(x_0)h^2+1/6f^{(3)}(x_0)h^3+1/24f^{(4)}(x_0)h^4+O(h^5)$$

and

$$f(x_0-h) = f(x_0)-f'(x_0)h+1/2f''(x_0)h^2-1/6f^{(3)}(x_0)h^3+1/24f^{(4)}(x_0)h^4-O(h^5)$$

Next, add the two equations and find the following centered difference formula for the second derivatives

$$d^2f/dx^2 = [f(x_0+h) - 2f(x_0) + f(x_0-h)]/h^2 + O(h^2).$$

Forward and backward finite difference formulas for the second derivatives are given, respectively, by

$$d^2f/dx^2 = [f(x_0+2h) - 2f(x_0+h) + f(x_0)]/h^2 + O(h),$$

and

$$d^2f/dx^2 = [f(x_0) - 2f(x_0-h) + f(x_0-2h)]/h^2 + O(h).$$

(see [57] in Bibliography for this section).

2

Asymptotic Formulas and Riesz Basis Property of Differential Operator in Space of Vector Functions

In this chapter we investigate the asymptotic formulas of the eigenvalues and eigenfunctions for the operator which is defined in (0.1) by the boundary conditions (0.2). The main theoretical results of this thesis are given at the end of this chapter.

Let us first consider the differential operator $L(q)$ generated in the space $L_2\,[0,1]$ by the differential expression

$$-\,y^{''}(x) + q\,(x)\,y(x) \tag{2.1}$$

where $q(x)$ is a summable function, and the boundary conditions are as defined in (0.3).

The eigenvalues of the operator $L(q)$ generated in the space $L_2\,[0,1]$ by the differential expression (2.1) and strongly regular boundary conditions (0.3), where q is a summable function, consist of the sequences

$$\{\rho_n^{(1)}(q)\} \;\;\&\;\; \{\rho_n^{(2)}(q)\} \tag{2.2}$$

satisfying

$$\rho_n^{(1)}(q) = (2n\pi + \gamma_1)^2 + O(1),\;\; \rho_n^{(2)}(q) = (2n\pi + \gamma_2)^2 + O(1);\;\; n \geq N >> 1, \tag{2.3}$$

where

$$\gamma_j = -i \ln \zeta_j, \operatorname{Re} \gamma_j \in (-\pi, \pi], \ \zeta_1 \neq \zeta_2, \tag{2.4}$$

and ζ_1, ζ_2 are the roots of the equation

$$\theta_1 \zeta^2 + \theta_0 \zeta + \theta_{-1} = 0. \tag{2.5}$$

By the help of Remark 1.2.1, when the conditons in Proposition 1.2.1(a) and Proposition 1.2.1(c) hold, remember that we find $\theta_1 = -1$, $\theta_0 = 0$ and $\theta_{-1} = 1$. If we substitute these values in (2.5), we have the equation

$$-\zeta^2 + 1 = 0,$$

which have the roots $\zeta_1 = 1$ and $\zeta_2 = -1$. Hence using (2.4), we have

$$\gamma_1 = 0, \gamma_2 = \pi. \tag{2.6}$$

In the condition of Proposition 1.2.1(b) since we obtain that θ_1 and θ_{-1} are equal, equation (2.5) has the form

$$\zeta^2 + \frac{b}{a}\zeta + 1 = 0,$$

that is, $\zeta_1 \zeta_2 = 1$ and by (2.4) $\zeta_1 \neq \zeta_2$ which implies that $\zeta_1 \neq \pm 1$ and $\zeta_2 \neq \pm 1$. Therefore, we have

$$\gamma_1 = -\gamma_2 \neq \pi k. \tag{2.7}$$

Theorem 2.0.1 *If the boundary conditions (0.3) are regular then the boundary conditions (0.2) are also regular.*

Proof. The conditions (0.3) are regular (see [33], p. 121) if the numbers Θ_{-m} , Θ_m defined by the identity

$$\Theta_{-m} s^{-m} + \Theta_{-m+1} s^{m-1} + ... + \Theta_m s^m = \det M(m) \tag{2.8}$$

are both different from zero, where

$$M(m) = \begin{bmatrix} (\alpha_1 + s\beta_1)\omega_1^{k_1} I & (\alpha_1 + \frac{1}{s}\beta_1)\omega_2^{k_1} I \\ (\alpha_2 + s\beta_2)\omega_1^{k_2} I & (\alpha_2 + \frac{1}{s}\beta_2)\omega_2^{k_2} I \end{bmatrix}$$

and I is $m \times m$ identity matrix. One can easily see that the intersection of the first and $(m+1)$-th rows and columns forms the matrix

$$M(1) = \begin{bmatrix} (\alpha_1 + s\beta_1)\omega_1^{k_1} & (\alpha_1 + \frac{1}{s}\beta_1)\omega_2^{k_1} \\ (\alpha_2 + s\beta_2)\omega_1^{k_2} & (\alpha_2 + \frac{1}{s}\beta_2)\omega_2^{k_2} \end{bmatrix}$$

and its complementary minor is $M(m-1)$. Moreover, the determinant of the minors of $M(m)$ formed by intersection of the first and $(m+1)$-th rows and other pairs of columns is zero, since the $2 \times m$ matrix consisting of these rows has the form

$$\left[\begin{array}{cccccccccc} (\alpha_1 + s\beta_1)\omega_1^{k_1} & 0 & 0 & \cdots & 0 & (\alpha_1 + \frac{1}{s}\beta_1)\omega_2^{k_1} & 0 & 0 & ... & 0 \\ (\alpha_2 + s\beta_2)\omega_1^{k_2} & 0 & 0 & ... & 0 & (\alpha_2 + \frac{1}{s}\beta_2)\omega_2^{k_2} & 0 & 0 & ... & 0 \end{array}\right].$$

Therefore using the Laplace's cofactor expansion along the first and $(m+1)$-th rows we obtain

$$\det M(m) = \det M(1) \det M(m-1). \tag{2.9}$$

By induction the formula (2.9) implies that

$$\det M(m) = (\det M(1))^m . \tag{2.10}$$

Now it follows from (2.2) and the results of Proposition 1.2.1 that

$$\Theta_m = (\theta_1)^m , \Theta_{-m} = (\theta_{-1})^m$$

which implies that the boundary conditions (0.2) are regular if (0.3) are regular.
■

By (2.5), (2.8) and (2.10), ζ_1 and ζ_2 are the roots of the equation

$$\Theta_{-m}\zeta^{-m} + \Theta_{-m+1}\zeta^{m-1} + ... + \Theta_m\zeta^m = 0$$

with multiplicity m. Therefore it follows from the theorem (see [44], Theorem 2 in p.123) that to each root ζ_1 and ζ_2 correspond m sequences, denoted by

$$\{\lambda_{k,1}^{(1)} : k = N, N{+}1, ...\}, \{\lambda_{k,2}^{(1)} : k = N, N{+}1, ...\}, ..., \{\lambda_{k,m}^{(1)} : k = N, N{+}1, ...\}$$

and

$$\{\lambda_{k,1}^{(2)} : k = N, N{+}1, ...\}, \{\lambda_{k,2}^{(2)} : k = N, N{+}1, ...\}, ..., \{\lambda_{k,m}^{(2)} : k = N, N{+}1, ...\}$$

respectively, satisfying

$$\lambda_{k,j}^{(1)} = (2k\pi + \gamma_1)^2 + O(k^{1-\frac{1}{m}}), \lambda_{k,j}^{(2)} = (2k\pi + \gamma_2)^2 + O(k^{1-\frac{1}{m}}) \tag{2.11}$$

for $k = N, N+1, ...$ and $j = 1, 2, ..., m$, where $N \gg 1$.

Now to analyze the operators $L_m(0)$, $L_m(C)$ and $L_m(Q)$, we introduce the following notations. To simplify the notations we omit the upper indices in $\rho_n^{(1)}(0)$, $\rho_n^{(2)}(0)$, $\lambda_{k,j}^{(1)}$, $\lambda_{k,j}^{(2)}$ (see (2.3) and (2.11)) and enumerate these eigenvalues in the following way

$$\rho_n^{(1)}(0) =: \rho_n,\ \rho_n^{(2)}(0) =: \rho_{-n},\ \lambda_{k,j}^{(1)} =: \lambda_{k,j}, \lambda_{k,j}^{(2)} =: \lambda_{-k,j} \tag{2.12}$$

for $n > 0$ and $k \geq N \gg 1$. We remark that there is one-to-one correspondence between the eigenvalues (counting with multiplicities) of the operator $L_1(0)$ and integers which preserve asymptotic (2.3). This statement can easily be proved in a standard way by using Rouche's theorem (we omit the proof of this fact, since it is used only to simplify the notations). Denote the normalized eigenfunction of the operator $L_1(0)$ corresponding to the eigenvalue ρ_n by φ_n. Clearly,

$$\varphi_{n,1} = (\varphi_n, 0, 0, ...0)^T, \ \varphi_{n,2} = (0, \varphi_n, 0, ...0)^T, ..., \ \varphi_{n,m} = (0, 0, ...0, \varphi_n)^T \tag{2.13}$$

are the eigenfunctions of the operator $L_m(0)$ corresponding to the eigenvalue ρ_n. Similarly,

$$\varphi^*_{n,1} = (\varphi^*_n, 0, 0, ...0)^T, \ \varphi^*_{n,2} = (0, \varphi^*_n, 0, ...0)^T, ..., \ \varphi^*_{n,m} = (0, 0, ...0, \varphi^*_n)^T \tag{2.14}$$

are the eigenfunctions of the operator $L^*_m(0)$ corresponding to the eigenvalue $\overline{\rho_n}$, where φ^*_n is the eigenfunction of $L^*_1(0)$ corresponding to the eigenvalue $\overline{\rho_n}$.

Since the boundary conditions (0.3) are strongly regular, all eigenvalues of sufficiently large modulus of $L_1(q)$ are simple (see, [44], the end of Theorem 2 of p.65). Therefore, there exists n_0 such that the eigenvalues ρ_n of $L_1(0)$ are simple for $|n| > n_0$. However, the operator $L_1(0)$ may have associated functions $\varphi^{(1)}_n, \varphi^{(2)}_n, ..., \varphi^{(t(n))}_n$ corresponding to the eigenfunction φ_n for $|n| \leq n_0$. Then, it is not hard to see that $L_m(0)$ has associated functions

$$\varphi_{n,1,p} = (\varphi^{(p)}_n, 0, 0, ...0)^T, \varphi_{n,2,p} = (0, \varphi^{(p)}_n, 0, ...0)^T, ..., \varphi_{n,m,p} = (0, 0, ...0, \varphi^{(p)}_n)^T,$$

for $p = 1, 2, ..., t(n)$ corresponding to ρ_n for $|n| \leq n_0$, that is,

$$\begin{aligned}
&(L_m(0) - \rho_n)\varphi_{n,i,0} = 0, \\
&(L_m(0) - \rho_n)\varphi_{n,i,p} = \varphi_{n,i,p-1}, \ \ p = 1, 2, ..., t(n),
\end{aligned}$$

where $\varphi_{n,i,0}(x) =: \varphi_{n,i}(x)$. Since the system of the root functions of $L_1(0)$ forms Riesz basis in $L_2\,(0, 1)$ (see [43]), the system

$$\{\varphi_{n,i,p} : n \in \mathbb{Z}, \ i = 1, 2, ..., m, \ p = 1, 2, ..., t(n)\} \tag{2.15}$$

forms a Riesz basis in $L^m_2\,(0, 1)$. The system,

$$\{\varphi^*_{n,i,p} : n \in \mathbb{Z}, \ i = 1, 2, ..., m, \ p = 1, 2, ..., t(n)\} \tag{2.16}$$

which is biorthogonal to $\{\varphi_{n,i,p}\}$ is the system of the eigenfunctions and the associated functions of the adjoint operator $L^*_m(0)$. Clearly, (2.16) can be constructed by repeating the construction of (2.15) and replacing everywhere φ_n by φ^*_n. Thus

$$(L^*_m(0) - \overline{\rho_n})\varphi^*_{n,i,0} = 0, \tag{2.17}$$

$$(L^*_m(0) - \overline{\rho_n})\varphi^*_{n,i,p} = \varphi^*_{n,i,p-1}, \ \ p = 1, 2, ..., t(n). \tag{2.18}$$

To prove the main results, we need the following properties of the eigenfunctions φ_n and φ^*_n.

Proposition 2.0.1 *If the boundary conditions (0.3) are strongly regular then there exists a positive constant M such that*

$$\sup_{x\in[0,1]} |\varphi_n(x)| \leq M, \quad \sup_{x\in[0,1]} |\varphi_n^*(x)| \leq M, \tag{2.19}$$

$$\sup_{x\in[0,1]} |\varphi_{n,i,p}(x)| \leq M, \quad \sup_{x\in[0,1]} \left|\varphi_{n,i,p}^*(x)\right| \leq M, \tag{2.20}$$

for all n, i, p, where φ_n^* is the eigenfunctions of $L_1^*(0)$, satisfying

$$(\varphi_n, \varphi_n^*) = 1 \tag{2.21}$$

for $|n| > n_0$. Moreover, the following asymptotic formulas hold

$$\overline{\varphi_n^*(x)}\varphi_n(x) = 1 + A_1 e^{i(4\pi n+2\gamma_1)x} + B_1 e^{-i(4\pi n+2\gamma_1)x} + O(\frac{1}{n}), n > 0, \tag{2.22}$$
$$\overline{\varphi_n^*(x)}\varphi_n(x) = 1 + A_2 e^{i(4\pi n+2\gamma_2)x} + B_2 e^{-i(4\pi n+2\gamma_2)x} + O(\frac{1}{n}), n < 0,$$

where A_j and B_j for $j = 1, 2$ are constants.

Proof. When the condition in Proposition 1.2.1(c) holds, we have

$$\varphi_n(x) = \varphi_n^*(x) = \sqrt{2}\sin 2nx, \ \varphi_{-n}(x) = \varphi_{-n}^*(x) = \sqrt{2}\sin(2n+1)x, \tag{2.23}$$

where $n = 1, 2, ...$ In (1.11), using the well known expression for eigenfunction

$$\begin{vmatrix} e^{i\rho_n x} & e^{-i\rho_n x} \\ U_1(e^{i\rho_n x}) & U_1(e^{-i\rho_n x}) \end{vmatrix}$$

and (2.3), (2.12), and also taking into account that $\gamma_1 = 0$ and $\gamma_2 = \pi$ (see the proof of Proposition 1.2.1(a) and (2.6)), we obtain

$$\varphi_n(x) = \sqrt{2}\cos 2nx + O(\frac{1}{n}), \ \varphi_{-n}(x) = \sqrt{2}\cos(2n+1)x + O(\frac{1}{n}), \tag{2.24}$$

and

$$\varphi_n^*(x) = \sqrt{2}\cos 2nx + O(\frac{1}{n}), \ \varphi_{-n}^*(x) = \sqrt{2}\cos(2n+1)x + O(\frac{1}{n}). \tag{2.25}$$

In the same way in (1.12) we get the formulas

$$\varphi_n(x) = a^+ e^{i(2\pi n+\gamma_1)x} + b^+ e^{-i(2\pi n+\gamma_1)x} + O(\frac{1}{n}), \tag{2.26}$$
$$\varphi_n^*(x) = c^+ e^{i(2\pi n+\gamma_1)x} + d^+ e^{-i(2\pi n+\gamma_1)x} + O(\frac{1}{n}),$$

for $n > 0$ and

$$\varphi_n(x) = a^- e^{i(2\pi n+\gamma_2)x} + b^- e^{-i(2\pi n+\gamma_2)x} + O(\frac{1}{n}), \tag{2.27}$$
$$\varphi_n^*(x) = c^- e^{i(2\pi n+\gamma_2)x} + d^- e^{-i(2\pi n+\gamma_2)x} + O(\frac{1}{n}),$$

for $n < 0$. Thus in any case inequality (2.19) holds. Equality (2.20) follows from (2.19) and equality (2.22) follows from (2.21), (2.23)-(2.27). ■

As it is noted in the introduction, we obtain asymptotic formulas for the eigenvalues and eigenfunctions of $L_m(Q)$ in term of the eigenvalues and eigenfunctions of $L_m(C)$. Therefore first we analyze the eigenvalues and eigenfunctions of $L_m(C)$. Suppose that the matrix C has m simple eigenvalues $\mu_1, \mu_2, ..., \mu_m$. The normalized eigenvector corresponding to the eigenvalue μ_j is denoted by v_j. In these notations the eigenvalues and eigenfunctions of $L_m(C)$ are

$$\mu_{k,j} = \rho_k + \mu_j \quad \& \quad \Phi_{k,j}(x) = v_j \varphi_k(x) \tag{2.28}$$

respectively. It can be easily verified since

$$L_m(C) = L_m(0) + C$$

and multiplying both sides by $\Phi_{k,j}(x)$. Indeed,

$$L_m(C)\Phi_{k,j}(x) = L_m(0)\Phi_{k,j}(x) + C\Phi_{k,j}(x),$$

hence using the equalities needed, we obtain,

$$L_m(C)\Phi_{k,j}(x) = L_m(0)v_j\varphi_k(x) + Cv_j\varphi_k(x),$$

and

$$L_m(C)\Phi_{k,j}(x) = \rho_k v_j\varphi_k(x) + \mu_j v_j\varphi_k(x).$$

Finally, it can be easily seen that

$$L_m(C)\Phi_{k,j}(x) = (\rho_k + \mu_j)\Phi_{k,j}(x).$$

Similarly, the eigenvalues and eigenfunctions of $(L_m(C))^*$ are $\overline{\mu_{k,j}}$, $\Phi_{k,j}^*(x) = v_j^*\varphi_k^*$, where v_j^* is the eigenvector of C^* corresponding to $\overline{\mu_j}$ such that $\left(v_j^*, v_j\right) = 1$. To obtain the asymptotic formulas for the eigenvalues and eigenfunctions of $L_m(Q)$, we use the following formula

$$(\lambda_{k,j} - \mu_{k,i})(\Psi_{k,j}, \Phi_{k,i}^*) = ((Q - C)\Psi_{k,j}, \Phi_{k,i}^*) \tag{2.29}$$

obtained from

$$L_m(Q)\Psi_{k,j}(x) = \lambda_{k,j}\Psi_{k,j}(x) \tag{2.30}$$

by multiplying both sides of (2.29) with $\Phi^*_{k,i}(x)$ and using $L_m(Q) = L_m(C) + (Q - C)$. Indeed, we start with the equation

$$(L_m(Q)\Psi_{k,j}(x), \Phi^*_{k,j}(x)) = (\lambda_{k,j}\Psi_{k,j}(x), \Phi^*_{k,j}(x)),$$

and using the equalities for $L_m(Q)$, we obtain

$$(L_m(C) + (Q - C)\Psi_{k,j}(x), \Phi^*_{k,j}(x)) = (\lambda_{k,j}\Psi_{k,j}(x), \Phi^*_{k,j}(x)).$$

Now using the properties of inner product and adjoint operators, we have

$$(L_m(C)\Psi_{k,j}(x), \Phi^*_{k,j}(x)) + ((Q-C)\Psi_{k,j}(x), \Phi^*_{k,j}(x)) = (\lambda_{k,j}\Psi_{k,j}(x), \Phi^*_{k,j}(x)),$$

and

$$(\Psi_{k,j}(x), L^*_m(C)\Phi^*_{k,j}(x)) + ((Q-C)\Psi_{k,j}(x), \Phi^*_{k,j}(x)) = (\lambda_{k,j}\Psi_{k,j}(x), \Phi^*_{k,j}(x)).$$

Since

$$(\Psi_{k,j}(x), L^*_m(C)\Phi^*_{k,j}(x)) = (\Psi_{k,j}(x), \overline{\mu_{k,j}}\Phi^*_{k,j}(x)) = (\mu_{k,j}\Psi_{k,j}(x), \Phi^*_{k,j}(x)),$$

we obtain (2.29). To prove that $\lambda_{k,j}$ is close to $\mu_{k,j}$, we first show that the right-hand side of (2.29) is a small number for all j and i (see Lemma 2.0.1) and then we prove that for each eigenfunction $\Psi_{k,j}$ of $L_m(Q)$, where $|k| \geq N$, there exists a root function of $(L_m(C)))^*$ denoted by $\Phi^*_{k,j}$ such that $(\Psi_{k,j}, \Phi^*_{k,j})$ is a number of order 1 (see Lemma 2.0.2). Before the proof of these lemmas, we need the following preparations: Multiplying both sides of (2.30) by $\varphi^*_{n,i,0}$, using $L_m(Q) = L_m(0) + Q$ and (2.17) we get

$$(\lambda_{k,j} - \rho_n)(\Psi_{k,j}, \varphi^*_{n,i,0}) = (Q\Psi_{k,j}, \varphi^*_{n,i,0}),$$

$$(\Psi_{k,j}, \varphi^*_{n,i,0}) = \frac{(Q\Psi_{k,j}, \varphi^*_{n,i,0})}{\lambda_{k,j} - \rho_n}, \tag{2.31}$$

for $i, j = 1, 2, ..., m$ and $\lambda_{k,j} \neq \rho_n$. Now multiplying (2.30) by $\varphi^*_{n,i,1}$ and using (2.18), (2.31), we get

$$(\lambda_{k,j} - \rho_n)(\Psi_{k,j}, \varphi^*_{n,i,1}) = (Q\Psi_{k,j}, \varphi^*_{n,i,1}) + \frac{(Q\Psi_{k,j}, \varphi^*_{n,i,0})}{\lambda_{k,j} - \rho_n},$$

$$(\Psi_{k,j}, \varphi^*_{n,i,1}) = \frac{(Q\Psi_{k,j}, \varphi^*_{n,i,1})}{\lambda_{k,j} - \rho_n} + \frac{(Q\Psi_{k,j}, \varphi^*_{n,i,0})}{(\lambda_{k,j} - \rho_n)^2}.$$

In this way one can deduce the formulas

$$(\Psi_{k,j}, \varphi^*_{n,i,s}) = \sum_{p=0}^{s} \frac{(Q\Psi_{k,j}, \varphi^*_{n,i,p})}{(\lambda_{k,j} - \rho_n)^{s+1-p}}, \tag{2.32}$$

for $s = 0, 1, ..., t(n)$.

Since, $(Q\Psi_{k,j}, \varphi^*_{n,i,p}) = (\Psi_{k,j}, Q^*\varphi^*_{n,i,p})$, $\|\Psi_{k,j}\| = 1$, and the entries of Q are the elements of $L_2(0,1)$, it follows from (2.20) and Cauchy-Schwarz inequality that there exists a positive constant c_1 such that

$$\left|(Q\Psi_{k,j}, \varphi^*_{n,i,p})\right| < c_1. \tag{2.33}$$

In the subsequent estimates we denote by c_m for $m = 1, 2, ...$, the positive constants whose exact value are inessential. On the other hand, it follows from (2.3), (2.11) and (2.12) that if k is a sufficiently large number then

$$\mid \lambda_{k,j} - \rho_p \mid > c_2 k^2, \forall p \leq n_0, \tag{2.34}$$

$$|\lambda_{k,j} - \rho_p| > c_3 \left||p| - |k|\right| (|p| + |k|), \forall p \neq \pm k, \tag{2.35}$$

and

$$|\lambda_{k,j} - \rho_{-k}| > c_4 |k|. \tag{2.36}$$

Therefore by (2.31)-(2.36) we have

$$\left|(\Psi_{k,j}, \varphi^*_{p,q,s})\right| < c_5 k^{-2}, \forall p \leq n_0, \tag{2.37}$$

$$\mid (\Psi_{k,j}, \varphi^*_{p,q}) \mid \leq \frac{c_6}{\left||p| - |k|\right| (|p| + |k|)}, \ \mid (\Psi_{k,j}, \varphi^*_{-k,q}) \mid \leq \frac{c_7}{|k|} \tag{2.38}$$

for all $|k| \gg 1$, $p \neq \pm k$, $s = 0, 1, ..., t(p)$ and $q, j = 1, 2, ..., m$.

Now, we are ready to prove the lemmas.

Lemma 2.0.1 *For any $i = 1, 2, ..., m$ and $j = 1, 2, ..., m$ the following estimation holds*

$$\left(\Psi_{k,j}, (Q^* - C^*)\Phi^*_{k,i}\right) = O(\alpha_k) + O(\frac{\ln|k|}{k}), \tag{2.39}$$

where $\alpha_k = \max\left\{\mid b^+_{s,i,2k,r} \mid, \mid b^-_{s,i,2k,r} \mid : s, i = 1, 2, ..., m; \ r = 1, 2\right\}$,

$$b^{\pm}_{s,i,2k,r} = \int_0^1 b_{s,i}(x) e^{\pm i(4\pi k + 2\gamma_r)x}, \tag{2.40}$$

$b_{s,i} \in L_2(0,1)$ *are the entries of the matrix* Q, *and* γ_r *is defined in (2.4).*

Proof. Since $\Phi^*_{k,i}(x) = v^*_i \varphi^*_k(x)$, it is enough to prove that

$$\left(\Psi_{k,j}, (Q^* - C^*)\varphi^*_{k,s}\right) = O(\alpha_k) + O(\frac{\ln|k|}{k}) \tag{2.41}$$

for $s = 1, 2, ..., m$. The decomposition of $(Q^* - C^*)\varphi^*_{k,s}$ by the basis (2.16) has the form

$$(Q^* - C^*)\varphi^*_{k,s} = \sum_{q=1}^{m} \sum_{p:|p| \leq n_0} \sum_{v=1}^{t(p)} c(k, s, p, q, v)\varphi^*_{p,q,v} +$$

$$\sum_{q=1,2,\dots m}\ \sum_{p:|p|>n_0}^{\infty}((Q^*-C^*)\varphi^*_{k,s},\varphi_{p,q})\varphi^*_{p,q}$$

Therefore

$$(\Psi_{k,j},(Q^*-C^*)\varphi^*_{k,s})=\sum_{q=1}^{m}\sum_{p:|p|\le n_0}\sum_{v=1}^{t(p)}\overline{c(k,s,p,q,v)}(\Psi_{k,j},\varphi^*_{p,q,v})+ \tag{2.42}$$
$$\sum_{q=1}^{m}\sum_{p:|p|>n_0}\overline{((Q^*-C^*)\varphi^*_{k,s},\varphi_{p,q})}(\Psi_{k,j},\varphi^*_{p,q})$$

Since $c(k,s,p,q,v)=O(1)$, by (2.37) the first summation of the right hand side of (2.42) is $O(k^{-2})$.

Now let us estimate the second summation S of the right hand side of (2.42). It can be written in the form

$$S=S_1+S_2, \tag{2.43}$$

where

$$S_1=\sum_{q=1}^{m}\overline{((Q^*-C^*)\varphi^*_{k,s},\varphi_{k,q})}(\Psi_{k,j},\varphi^*_{k,q}),$$
$$S_2=\sum_{q=1}^{m}\sum_{p\neq k}\overline{((Q^*-C^*)\varphi^*_{k,s},\varphi_{p,q})}(\Psi_{k,j},\varphi^*_{p,q}).$$

Using (2.13), (2.14), (2.20) and (2.22), one can easily verify that

$$S_1=O\left(\alpha_k\right)+O(\frac{1}{k}) \tag{2.44}$$

On the other hand, by (2.38), we have

$$S_2=O(\frac{\ln|k|}{k}) \tag{2.45}$$

Therefore, (2.41) follows from (2.42)-(2.45). The lemma is proved. ■

Lemma 2.0.2 *For each eigenfunction $\Psi_{k,j}$ of $L_m(Q)$, where $|k|\ge N$, there exists an eigenfunction of $(L_m(C))^*$ denoted by $\Phi^*_{k,j}$ such that*

$$\left|\left(\Psi_{k,j},\Phi^*_{k,j}\right)\right|>c_8. \tag{2.46}$$

Proof. Since (2.15) is a Riesz basis of $L_2^m\left[0,1\right]$, we have

$$\Psi_{k,j}=\sum_{q=1}^{m}\sum_{p:|p|\le n_0}\sum_{v=1}^{t(p)}c(k,j,p,q,v)\varphi_{p,q,v}+\sum_{q=1}^{m}\sum_{p:|p|>n_0}\left(\Psi_{k,j},\varphi^*_{p,q}\right)\varphi_{p,q}. \tag{2.47}$$

It follows from (2.20) and (2.38) that

$$\sum_{q=1}^{m}\sum_{\substack{p\neq k\\|p|>n_0}}\left\|\left(\Psi_{k,j},\varphi_{p,q}^{*}\right)\varphi_{p,q}\right\|=O(\frac{\ln|k|}{k}).\tag{2.48}$$

On the other hand, arguing as in the estimation for the first summation of (2.42) we get

$$\sum_{q=1}^{m}\sum_{p:|p|\le n_0}\sum_{j=1}^{t(p)}\|c(k,s,p,q,j)\varphi_{p,q,j}\|=O(\frac{1}{k^2}).$$

Therefore using (2.48), (2.47), we obtain

$$\Psi_{k,j}=\sum_{q=1}^{m}\left(\Psi_{k,j},\varphi_{k,q}^{*}\right)\varphi_{k,q}+O(\frac{\ln|k|}{k})\tag{2.49}$$

Since $\{\varphi_{k,1},\varphi_{k,2},....,\varphi_{k,m}\}$ is orthonormal system and $\|\Psi_{k,j}\|=1$, there exists an index q such that

$$\left|\left(\Psi_{k,j},\varphi_{k,q}^{*}\right)\right|>c_9.\tag{2.50}$$

On the other hand

$$\varphi_{k,q}^{*}=\sum_{j=1}^{m}\left(\varphi_{k,q}^{*},\Phi_{k,j}\right)\Phi_{k,j}^{*}\tag{2.51}$$

because $\Phi_{k,j}^{*}=v_j^{*}\varphi_k^{*}$, and the vectors v_j^{*}, $j=1,2,...m$ form a basis in $\mathbb{C}^m$. Now, using (2.51) in (2.50), we get the proof of the lemma. ■

Here comes the main and new result of this chapter:

Theorem 2.0.2 *Suppose that all eigenvalues $\mu_1,\mu_2,...,\mu_m$ of the matrix C are simple. Then, there exists a number N such that all eigenvalues $\lambda_{k,1},\lambda_{k,2},...,\lambda_{k,m}$ of $L_m(Q)$ for $|\ k\ |\ge N$ are simple and satisfy the asymptotic formula*

$$\lambda_{k,j}=\mu_{k,j}+O(\alpha_k)+O(\frac{\ln|k|}{k}),\tag{2.52}$$

where $\mu_{k,j}$ is the eigenvalue of $L_m(C)$ and α_k is defined in Lemma 2.0.1. The normalized eigenfunction $\Psi_{k,j}(x)$ of $L_m(Q)$ corresponding to $\lambda_{k,j}$ satisfies

$$\Psi_{k,j}(x)=\Phi_{k,j}(x)+O(\alpha_k)+O(\frac{\ln|k|}{k}),\tag{2.53}$$

where $\Phi_{k,j}(x)$ is the normalized eigenfunction of $L_m(C)$ corresponding to $\mu_{k,j}$. The root functions of $L_m(Q)$ form a Riesz basis in $L_2^m(0,1)$.

Proof. In (2.29) replacing i by j, and then dividing the both sides of the obtained equality by $(\Psi_{k,j}, \Phi_{k,j}^*)$ and using Lemma 2.0.1 and Lemma 2.0.2, we see all large eigenvalues of $L_m(Q)$ lie in r_k neighborhood of the eigenvalues $\mu_{k,j}$ for $\mid k \mid \geq N$, $j = 1, 2, ..., m$ of $L_m(C)$, where

$$r_k = O(\alpha_k) + O(\frac{\ln |k|}{k}). \tag{2.54}$$

Now we prove that these eigenvalues are simple. Let $\lambda_{k,j}$ be an eigenvalue of $L_m(Q)$ lying in $\frac{1}{2}a_j$ neighborhood of $\mu_{k,j} = \rho_k + \mu_j$ (see (2.28)), where $a_j = \min_{i \neq j} \mid \mu_j - \mu_i \mid$. Then, by triangle inequality

$$\mid \lambda_{k,j} - \mu_{k,i} \mid > \mid \mu_{k,j} - \mu_{k,i} \mid - \mid \lambda_{k,j} - \mu_{k,j} \mid \geq a_j - \frac{1}{2}a_j = \frac{1}{2}a_j$$

for $i \neq j$. Therefore using (2.29) and Lemma 2.0.1 we get

$$(\Psi_{k,j}, \Phi_{k,i}^*) = O(\alpha_k) + O(\frac{\ln |k|}{k})$$

for $i \neq j$. This and (2.49) imply that (2.53) holds for any normalized eigenfunction corresponding to $\lambda_{k,j}$, since

$$span\{\varphi_{k,1}, \varphi_{k,2}, ..., \varphi_{k,m}\} = span\{\Phi_{k,1,}, \Phi_{k,2,}, ..., \Phi_{k,m,}\}.$$

Using this, let us prove that $\lambda_{k,j}$ is a simple eigenvalue. Suppose to the contrary that $\lambda_{k,j}$ is a multiple eigenvalue. If there are two linearly independent eigenfunctions corresponding to $\lambda_{k,j}$, then one can find two orthogonal eigenfunctions satisfying (2.53), which is impossible. Hence there exists a unique eigenfunction $\Psi_{k,j}$ corresponding to $\lambda_{k,j}$. If there exists an associated function $\Psi_{k,j,1}$ belonging to the eigenfunction $\Psi_{k,j}$, then

$$(L_m(Q) - \lambda_{k,j})\Psi_{k,j,1}(x) = \Psi_{k,j}(x).$$

Multiplying both sides of this equality by $\Psi_{k,j}^*(x)$, where $\Psi_{k,j}^*(x)$ is the normalized eigenfunction of $(L_m(Q))^*$ corresponding to the eigenvalue $\overline{\lambda_{k,j}}$,we obtain

$$(\Psi_{k,j}, \Psi_{k,j}^*) = (\Psi_{k,i,1}, ((L_m(Q))^* - \overline{\lambda_{k,j}}))\Psi_{k,j}^*) = 0. \tag{2.55}$$

Since the proved statements are also applicable for the adjoint operator $(L_m(Q))^*$, formula (2.53) holds for this operator too, that is, we have

$$\Psi_{k,j}^*(x) = \Phi_{k,j}^*(x) + O(\alpha_k) + O(\frac{\ln |k|}{k}). \tag{2.56}$$

This formula, (2.53) and the obvious relation $(\Phi_{k,j}, \Phi_{k,j}^*) = 1$ contradict with (2.55). Thus, $\lambda_{k,j}$ is a simple eigenvalue.

We proved that all large eigenvalues of $L_m(Q)$ lie in the disk

$$\Delta_{k,j} = \{z : |z - \mu_{k,j}| < r_k\}$$

for $\mid k \mid \geq N$, $j = 1, 2, ..., m$, where r_k is defined in (2.54). Clearly, the disks $\Delta_{k,j}$ for $j = 1, 2, ..., m$ and $\mid k \mid \geq N$ are pairwise disjoint. Let us prove that each of these disks does not contain more than one eigenvalue of $L_m(Q)$. Suppose to the contrary that, two different eigenvalues Λ_1 and Λ_2 lie in $\Delta_{k,j}$. Then it has already been proven that these eigenvalues are simple and the corresponding eigenfunctions Ψ_1 and Ψ_2 satisfy

$$\Psi_p(x) = \Phi_{k,j}(x) + O(\alpha_k) + O(\frac{\ln|k|}{k})$$

for $p = 1, 2$. Similarly, the eigenfunctions Ψ_1^* and Ψ_2^* of $(L_m(Q))^*$ corresponding to the eigenvalues $\overline{\Lambda_1}$ and $\overline{\Lambda_2}$ satisfy

$$\Psi_p^*(x) = \Phi_{k,j}^*(x) + O(\alpha_k) + O(\frac{\ln|k|}{k}).$$

for $p = 1, 2$. Since $\Lambda_1 \neq \Lambda_2$, we have

$$0 = (\Psi_1, \Psi_2^*) = 1 + O(\alpha_k) + O(\frac{\ln|k|}{k})$$

which is impossible. Hence the pairwise disjoint disks $\Delta_{k,1}$, $\Delta_{k,2}$, ..., $\Delta_{k,m}$, where $|k| \geq N$, contain m eigenvalues of $L_m(Q)$ and each of these disks does not contain more than one eigenvalue. Therefore, there exists a unique eigenvalue $\lambda_{k,j}$ of $L_m(Q)$ lying in $\Delta_{k,j}$, where $j = 1, 2, ..., m$ and $|k| \geq N$. Thus the eigenvalues $\lambda_{k,j}$ for $|k| \geq N$ are simple and the formulas (2.52) and (2.53) hold.

It remains to prove that the root functions of $L_m(Q)$ form a Riesz basis in $L_2^m(0, 1)$. For this, let us prove that for $f \in L_2^m(0, 1)$, the following series is convergent

$$\sum_{j=1}^{m} \sum_{k=N+1}^{\infty} |(f, \Psi_{k,j})|^2 < \infty, \tag{2.57}$$

where N is a large positive number. By the asymptotic formula (2.53), we have

$$\sum_{k=N+1}^{\infty} |(f, \Psi_{k,j})|^2 \leq 3(\sum_{k=N+1}^{\infty} |(f, \Phi_{k,j})|^2 + \sum_{k=N+1}^{\infty} |(f, g_k)|^2 + \sum_{k=N+1}^{\infty} |(f, h_k)|^2) \tag{2.58}$$

where $\|g_k\| = O(\alpha_k)$ and $\|h_k\| = O(\frac{\ln|k|}{k})$. The first series in the right side of (2.58) converges, since the root functions of $L_m(C)$ is a Riesz basis in $L_2^m(0, 1)$. Using the Cauchy-Schwarz inequality we get

$$\sum_{k=N+1}^{\infty} |(f, g_k)|^2 \leq c_{10} \|f\|^2 \sum_{k=N+1}^{\infty} |\alpha_k|^2 . \tag{2.59}$$

On the other hand, using the definition of α_k (see Lemma 2.0.1) and taking into account that the entries of the matrix Q are the element of $L_2\left(0,1\right)$, we obtain

$$\sum_{k=N+1}^{\infty} \left|\alpha_k\right|^2 < \infty.$$

Therefore, by (2.59), the second series in the right side of (2.58) converges too. In the same way, we prove that the third series in the right side of (2.58) converges. Thus, the series of the left-hand side of (2.58) converges, that is, (2.57) is proved. By Bari's definition, this implies that the system of eigenfunctions of the operator under consideration is Bessel. Since the system of root functions of the adjoint operator has the asymptotics (2.56), in the same way, we obtain that it is also Bessel. Moreover, the equality

$$(\Psi_{k,j}, \Psi_{k,j}^{*}) = 1 + O\left(\alpha_k\right) + O(\frac{\ln|k|}{k})$$

(see (2.53) and (2.56)) implies that the system of the root functions of $\left(L_m(Q)\right)^*$, which is biorthogonal to the system of the root functions of $L_m(Q)$, is also Bessel. As it is noted in [52] and [53], the system of root functions of the operators $L_m(Q)$ and $\left(L_m(Q)\right)^*$ are complete in the space $L_2^m(0,1)$. These arguments and Bari's theorem (if two biorthogonal systems are complete and Bessel, then they both are Riesz bases, see Theorem 1.4.2) conclude the proof of the theorem. ■

These results are given in [47] and published in 2014.

3

Numerical Estimation of Small Eigenvalues

3.1 System of Sturm Liouville Operator with Dirichlet Boundary Conditions

In this chapter, we consider the differential operator $D_d(Q)$ generated in the space $L_2^d[0,1]$ by the differential expression (0.1) and the Dirichlet boundary conditions

$$\mathbf{y}(0) = \mathbf{y}(1) = 0. \tag{3.1}$$

First, for simplicity, let us consider the case $d = 2$. Then, we will investigate the case $d = 3$ and general d.

To estimate the small eigenvalues of $D_d(Q)$, take an equally spaced mesh

$$0 = x_0 < x_1 < ... < x_{m+1} = 1,$$

where

$$x_j = jh, \qquad h = \frac{1}{m+1}.$$

Substituting

$$\mathbf{y}(x_j) = \mathbf{y}_j = (y_{j,1}, y_{j,2})^T, \; Q(x_j) = \begin{bmatrix} b_{1,1}(x_j) & b_{1,2}(x_j) \\ b_{2,1}(x_j) & b_{2,2}(x_j) \end{bmatrix}, \; \mathbf{y}''(x_j) = \mathbf{y}_j'', \tag{3.2}$$

and using the centered difference approximation

$$-\mathbf{y}_j^{''} = \frac{-\mathbf{y}_{j-1} + 2\mathbf{y}_j - \mathbf{y}_{j+1}}{h^2} + O(h^2),$$

in the equation

$$-\mathbf{y}^{''}(x) + Q(x)\mathbf{y}(x) = \lambda \mathbf{y}, \tag{3.3}$$

we obtain the approximating scheme

$$\frac{-\mathbf{y}_{j-1} + 2\mathbf{y}_j - \mathbf{y}_{j+1}}{h^2} + \mathbf{O}(h^2) + Q_j\mathbf{y}_j = \lambda \mathbf{y}_j \qquad j = 1, 2, ..., m. \tag{3.4}$$

Incorporating the boundary conditions (3.1), we get

$$\mathbf{y}_0 = 0, \text{ and } \mathbf{y}_{m+1} = 0.$$

One can easily notice that, (3.4) is a system of $2m$ equation with respect to the $2m$ unknown $y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}, ..., y_{m,1}, y_{m,2}$. Now, let us write all equations which form this system with these unknown functions.

For $j = 1$,

$$\frac{-\mathbf{y}_0 + 2\mathbf{y}_1 - \mathbf{y}_2}{h^2} + Q(x_1)\mathbf{y}_1 - \lambda \mathbf{y}_1 = \mathbf{O}(h^2).$$

Using (3.2), we obtain

$$\frac{2\begin{bmatrix} y_{1,1} \\ y_{1,2} \end{bmatrix} - \begin{bmatrix} y_{2,1} \\ y_{2,2} \end{bmatrix}}{h^2} + \begin{bmatrix} b_{1,1}(x_1) & b_{1,2}(x_1) \\ b_{2,1}(x_1) & b_{2,2}(x_1) \end{bmatrix} \begin{bmatrix} y_{1,1} \\ y_{1,2} \end{bmatrix} - \lambda \begin{bmatrix} y_{1,1} \\ y_{1,2} \end{bmatrix} = \mathbf{O}(h^2),$$

and with the operations on matrices, we have

$$\frac{\begin{bmatrix} 2y_{1,1} - y_{2,1} \\ 2y_{1,2} - y_{2,2} \end{bmatrix}}{h^2} + \begin{bmatrix} b_{1,1}(x_1)y_{1,1} + b_{1,2}(x_1)y_{1,2} \\ b_{2,1}(x_1)y_{1,1} + b_{2,2}(x_1)y_{1,2} \end{bmatrix} - \lambda \begin{bmatrix} y_{1,1} \\ y_{1,2} \end{bmatrix} = \mathbf{O}(h^2).$$

Finally, when we simplify, we obtain the first two equations as:

$$\begin{aligned} \frac{2y_{1,1} - y_{2,1}}{h^2} + b_{1,1}(x_1)y_{1,1} + b_{1,2}(x_1)y_{1,2} - \lambda y_{1,1} = O(h^2), \\ \frac{2y_{1,2} - y_{2,2}}{h^2} + b_{2,1}(x_1)y_{1,1} + b_{2,2}(x_1)y_{1,2} - \lambda y_{1,2} = O(h^2). \end{aligned} \tag{3.5}$$

For $j = 2$, we have the equation

$$\frac{-\mathbf{y}_1 + 2\mathbf{y}_2 - \mathbf{y}_3}{h^2} + Q(x_2)\mathbf{y}_2 - \lambda \mathbf{y}_2 = \mathbf{O}(h^2).$$

After the operations of multiplication and addition on matrices,

$$\frac{-\begin{bmatrix} y_{1,1} \\ y_{1,2} \end{bmatrix} + 2\begin{bmatrix} y_{2,1} \\ y_{2,2} \end{bmatrix} - \begin{bmatrix} y_{3,1} \\ y_{3,2} \end{bmatrix}}{h^2} + \begin{bmatrix} b_{1,1}(x_2) & b_{1,2}(x_2) \\ b_{2,1}(x_2) & b_{2,2}(x_2) \end{bmatrix}\begin{bmatrix} y_{2,1} \\ y_{2,2} \end{bmatrix} - \lambda\begin{bmatrix} y_{2,1} \\ y_{2,2} \end{bmatrix} = \mathbf{O}(h^2),$$

it can be easily seen that we obtain,

$$\begin{aligned} \frac{-y_{1,1} + 2y_{2,1} - y_{3,1}}{h^2} + b_{1,1}(x_2)y_{2,1} + b_{1,2}(x_2)y_{2,2} - \lambda y_{2,1} = O(h^2), \\ \frac{-y_{1,2} + 2y_{2,2} - y_{3,2}}{h^2} + b_{2,1}(x_2)y_{2,1} + b_{2,2}(x_2)y_{2,2} - \lambda y_{2,2} = O(h^2). \end{aligned} \tag{3.6}$$

For $j = m$, we have the equation

$$\frac{-\mathbf{y}_{m-1} + 2\mathbf{y}_m - \mathbf{y}_{m+1}}{h^2} + Q(x_m)\mathbf{y}_m - \lambda\mathbf{y}_m = \mathbf{O}(h^2),$$

In the same way and using the boundary condition $\mathbf{y}_{m+1} = 0$, we obtain

$$\frac{-\begin{bmatrix} y_{m-1,1} \\ y_{m-1,2} \end{bmatrix} + 2\begin{bmatrix} y_{m,1} \\ y_{m,2} \end{bmatrix}}{h^2} + \begin{bmatrix} b_{1,1}(x_m) & b_{1,2}(x_m) \\ b_{2,1}(x_m) & b_{2,2}(x_m) \end{bmatrix}\begin{bmatrix} y_{m,1} \\ y_{m,2} \end{bmatrix} - \lambda\begin{bmatrix} y_{m,1} \\ y_{m,2} \end{bmatrix} = \mathbf{O}(h^2),$$

Hence the last two equations are,

$$\begin{aligned} \frac{-y_{m-1,1} + 2y_{m,1}}{h^2} + b_{1,1}(x_m)y_{m,1} + b_{1,2}(x_m)y_{m,2} - \lambda y_{m,1} = O(h^2), \\ \frac{-y_{m-1,2} + 2y_{m,2}}{h^2} + b_{2,1}(x_m)y_{m,1} + b_{2,2}(x_m)y_{m,2} - \lambda y_{m,2} = O(h^2). \end{aligned} \tag{3.7}$$

Therefore, the equation (3.5)-(3.7) can be written in the matrix form

$$(T_2 - \lambda I)Y = \mathbf{O}(h^2), \tag{3.8}$$

where $Y = (y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}, ..., y_{m,1}, y_{m,2})^T$ and $\mathbf{O}(h^2)$ is an $2m$ dimensional vector with components $O(h^2)$. Here T_2 is defined by

$$T_2 = \frac{1}{h^2}K_2 + B_2,$$

where

$$K_2 = \begin{pmatrix} 2 & 0 & -1 & & & \\ 0 & 2 & 0 & -1 & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & -1 & 0 & & 2 & 0 \\ & & -1 & & 0 & 2 \end{pmatrix},$$

and

$$B_2 = \begin{pmatrix} b_{1,1}(x_1) & b_{1,2}(x_1) & & & & \\ b_{2,1}(x_1) & b_{2,2}(x_1) & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & b_{1,1}(x_m) & b_{1,2}(x_m) \\ & & & & b_{2,1}(x_m) & b_{2,2}(x_m) \end{pmatrix}. \tag{3.9}$$

Similarly, for the case $d = 3$, substituting

$$\mathbf{y}(x_j) = \mathbf{y}_j = (y_{j,1}, y_{j,2}, y_{j,3})^T, \; Q(x_j) = \begin{bmatrix} b_{1,1}(x_j) & b_{1,2}(x_j) & b_{1,3}(x_j) \\ b_{2,1}(x_j) & b_{2,2}(x_j) & b_{2,3}(x_j) \\ b_{3,1}(x_j) & b_{3,2}(x_j) & b_{3,3}(x_j) \end{bmatrix}, \tag{3.10}$$

in the equation (3.3), and writing all equations obtained from the approximating scheme (3.4) for $j = 1, ..., m$, we get the following matrix form

$$(T_3 - \lambda I)Y = \mathbf{O}(h^2),$$

where

$$T_3 = \frac{1}{h^2} K_3 + B_3,$$

with the matrices K_3 and B_3 are defined as,

$$K_3 = \begin{pmatrix} 2 & 0 & 0 & -1 & & \\ 0 & 2 & 0 & 0 & -1 & \\ 0 & 0 & 2 & 0 & 0 & -1 \\ & & & \cdot & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & -1 & 0 & 0 & 2 & 0 & 0 \\ & & -1 & 0 & 0 & 2 & 0 \\ & & & -1 & 0 & 0 & 2 \end{pmatrix},$$

and

$$B_3 = \begin{pmatrix} b_{1,1}(x_1) & b_{1,2}(x_1) & b_{1,3}(x_1) & & & & \\ b_{2,1}(x_1) & b_{2,2}(x_1) & b_{2,3}(x_1) & & & & \\ b_{3,1}(x_1) & b_{3,2}(x_1) & b_{3,3}(x_1) & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & b_{1,1}(x_m) & b_{1,2}(x_m) & b_{1,3}(x_m) \\ & & & & b_{2,1}(x_m) & b_{2,2}(x_m) & b_{2,3}(x_m) \\ & & & & b_{3,1}(x_m) & b_{3,2}(x_m) & b_{3,3}(x_m) \end{pmatrix}. \tag{3.11}$$

In the same way, for general case d instead of (3.8), we obtain the following equation

$$(T_d - \lambda I)Y = \mathbf{O}(h^2), \tag{3.12}$$

where

$$T_d = \frac{1}{h^2}K_d + B_d,$$

and the matrices K_d, B_d are defined as follows:

$$K_d = \begin{bmatrix} M_1 & P_1 & & & & \\ N_2 & M_2 & P_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & N_{m-1} & M_{m-1} & P_{m-1} \\ & & & & N_m & M_m \end{bmatrix},$$

where $M_1, M_2, ..., M_m$ are $d \times d$ diagonal matrices with entries 2, $N_2, ..., N_m$ and $P_1, P_2, ..., P_{m-1}$ are $d \times d$ diagonal matrices which have entries -1, and

$$B_d = \begin{bmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_m \end{bmatrix}, \tag{3.13}$$

where $R_1, R_2, ..., R_m$ are $d \times d$ square matrices that have the form

$$R_1 = \begin{bmatrix} b_{1,1}(x_1) & ... & b_{1,d}(x_1) \\ . & . & . \\ b_{d,1}(x_1) & ... & b_{d,d}(x_1) \end{bmatrix}, ..., R_m = \begin{bmatrix} b_{1,1}(x_m) & ... & b_{1,d}(x_m) \\ . & . & . \\ b_{d,1}(x_m) & ... & b_{d,d}(x_m) \end{bmatrix},$$

Now, we prove that the eigenvalues of the operator $D_d(Q)$ are approximated by the eigenvalues of the matrix T_d.

Theorem 3.1.1 *Suppose that $Q(x)$ is a symmetric matrix for all $x \in [0, 1]$. Let $\lambda_1, \lambda_2, ..., \lambda_{dm}$ be eigenvalues of the matrix T_d. Then, for every small eigenvalue λ of $D_d(Q)$ there is an index j such that*

$$\lambda - \lambda_j = O(h^{\frac{3}{2}}). \tag{3.14}$$

Proof. To prove the theorem we use (3.12). Since $md = O(\frac{1}{h})$ and as we noted above that the right -hand side of (3.12) is a vector with components having the norm $O(h^2)$, we obtain

$$\| \mathbf{O}(h^2) \| = (mdO(h^4))^{\frac{1}{2}} = O(h^{\frac{3}{2}}),$$

since $md = O(\frac{1}{h})$. On the other hand , without loss of the generality it can be assumed that $\|Y\| = 1$. Using these we obtain $Y = (T_d - \lambda I)^{-1}\mathbf{O}(h^2)$ and

$$1 \leq \| (T_d - \lambda I)^{-1} \| O(h^{\frac{3}{2}}).$$

Since $(T_d - \lambda I)^{-1}$ is the symmetric matrix having the eigenvalues $(\lambda_i - \lambda)^{-1}$ for $i = 1, 2, ..., dm$, we have

$$\| (T_d - \lambda I)^{-1} \| = \max_{i=1,2,...,dm} | \lambda - \lambda_i |^{-1} = | \lambda - \lambda_j |^{-1},$$

for some j. The last two equalities give

$$1 \leq | \lambda - \lambda_j |^{-1} O(h^{\frac{3}{2}}),$$

which implies the equality (3.14). ■

3.2 System of Sturm Liouville Operator with Separated Boundary Conditions

In this section, we consider the differential operator $S_d(Q)$ generated in the space $L_2^d[0,1]$ by the differential expression (0.1) and separated boundary conditions

$$\begin{aligned} \alpha_1 \mathbf{y}'(0) - \alpha_2 \mathbf{y}(0) &= 0, \\ \beta_1 \mathbf{y}'(1) + \beta_2 \mathbf{y}(1) &= 0, \end{aligned} \tag{3.15}$$

where $\alpha_1 \neq 0$, $\beta_1 \neq 0$.

These boundary conditions (3.15) can be written in the form

$$\begin{aligned} \mathbf{y}'(0) &= a\mathbf{y}(0), \\ \mathbf{y}'(1) &= b\mathbf{y}(1). \end{aligned} \tag{3.16}$$

It easily follows from Proposition 1.2.1(a) that the boundary conditions (3.16) for the scalar case $d = 1$ are strongly regular. Again, as in the previous section, first let us consider the case $d = 2$, then for $d = 3$ and then for general d. Using the boundary conditions (3.16), Taylor series at $x = 0$ and taking into account the equality

$$\mathbf{y}''(0) = (Q(0) - \lambda)\mathbf{y}(0), \tag{3.17}$$

(see (3.3)), we obtain

$$\begin{aligned} y_{1,1} &= y_{0,1} + hy_{0,1}' + \frac{1}{2}h^2 y_{0,1}'' + O(h^3), \\ &= y_{0,1} + hay_{0,1} + \frac{1}{2}h^2(Q(0) - \lambda)y_{0,1} + O(h^3), \\ &= y_{0,1} + hay_{0,1} + \frac{1}{2}h^2((b_{1,1}(0) - \lambda)y_{0,1} + b_{1,2}(0)y_{0,2}) + O(h^3). \end{aligned} \tag{3.18}$$

First, let us estimate the expression

$$y_{0,1}^{''} = ((b_{1,1}(0) - \lambda)y_{0,1} + b_{1,2}(0)y_{0,2}). \tag{3.19}$$

First consider the case $y_{0,1} = c \neq 0$. By Definition 1.1.1 $y_1^{'}$ is an absolutely continuous function on the closed interval $[0, 1]$ and hence there exists a positive constant K such that $\left|y_1^{'}(x)\right| \leq K$ for all $x \in [0, 1]$. Then

$$\left|\int_0^x y_1^{'}(t)dt\right| \leq Kx.$$

This, with the equality

$$y_1(x) = y_1(0) + \int_0^x y_1^{'}(t)dt,$$

where $y_1(0) = y_{0,1}$, and the triangle inequality implies that

$$|y_1(x)| \geq |y_{0,1}| - \left|\int_0^x y_1^{'}(t)dt\right| \geq |y_{0,1}| - Kx$$

From this inequality we obtain that

$$|y_1(x)| \geq \frac{1}{2}|y_{0,1}|, \ \forall x \in [0, \frac{|c|}{2K}] \tag{3.20}$$

Let p be integer part of $m\frac{|c|}{2K}$. Then by (3.20) we have

$$|y_{k,1}| \geq \frac{1}{2}|y_{0,1}|, \forall k = 1, 2, ..., p$$

which implies that

$$|y_{0,1}|^2 \leq 4|y_{k,1}|^2, \forall k = 1, 2, ..., p$$

Therefore we have

$$p|y_{0,1}|^2 \leq 4\sum_{k=1}^{p}|y_{k,1}|^2 \leq 4\sum_{k=1}^{m}|y_{k,1}|^2$$

and hence

$$|y_{0,1}|^2 = O(h)\sum_{k=1}^{m}|y_{k,1}|^2, \tag{3.21}$$

since $\frac{1}{p} = O(h)$. On the other hand, if $y_1(x)$ is eigenfunction then $Cy_1(x)$, for any constant C, is also eigenfunction and (3.21) holds if $y_1(x)$ is replaced by $Cy_1(x)$. Therefore it can be assumed without loss of generality that

$$\sum_{k=1}^{m}|y_{k,1}|^2 = 1. \tag{3.22}$$

Then from (3.21) and (3.22) we obtain

$$y_{0,1} = O(h^{1/2}) \tag{3.23}$$

in the case $y_{0,1} \neq 0$. It is clear that (3.23) holds in the case $y_{0,1} = 0$ too.

In the same way we obtain

$$y_{0,2} = O(h^{1/2}). \tag{3.24}$$

Thus, by (3.19), (3.23) and (3.24) we have

$$((b_{1,1}(0) - \lambda)y_{0,1} + b_{1,2}(0)y_{0,2}) = O(h^{1/2}). \tag{3.25}$$

Using this in (3.18), we get

$$y_{1,1} = y_{0,1} + hay_{0,1} + O(h^{5/2}),$$

that is,

$$y_{0,1} = \frac{y_{1,1}}{1+ah} + O(h^{5/2}). \tag{3.26}$$

In the same way, we obtain

$$y_{0,2} = \frac{y_{1,2}}{1+ah} + O(h^{5/2}). \tag{3.27}$$

Moreover arguing as above and using the Taylor series at $x = 1$,we have the equality

$$y_{m,1} = y_{m+1,1} - hy'_{m+1,1} + \frac{1}{2}h^2 y''_{m+1,1} + O(h^3),$$

and therefore

$$y_{m+1,1} = \frac{y_{m,1}}{1-bh} + O(h^{5/2}), y_{m+1,2} = \frac{y_{m,2}}{1-bh} + O(h^{5/2}). \tag{3.28}$$

Now, let us define the matrix Z_2 with the similar operations that are done in the previous section.

For $j = 1$, we have

$$\frac{-\mathbf{y}_0 + 2\mathbf{y}_1 - \mathbf{y}_2}{h^2} + Q(x_1)\mathbf{y}_1 - \lambda \mathbf{y}_1 = \mathbf{O}(h^{1/2}).$$

Using (3.26), (3.27) and taking into account the assumptions (3.22)-(3.23), we obtain

$$\frac{-\begin{bmatrix} \frac{y_{1,1}}{1+ah} \\ \frac{y_{1,2}}{1+ah} \end{bmatrix} + 2\begin{bmatrix} y_{1,1} \\ y_{1,2} \end{bmatrix} - \begin{bmatrix} y_{2,1} \\ y_{2,2} \end{bmatrix}}{h^2} + \mathbf{O}(h^{1/2}) + \begin{bmatrix} b_{1,1}(x_1) & b_{1,2}(x_1) \\ b_{2,1}(x_1) & b_{2,2}(x_1) \end{bmatrix} \begin{bmatrix} y_{1,1} \\ y_{1,2} \end{bmatrix} = \lambda \begin{bmatrix} y_{1,1} \\ y_{1,2} \end{bmatrix},$$

simplifying the matrices by the operations of addition and multiplication,

$$\frac{\begin{bmatrix} 2y_{1,1}-y_{2,1}-\frac{y_{1,1}}{1+ah} \\ 2y_{1,2}-y_{2,2}-\frac{y_{1,2}}{1+ah} \end{bmatrix}}{h^2}+\mathbf{O}(h^{1/2})+\begin{bmatrix} b_{1,1}(x_1)y_{1,1}+b_{1,2}(x_1)y_{1,2} \\ b_{2,1}(x_1)y_{1,1}+b_{2,2}(x_1)y_{1,2} \end{bmatrix}=\lambda\begin{bmatrix} y_{1,1} \\ y_{1,2} \end{bmatrix},$$

and we get

$$\begin{aligned} \frac{(2-\frac{1}{1+ah})y_{1,1}-y_{2,1}}{h^2}+O(h^{1/2})+b_{1,1}(x_1)y_{1,1}+b_{1,2}(x_1)y_{1,2}=\lambda y_{1,1}, \\ \frac{(2-\frac{1}{1+ah})y_{1,2}-y_{2,2}}{h^2}+O(h^{1/2})+b_{2,1}(x_1)y_{1,1}+b_{2,2}(x_1)y_{1,2}=\lambda y_{1,2}. \end{aligned} \tag{3.29}$$

For $j=2$,

$$\frac{-\mathbf{y}_1+2\mathbf{y}_2-\mathbf{y}_3}{h^2}+\mathbf{O}(h^2)+Q(x_2)\mathbf{y}_2=\lambda\mathbf{y}_2.$$

Notice that the order of error is $O(h^2)$, because of centered difference formula for the second derivative. Hence

$$\frac{-\begin{bmatrix} y_{1,1} \\ y_{1,2} \end{bmatrix}+2\begin{bmatrix} y_{2,1} \\ y_{2,2} \end{bmatrix}-\begin{bmatrix} y_{3,1} \\ y_{3,2} \end{bmatrix}}{h^2}+\mathbf{O}(h^2)+\begin{bmatrix} b_{1,1}(x_2) & b_{1,2}(x_2) \\ b_{2,1}(x_2) & b_{2,2}(x_2) \end{bmatrix}\begin{bmatrix} y_{2,1} \\ y_{2,2} \end{bmatrix}=\lambda\begin{bmatrix} y_{2,1} \\ y_{2,2} \end{bmatrix},$$

and finally we have

$$\begin{aligned} \frac{-y_{1,1}+2y_{2,1}-y_{3,1}}{h^2}+O(h^2)+b_{1,1}(x_2)y_{2,1}+b_{1,2}(x_2)y_{2,2}=\lambda y_{2,1}, \\ \frac{-y_{1,2}+2y_{2,2}-y_{3,2}}{h^2}+O(h^2)+b_{2,1}(x_2)y_{2,1}+b_{2,2}(x_2)y_{2,2}=\lambda y_{2,2}. \end{aligned} \tag{3.30}$$

For $j=m$, the approximating scheme have the form

$$\frac{-\mathbf{y}_{m-1}+2\mathbf{y}_m-\mathbf{y}_{m+1}}{h^2}+\mathbf{O}(h^{1/2})+Q(x_m)\mathbf{y}_m=\lambda\mathbf{y}_m.$$

Taking into account the boundary conditions, we get

$$\frac{-\begin{bmatrix} y_{m-1,1} \\ y_{m-1,2} \end{bmatrix}+2\begin{bmatrix} y_{m,1} \\ y_{m,2} \end{bmatrix}-\begin{bmatrix} \frac{y_{m,1}}{1-bh} \\ \frac{y_{m,2}}{1-bh} \end{bmatrix}}{h^2}+\mathbf{O}(h^{1/2})+\begin{bmatrix} b_{1,1}(x_m) & b_{1,2}(x_m) \\ b_{2,1}(x_m) & b_{2,2}(x_m) \end{bmatrix}\begin{bmatrix} y_{m,1} \\ y_{m,2} \end{bmatrix}=\lambda\begin{bmatrix} y_{m,1} \\ y_{m,2} \end{bmatrix},$$

and therefore

$$\begin{aligned} \frac{-y_{m-1,1}+(2-\frac{1}{1-bh})y_{m,1}}{h^2}+O(h^{1/2})+b_{1,1}(x_m)y_{m,1}+b_{1,2}(x_m)y_{m,2}=\lambda y_{m,1}, \\ \frac{-y_{m-1,2}+(2-\frac{1}{1-bh})y_{m,2}}{h^2}+O(h^{1/2})+b_{2,1}(x_m)y_{m,1}+b_{2,2}(x_m)y_{m,2}=\lambda y_{m,2}. \end{aligned} \tag{3.31}$$

The equation (3.29)-(3.31) can be written in the matrix form

$$(Z_2 - \lambda I)Y = u, \tag{3.32}$$

where $Y = (y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}, ..., y_{m,1}, y_{m,2})^T$ and Z_2 is defined by

$$Z_2 = \frac{1}{h^2}V_2 + W_2, \tag{3.33}$$

with the matrices

$$V_2 = \begin{pmatrix} 2-\frac{1}{1+ah} & 0 & -1 & 0 & . & & & \\ 0 & 2-\frac{1}{1+ah} & 0 & -1 & 0 & . & & \\ -1 & 0 & 2 & 0 & -1 & 0 & . & . \\ . & . & . & \ddots & . & . & . & . \\ . & . & . & . & \ddots & . & . & . \\ . & . & . & . & . & \ddots & . & . \\ . & & & 0 & -1 & 0 & 2-\frac{1}{1-bh} & 0 \\ . & & & & 0 & -1 & 0 & 2-\frac{1}{1-bh} \end{pmatrix},$$

and W_2 is defined as in (3.9). Here, $u = (u_1, u_2, ..., u_{2m})$ is a vector where

$$u_1 = O(h^{1/2}), u_2 = O(h^{1/2}), u_{2m-1} = O(h^{1/2}), u_{2m} = O(h^{1/2}),$$

and $u_k = O(h^2)$ for $2 < k < 2m-1$. Therefore we have

$$\|u\| = O(h^{1/2}). \tag{3.34}$$

Similarly, in case $d = 3$, instead of (3.31) we have

$$(Z_3 - \lambda I)Y = u.$$

Here, Z_3 is defined by

$$Z_3 = \frac{1}{h^2}V_3 + W_3,$$

where

$$V_3 = \begin{pmatrix} 2-\frac{1}{1+ah} & 0 & 0 & -1 & & & & & & \\ 0 & 2-\frac{1}{1+ah} & 0 & 0 & -1 & & & & & \\ 0 & 0 & 2-\frac{1}{1+ah} & 0 & 0 & -1 & & & & \\ -1 & 0 & 0 & 2 & 0 & 0 & -1 & & & \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ & & & & -1 & 0 & 0 & 2-\frac{1}{1-bh} & 0 & 0 \\ & & & & & -1 & 0 & 0 & 2-\frac{1}{1-bh} & 0 \\ & & & & & & -1 & 0 & 0 & 2-\frac{1}{1-bh} \end{pmatrix},$$

and W_3 is defined as in (3.11). Moreover in this case, (3.34) also holds.

In the same way, in the general case we obtain

$$(Z_d - \lambda I)Y = u,$$

and Z_d is defined by

$$Z_d = \frac{1}{h^2}V_d + W_d.$$

Now, V_d and W_d are defined as in the following:

$$V_d = \begin{bmatrix} E_1 & F_1 & & & & \\ G_2 & E_2 & F_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & G_{m-1} & E_{m-1} & F_{m-1} \\ & & & & G_m & E_m \end{bmatrix},$$

where

$$E_1 = \begin{bmatrix} 2 - \frac{1}{1+ah} & & \\ & \ddots & \\ & & 2 - \frac{1}{1+ah} \end{bmatrix},$$

$E_2, ..., E_{m-1}$ are $d \times d$ diagonal matrices with entries 2,

$$E_m = \begin{bmatrix} 2 - \frac{1}{1-bh} & & \\ & \ddots & \\ & & 2 - \frac{1}{1-bh} \end{bmatrix},$$

$G_2, ..., G_m$ and $F_1, F_2, ..., F_{m-1}$ are $d \times d$ diagonal matrices which have the entries -1. One can easily notice that there is no change for the matrix W_d. Therefore, in the general case, W_d is again defined as (3.13). Besides (3.34) holds in general case also.

Now using (3.32) and repeating the proof of Theorem 3.1.1 we state the following theorem:

Theorem 3.2.1 *Suppose that $Q(x)$ is a symmetric matrix for all $x \in [0, 1]$. Let $\lambda_1, \lambda_2, ..., \lambda_{dm}$ be eigenvalues of the matrix Z_d. If a and b are real numbers, then for every small eigenvalue λ of $S_d(Q)$ then there is an index j such that*

$$\lambda - \lambda_j = O(h^{\frac{1}{2}}). \tag{3.35}$$

4

Some Examples and Conclusions

In this chapter, we will give three examples about finite difference approximations applied to eigenvalue problem in the space of vector functions. With these examples, we show the combination of the theoretical facts which are proved in Chapter 2 and the numerical approach which are given in Chapter 3.

First, we want to show that the eigenvalue problem in vectoral case can be reduced to the eigenvalue problem in scalar case. To do this, let us consider the operator $L_d(Q)$ generated by the differential expression (0.1) and the Dirichlet boundary conditions (3.1) with the potential $Q(x) = q(x)I + A$, where $q(x)$ is the complex-valued square intregrable function, I is d by d unit matrix, A is a d by d constant matrix. In this case, we present the following proposition:

Proposition 4.0.1 *The eigenvalues and eigenfunctions of the operator $L_d(qI + A)$ are*

$$\rho_{k,j} = \lambda_k(q) + \rho_j \ \ \& \ \ \Phi_{k,j} = v_j \varphi_k(x)$$

for $k = 1, 2, ...$ and $j = 1, 2, ..., d$, where $\lambda_k(q)$ and $\varphi_k(x)$ are the eigenvalue and eigenfunctions of boundary value problem

$$L(q) = -y^{''}(x) + q(x)y(x)$$

with Dirichlet boundary conditions, ρ_j and v_j are the eigenvalue and eigenvector of the matrix A.

Proof. Indeed, using $Av_j = \rho_j v_j$ and $L(q)\varphi_k(x) = \lambda_k \varphi_k(x)$, we obtain,

$$L_d(q(x)I + A)\Phi_{k,j} = L_d(q(x)I)\Phi_{k,j} + A\Phi_{k,j},$$

$$= L_d(q(x)I)v_j\varphi_k(x) + Av_j\varphi_k(x) = \lambda_k(q)v_j\varphi_k(x) + \rho_j v_j\varphi_k(x),$$

$$= (\lambda_k(q) + \rho_j)v_j\varphi_k(x) = (\lambda_k(q) + \rho_j)\Phi_{k,j}.$$

which gives the proof. ■

Hence, for an ordinary eigenvalue problem in vectoral case, which has the potential in the form $Q(x) = q(x)I + A$, one can determine the eigenvalues of the operator $L_d(qI + A)$, using the results in scalar case (i.e. the eigenvalues of $L(q)$) and computing the eigenvalues of the matrix A.

Now we are ready to give the example:

Example 4.0.1 *Let $L_2^{(1)}(Q)$ be the differential operator generated in the space $L_2^2\,[0,1]$ by the differential expression (0.1) and the Dirichlet boundary conditions (3.1), where*

$$Q(x) = \begin{bmatrix} x^2+4 & 2 \\ -1 & x^2+1 \end{bmatrix}.$$

The eigenvalues of the matrix,

$$A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix},$$

are $\rho_1 = 2$ and $\rho_2 = 3$.

The eigenvalues of the boundary value problem with matrix potential obtained by finite difference method, are summarized in Table 4.1. Here the spacing is taken $h = \frac{1}{100}, h = \frac{1}{300}$ and $h = \frac{1}{500}$ respectively.

Table 4.1

k	$\rho_{k,j}(h = \frac{1}{100})$	$\rho_{k,j}(h = \frac{1}{300})$	$\rho_{k,j}(h = \frac{1}{500})$
1	12.150351845	12.151073785	12.151131540
2	13.150351845	13.151073785	13.151131540
3	41.786406415	41.797949881	41.798873475
4	42.786406415	42.797949881	42.798873475
5	91.088610368	91.147036971	91.151712425
6	92.0886610368	92.147036971	92.151715425
7	160.036264456	160.220873452	160.235649623
8	161.036264456	161.220873452	161.235649623
9	248.564577836	249.015134334	249.051207319
10	249.564577836	250.015134334	250.051207319
19	981.202522196	988.391682483	988.968634667
20	982.202522196	989.391682483	989.968634667
29	2182.202633316	2218.431810922	2221.350816542
30	2183.202633316	2219.431810922	2222.350816542
39	3821.993322956	3935.765080140	3944.982555028
40	3822.993322956	3936.765080140	3945.982555028
49	5860.197629400	6135.684823767	6158.162958280
50	5861.197629400	6136.684823767	6159.162958280

Now, Table 4.2 shows the eigenvalues of the boundary value problem with the potential q obtained by finite difference method. Here, again h is taken $\frac{1}{100}$, $\frac{1}{300}$ and $\frac{1}{500}$ respectively.

Table 4.2

k	$\lambda_k(h = \frac{1}{100})$	$\lambda_k(h = \frac{1}{300})$	$\lambda_k(h = \frac{1}{500})$
1	10.150351840	10.151073785	10.151131542
2	39.786406415	39.797949881	39.798873475
3	89.088610368	89.147036971	89.151712426
4	158.036264456	158.220873452	158.235649624
5	246.564577836	247.015134334	247.051207319
10	979.202522196	986.391682483	986.968634667
15	2180.202633162	2216.431810922	2219.350816542
20	3819.993322956	3933.765080141	3942.982555028
25	5858.197629400	6133.684523767	6156.162958281

Therefore, if we add the eigenvalues of the matrix A; i.e. $\rho_1 = 2$ and $\rho_2 = 3$, to the each eigenvalue given in Table 4.2, we obtain the eigenvalues given in Table 4.1, which satisfies Proposition 4.0.1.

Now, the next two examples concern with the estimation of small eigenvalues of the operators generated in the space $L_2^2[0,1]$ and $L_2^3[0,1]$ respectively.

Since we want to show the effect of h to the approximation of the solution, in these two examples, h is taken as $\frac{1}{100}$, $\frac{1}{900}$ and $\frac{1}{2500}$ respectively.

Moreover, since we are interested in the estimate of small eigenvalues, we list the first 100 eigenvalues in tables.

The whole results are in the end of the chapter.

Example 4.0.2 *We consider the differential operator $L_2^{(2)}(Q)$ generated in the space $L_2^2[0,1]$ by the differential expression (0.1) and the Dirichlet boundary conditions (3.1), where*

$$Q(x) = \begin{bmatrix} \cos 2\pi x & 3x^2 \\ 3x^2 & \cos 2\pi x \end{bmatrix}$$

is a symmetric matrix. The matrix C defined by (0.4) has the form,

$$C = \int_0^1 Q(x)\,dx = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues of C are $\mu_1 = -1$ and $\mu_2 = 1$. Besides, let

$$\Lambda_1 < \Lambda_2 < \Lambda_3 < ... < \Lambda_{2n-1} < \Lambda_{2n} < ....,$$

be eigenvalues of the operator $L_2^{(2)}(C)$ numerated in the increasing order, that is

$$\Lambda_{2n-1} = (\pi n)^2 - 1,\ \Lambda_{2n} = (\pi n)^2 + 1,$$

for $n = 1, 2, ...$ Moreover, let

$$\lambda_1(h) < \lambda_2(h) < \lambda_3(h) < ...$$

be eigenvalues of $L_2^{(2)}(Q)$ found by finite difference method.

In Table 4.3, first coloumn shows the eigenvalues of the operator $L_2^{(2)}(C)$, the second,third and fourth coloumns show the eigenvalues of $L_2^{(2)}(Q)$ evaluated via Finite Difference Method written in MATLAB software.

Table 4.4 shows the comparison for eigenvalues of the operator $L_2^{(2)}(C)$ with $L_2^{(2)}(Q)$.

Table 4.3

n	Λ_n	$\lambda_n(\frac{1}{100})$	$\lambda_n(\frac{1}{900})$	$\lambda_n(\frac{1}{2500})$
1	8.869604401	8.509249116	8.510056114	8.510064908
2	10.86960441	10.202562763	10.203371939	10.203380770
3	38.478417604	38.504985195	38.517816377	38.517956000
4	40.478417604	40.426580707	40.439415761	40.439555428
5	87.826439609	87.780420305	87.845344092	87.846050671
6	89.826439609	89.747687394	89.812614081	89.813320692
7	156.913670417	156.717034596	156.922169697	156.924402741
8	158.913670417	158.698835403	158.903973388	158.906206459
9	245.740110027	245.240264191	245.740926585	245.746378269
10	247.740110027	247.228660062	247.72932549	247.734777203
19	985.960440108	977.871442095	985.862000298	985.949223577
20	987.960440108	979.868548709	987.859111508	987.946334814
29	2219.660990245	2178.870301484	2219.154484972	2219.596027309
30	2221.660990245	2180.8690110385	2221.153201835	2221.594744200
39	3946.841760435	3818.660554919	3945.239022007	3946.634401715
40	3948.841760435	3820.659821994	3947.238300376	3948.633680113
49	6167.502750680	5856.864661462	6163.589374837	6166.995711857
50	6169.502750680	5858.864182880	6165.588913025	6168.995250075
59	8881.643960980	8243.295155568	8873.529700239	8880.592189513
60	8883.643960980	8245.294810936	8875.29379540	8882.591868845
69	12089.265391334	10919.190158386	12074.234485666	12087.316668083
70	12091.265391334	10921.189889174	12076.234250048	12089.31643249
79	15790.367041742	13818.660236841	15764.728749361	15787.042535528
80	15792.367041742	13820.660010353	15766.728568961	15789.042355159
89	19984.948912205	16870.310806950	19943.888323338	19979.623725316
90	19986.948912205	16872.310601858	19945.888180792	19981.623582804
99	24673.011002723	19999.000100000	24610.440191059	24664.894717485
100	24675.011002723	20000.999900000	24612.440075589	24666.894602051

The eigenvalues of $L_2^{(2)}(C)$ and $L_2^{(2)}(Q)$

Table 4.4

n	$\Lambda_n - \lambda_n(\frac{1}{100})$	$\Lambda_n - \lambda_n(\frac{1}{900})$	$\Lambda_n - \lambda_n(\frac{1}{2500})$
1	0.360355284	0.359548287	0.359539493
2	0.667041164	0.667041646	0.66622364
3	−0.265675917	−0.039398773	−0.039538396
4	0.051836896	0.039001843	0.038862176
5	0.046019303	−0.018904483	−0, 019611062
6	0.078752214	0.013825528	0.013118917
7	0.196635820	−0.00849928	−0.010732324
8	0.214835013	0.009697029	0.007463958
9	0.499845835	−0.000816558	−0.006268242
10	0.511449964	0.010784534	0.005332824
19	8.088998012	0.09843981	0.011216531
20	8.091891398	0.1013286	0.014105294
29	40.790688760	0.506505273	0.064962936
30	40.791979206	0.50778841	0.066246045
39	128.181205515	1.602738428	0.207358720
40	128.181938441	1.603460059	0.208080322
49	310.638089217	3.913375843	0.507038823
50	310.638567799	3.913837655	0.507500605
59	638.348805411	8.114260741	1.051770585
60	638.349150043	8.35016558	1.052092135
69	1170.075232947	15.030905668	1.948723251
70	1170.075502159	15.031141286	1.948958837
79	1971.706804900	25.638292381	3.324506214
80	1971.707031388	25.638472781	3.324686583
89	3114.638105254	41.060588867	5.325186889
90	3114.638310346	41.060731413	5.325329401
99	4674.010902722	62.570811664	8.116285238
100	4674.011102722	62.570927134	8.116400672

The difference table for the eigenvalues $L_2^{(2)}(C)$ and $L_2^{(2)}(Q)$

Figure 4.1 shows the results given in Table 4.4 graphically:

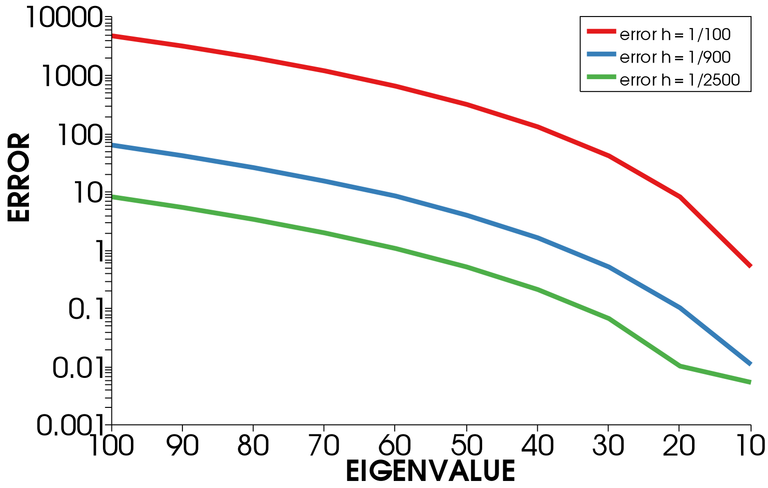

Figure 4.1

Example 4.0.3 *We consider the differential operator $L_3^{(3)}(Q^*)$ generated in the space $L_2^3[0,1]$ by the differential expression (0.1) and the Dirichlet boundary conditions (3.1), where*

$$Q^*(x) = \begin{bmatrix} \cos 2\pi x & \sin 2\pi x & 2x \\ \sin 2\pi x & \cos 2\pi x & \sin 4\pi x \\ 2x & \sin 4\pi x & \cos 4\pi x \end{bmatrix}$$

is a symmetric matrix. The matrix is defined by (0.4) has the form

$$C = \int_0^1 Q^*(x)\, dx = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The eigenvalues are $\mu_1 = -1$, $\mu_2 = 0$ and $\mu_2 = 1$. Let

$$\Lambda_1 < \Lambda_2 < \Lambda_3 < ... < \Lambda_{3n-2} < \Lambda_{3n-1} < \Lambda_{3n} < ....,$$

be eigenvalues of the operator $L_3^{(3)}(C)$ numerated in the increasing order, that is

$$\Lambda_{3n-2} = (\pi n)^2 - 1,\ \Lambda_{3n-1} = (\pi n)^2,\ \Lambda_{3n} = (\pi n)^2 + 1,$$

for $n = 1, 2, ...$ Let

$$\lambda_1(h) < \lambda_2(h) < \lambda_3(h) < ...$$

be eigenvalues of $L_3^{(3)}(Q^*)$ found by finite difference method.

In Table 4.5, first coloumn shows the eigenvalues of the operator $L_3^{(3)}(C)$, the second,third and fourth coloumns show the eigenvalues of $L_3^{(3)}(Q^*)$ evaluated via Finite Difference Method written in MATLAB software. Here, h is taken as $\frac{1}{100}$, $\frac{1}{900}$ and $\frac{1}{2500}$ respectively. These eigenvalues are the small ones.

Table 4.6 shows the comparison of the eigenvalues of the operator $L_3^{(3)}(C)$ with the eigenvalues of the operator $L_3^{(3)}(Q^*)$.

Table 4.5

n	Λ_n	$\lambda_n(\frac{1}{100})$	$\lambda_n(\frac{1}{900})$	$\lambda_n(\frac{1}{2500})$
1	8.869604401	8.564210723	8.565025800	8.565034694
2	9.869604401	9.345652519	9.346468170	9.346477024
3	10.869604401	10.637759053	10.638569295	10.638578137
4	38.478417604	38.194664012	38.207492497	38.207632095
5	39.4784176043	39.469419395	39.482254885	39.482394539
6	40.478417604	40.242353066	40.255191312	40.255331022
7	87.826439609	87.762469547	87.827395647	87.828102267
8	88.826439609	88.766537125	88.831460817	88.832167399
9	89.826439609	89.768518714	89.833440305	89.834146876
10	156.913670417	156.708353950	156.913489843	156.915722909
11	157.913670417	157.710323851	157.915458100	157.917691129
12	158.913670417	158.082863242	158.913422576	158.915655647
13	245.740110027	245.234639585	245.735303147	245.740754849
14	246.740110027	246.348706428	246.735534636	246.740986329
15	247.740110027	247.234602815	247.735266681	247.740718387
16	354.305758843	353.255970373	354.293760870	354.305065391
17	355.305758843	354.256115437	355.293906454	355.305210971
18	356.305758843	355.255941193	356.2937320193	356.305036544

The small eigenvalues of $L_3^{(3)}(C)$ and $L_3^{(3)}(Q^*)$

Table 4.6

n	$\Lambda_n - \lambda_n(\frac{1}{100})$	$\Lambda_n - \lambda_n(\frac{1}{900})$	$\Lambda_n - \lambda_n(\frac{1}{2500})$
1	0.305393677	0.304578601	0.304569707
2	0.523952211	0.523136230	0.523127377
3	0.231845348	0.231035105	0.231026264
4	0.283753592	0.270925107	0.2707855093
5	0.008998209	−0.003837281	−0.003976934
6	0.236064538	0.223226292	0.2230865823
7	0.063970062	−0.000956038	−0.001662657
8	0.059902484	−0.005021207	−0.005727789
9	0.057920895	−0.007000695	−0.007707266
10	0.205316467	0.000180573	−0.002052491
11	0.203346566	−0.001787683	−0.004020711
12	0.830807175	0.000247841	−0.001985229
13	0.505470442	0.004806879	−0.000644821
14	0.391403599	0.004575390	−0.000876301
15	0.505507212	0.004843346	−0.000608359
16	1.049788470	0.011997973	0.000693452
17	1.049643406	0.011852389	0.000547872
18	1.049817650	0.012026824	0.000722299

The difference table for the small eigenvalues of $L_3^{(3)}(C)$ and $L_3^{(3)}(Q^*)$

In Table 4.7, first coloumn shows the eigenvalues of the operator $L_3^{(3)}(C)$, the second,third and fourth coloumns show the eigenvalues of $L_3^{(3)}(Q^*)$ evaluated via Finite Difference Method written in MATLAB software. Here, h is taken as $\frac{1}{100}$, $\frac{1}{900}$ and $\frac{1}{2500}$ respectively. These eigenvalues are the middle ones.

Table 4.8 shows the approximate eigenvalues $L_3^{(3)}(C)$ obtained by the asymptotic method and their comparison with $L_3^{(3)}(Q^*)$.

Table 4.7

n	Λ_n	$\lambda_n(\frac{1}{100})$	$\lambda_n(\frac{1}{900})$	$\lambda_n(\frac{1}{2500})$
28	985.960440108	977.870014024	985.860574494	985.947797790
29	986.960440108	978.870060724	986.860621795	986.947845084
30	987.960440108	979.870003816	987.860564609	987.947787908
43	2219.660990245	2178.86966255	2219.153849728	2219.595392088
44	2220.660990245	2179.86968226	2220.153870083	2220.595412435
45	2221.660990245	2180.86965789	2221.153845397	2221.595387760
58	3946.841760435	3818.6601915	3945.238664375	3946.634044102
59	3947.841760435	3819.6602022	3946.238675691	3947.63405541
60	3948.841760435	3820.6601888	3947.238661950	3948.634041678
73	6167.502750680	5856.8644240	6163.589145858	6166.995482909
74	6168.502750680	5857.8644304	6164.589153059	6167.995490098
75	6169.502750680	5858.8644221	6165.589144308	6168.995481363
88	8881.643960980	8243.2949845	8873.529541184	8880.592030504
89	8882.643960980	8244.2949886	8874.529546170	8881.592035464
90	8883.643960980	8245.2949830	8875.539540108	8882.592029431
103	12089.265391334	10919.190024	12074.234368789	12087.316551242
104	12090.265391334	10920.190027	12075.234372444	12088.316554883
105	12091.265391334	10921.190023	12076.234367997	12089.316550455

The middle and large eigenvalues of $L_3^{(3)}(C)$ *and* $L_3^{(3)}(Q^*)$

Table 4.8

n	$\Lambda_n - \lambda_n(\frac{1}{100})$	$\Lambda_n - \lambda_n(\frac{1}{900})$	$\Lambda_n - \lambda_n(\frac{1}{2500})$
28	8.090426084	0.099865614	0.012642318
29	8.090379384	0.099818313	0.012595024
30	8.090436292	0.099875499	0.0126522
43	40.791327695	0.507140516	0.065598157
44	40.791307985	0.507120161	0.06557781
45	40.791332355	0.507144847	0.065602485
58	128.181568935	1.603096059	0.207716333
59	128.181558235	1.603084743	0.207705023
60	128.181571635	1.603098484	0.207718757
73	310.63832668	3.9136048215	0.507267771
74	310.63832028	3.9135976206	0.507260582
75	310.63832858	3.9136063718	0.507269317
88	638.34897648	8.1144197955	1.051930476
89	638.34897238	8.1144148079	1.051925516
90	638.34897798	8.1044208719	1.051931548
103	1170.075367334	15.031022544	1.948840092
104	1170.075364334	15.031018889	1.948836451
105	1170.075368334	15.031023336	1.948840879

The difference table for middle and large eigenvalues of $L_3^{(3)}(C)$ and $L_3^{(3)}(Q^*)$

Figure 4.2 shows the results given in Table 4.8 graphically.

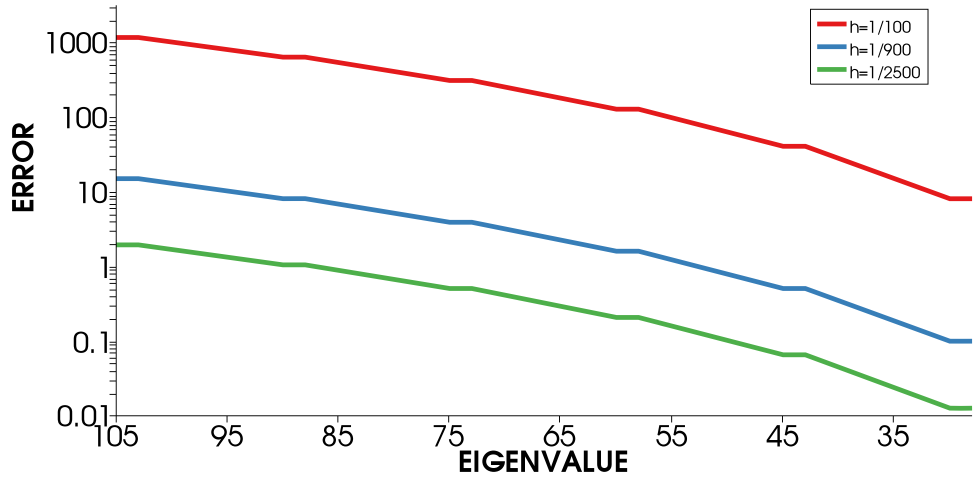

Figure 4.2

These calculations show that the study of finite difference method on approximating the eigenvalues for Sturm-Liouville problems gives sufficient accuracy when h is chosen very small. Moreover, we have presented the effectiveness of two methods: asymptotic formulas and numerical methods. Thus, asymptotic formulas are better to determine large eigenvalues, whereas numerical methods are preferable to calculate small and middle eigenvalues.

The numerical method, in general, gives better results for smaller eigenvalues. Table 4.4, Table 4.6, Figure 4.1 and Figure 4.2 show that the results of the asymptotic formulas are not at all bad for small eigenvalues.

The main result in this work is to determine a condition on the potential for which the root functions of the operator form a Riesz basis with the help of asymptotic formulas for the eigenvalues and eigenfunctions which we have derived in Chapter 2. We have focused on boundary conditions which are strongly regular in scalar case, while we have studied Sturm-Liouville operator in the space of vector functions.

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COMMUNAUTÉ D'UNIVERSITÉS ET ÉTABLISSEMENTS

UNIVERSITÉ DE NANTES

Thèse de Doctorat

Fulya Seref

Sur les propriétés spectrales des opérateurs générés par un système d'équations différentielles

On the Spectral Properties of the Operators Generated by a System of Differential Equations.

Résumé
Nous considérons un opérateur non-autoadjoint $L_m(Q)$, généré dans $L_2^m[0,1]$ par l'équation de Sturm-Liouville, munie d'un potentiel matriciel de taille $m \times m$ et de ses conditions de bord. Le cas scalaire $(m = 1)$ est fortement régulier. Nous obtenons des formulations asymptotiques des valeurs et des vecteurs propres de cet opérateur. Nous trouvions une condition sur le potentiel pour lequel les fonctions propres et les fonctions associées de cet opérateur forment une base de Riesz.
Nous évoquons aussi l'approximation par la méthode des différences finies de valeurs propres du opérateur non-autoadjoint $L_m(Q)$.

Abstract
We consider non-self-adjoint operator $L_m(Q)$ generated in $L_2^m[0,1]$ by the Sturm-Liouville equation with $m \times m$ matrix potential and the boundary conditions, whose scalar case $(m = 1)$ are strongly regular.First we obtain asymptotic formulas for the eigenvalues and eigenfunctions of $L_m(Q)$ and then find a condition on the potential for which the root functions of the operator form a Riesz basis.
We also study the approximation of eigenvalues of $L_m(Q)$ by finite difference method.

Mots clés
opérateurs différentiels, potentiel matriciel, base de Riesz, formules asymptotiques, valeurs propres, méthode des différences finies

Key Words
Differential operators, matrix potential, Riesz basis, asymptotic formulas, eigenvalues, finite difference method