UNIVERSITÉ DE NANTES
FACULTÉ DES SCIENCES ET TECHNIQUES

ÉCOLE DOCTORALE
MATÉRIAUX MATIÈRE MOLÉCULE EN PAYS DE LOIRE

Année : 2012

N° attribué par la bibliothèque

Supersymétrie étendue et ses applications dans des modèles de mécanique quantique associés aux champs de jauge auto-duaux

THÈSE DE DOCTORAT
Discipline : Physique
Spécialité : Physique Subatomique

Présentée et soutenue publiquement par

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Le 29 Mars 2012, devant le jury ci-dessous

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N° ED 500

Acknowledgment

Tout au long des différentes étapes de réalisation de cette thèse et pendant mes études supérieures dans leur ensemble, j'ai grandement bénéficié de discussions avec Andrei Smilga, mon directeur de thèse. Je souhaite vivement le remercier pour le partage de sa perspicacité, offrant des conseils, de maintenir l'intérêt et de prodiguer des encouragements.

L'étude a impliqué dans cette thèse a été réalisée dans une collaboration aimable et agréable avec Evgeny Ivanov. Je suis profondément reconnaissant sa participation à mon éducation, son aide et ses conseils pendant la rédaction du manuscrit.

Je voudrais également exprimer ma profonde gratitude à mon épouse Olga Driga. Je voudrais ne pas avoir terminé ma thèse, sans son appui inconditionnel.

Je suis aussi redevable à L. Alvarez-Gaume, F. Delduc, S. Fedoruk, A. Gorsky, O. Lechtenfeld, M. Shifman et A. Wipf pour leurs discussions éclairantes.

Throughout the various stages of completion of this thesis and during my graduate education as a whole, I have greatly benefited from discussions with Andrei Smilga, my research advisor. I warmly thank him for sharing his insight, offering advice, maintaining interest and providing encouragement.

The study involved in this thesis was done in an amiable and pleasant collaboration with Evgeny Ivanov. I deeply appreciate his participation in my education, his help and his advices during the writing of the manuscript.

Let me also express my deep gratitude to my wife Olga Driga. I would not have completed my thesis without her unconditional support.

I am also indebted to L. Alvarez-Gaume, F. Delduc, S. Fedoruk, A. Gorsky, O. Lechtenfeld, M. Shifman and A. Wipf for their illuminating discussions.

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Chapitre 1 Résumé

1.1 L'introduction

La mécanique quantique supersymétrique (MQS) offre un cadre approprié pour l'exploration et la modélisation des caractéristiques saillantes des théories des champs supersymétriques dans les diverses dimensions [1]. Certains modèles de MQS, sont obtenues comme des réductions unidimensionnelles des théories supersymétriques des dimensions supérieures. En même temps, de nombreux modèles intéressants de ce type peuvent être construits directement dans (0+1) dimensions, sans aucune référence à la procédure de réduction dimensionnelle. Ils présentent des propriétés surprenantes et des particularités liées à la supersymétrie unidimensionnelle. Pour tous les modèles de MQS (comme pour toutes les théories des champs supersymétriques), il est souhaitable, en dehors de la description en composantes hamiltonienne et lagrangienne, d'avoir la formulation appropriée des superchamps. Elle a la supersymétrie manifeste, et cela rend possibles des généralisations du modèle et permet de révéler les liens avec d'autres théories apparentées.

Ce manuscrit est consacré à l'hamiltonien et le lagrangien ainsi que à la formulation du lagrangien de superchamp pour une large classe de $\mathcal{N} = 4$ modèles MQS [1] avec champs de jauge auto-duaux ou anti-auto-duaux abéliennes ou non-abéliennes [2, 3, 4]. Étonnamment, ces systèmes n'ont pas attiré beaucoup d'attention jusqu'à maintenant. Le dispositif naturel et nécessaire pour cette formulation s'avère être l'approche du superespace harmonique (HSS) [5] adaptée au cas unidimensionnel [6].

Les modèles de la mécanique quantique supersymétrique, avec champs de jauge de fond sont d'un intérêt évident pour plusieurs raisons. Une des raisons est la relation étroite de ces systèmes au problème de type de Landau et ses généralisations (voir par exemple [7]). Les modèles de Landau constituent une base de la description théorique de l'effet Hall quantique (EHQ), et il est naturel de s'attendre à ce que leurs extensions supersymétriques, avec des variables supplémentaires fermioniques, peuvent être pertinents pour des versions spinorielles de EHQ. En outre, ces systèmes peuvent fournir une réalisation en mécanique quantique des applications de Hopf diverses étroitement liés à EHQ des dimensions supérieures (voir par exemple [8] et les références associées). Enfin, ils présentent une dimension prototype de couplage des formes d'ordre p dans la théorie superbranes.

Le première type de modèles de MQS considéré dans le manuscrit représente une sous-classe des systèmes bien connus qui décrivent le mouvement d'un fermion sur une variété

[1] En mécanique quantique, $\mathcal{N}$ va toujours compte le nombre des supercharge *réeles*.

avec un champ de jauge de fond arbitraire. Il a été observé il y a plusieurs années que l'on peut traiter ces systèmes comme supersymétriques. Par exemple, l'indice d'Atiyah-Singer d'un opérateur sans masse de Dirac $\not{\mathcal{D}}$ peut être interprété comme l'indice de Witten d'un certain hamiltonien supersymétrique [9]. La supercharge correspondante et l'hamiltonien sont

$$Q = \not{\mathcal{D}}(1+\gamma_5), \qquad \bar{Q} = \not{\mathcal{D}}(1-\gamma_5), \qquad H = \not{\mathcal{D}}^2,$$

où γ_5 correspond à "cinquième matrice gamma" avec $\gamma_5^2 = 1$. Elle anticommute avec l'opérateur de Dirac, $\{\gamma_5, \not{\mathcal{D}}\} = 0$. En effet, pour tout état propre Ψ de l'opérateur de Dirac sans masse $\not{\mathcal{D}}$ sa valeur propre λ est non nulle, et l'état $\gamma^5\Psi$ est aussi un état propre de $\not{\mathcal{D}}$ avec la valeur propre $-\lambda$. Ainsi, tous les états excités de H sont doublement dégénérés.

Il s'avère que pour une variété plat quadridimensionnelle et un champ de jauge auto-dual ou anti-auto-dual, abélien ou non-abélien, le spectre de H est 4 fois dégénèressant ce qu'implique l'étendue de la supersymétrie $\mathcal{N} = 4$. Pour un opérateur plat de Dirac dans le champ d'instanton, cela peut être retracée à la Réf. [10].

Les modèles de MQS $\mathcal{N} = 4$ avec des champs de jauge de fond abéliens ont été traités dans les articles pionniers [11, 12] et, plus récemment, par exemple, dans [13, 6, 14, 2]. En particulier, le formulation d'un cadre de lagrangien off-shell de superchamp des modèles généraux associés aux multiplets $(\mathbf{4},\mathbf{4},\mathbf{0})$ et $(\mathbf{3},\mathbf{4},\mathbf{1})$ dans le cas de superespace harmonique $\mathcal{N} = 4$, $d = 1$ a été présenté dans [6] [2]. On a constaté que la supersymétrie $\mathcal{N} = 4$ nécessite que le champ de jauge soit (anti)auto-dual dans les cas de quatre dimensions $(\mathbf{4},\mathbf{4},\mathbf{0})$, où qu'il satisfait un version "statique" de la condition d'(anti)auto-dualité dans le cas de trois dimensions $(\mathbf{3},\mathbf{4},\mathbf{1})$. Dans les articles [14, 2], en utilisant une approche de hamiltonien on observent que la MQS abélienne $(\mathbf{4},\mathbf{4},\mathbf{0})$ $\mathcal{N} = 4$ admet une simple généralisation à champ de fond arbitraire auto-dual non-abélien. Dans [3] il a été démontré que la formulation off-lagrangienne existe pour une catégorie particulière des modèles de MQS $\mathcal{N} = 4$ non-abéliens avec le groupe de jauge auto-dual SU(2) et ansatz de 't Hooft [16] pour le champ de jauge SU(2) auto-dual (voir aussi [17]). Comme dans le cas abélien, l'utilisation de superespace harmonique $\mathcal{N} = 4$, $d = 1$ nous a permis de construire cette formulation off-shell. Une nouvelle propriété non-triviale de la construction de [3] est l'implication de multiplet "semi-dynamique" auxiliaire $(\mathbf{4},\mathbf{4},\mathbf{0})$ avec l'action de type de Wess-Zumino qui possède une symétrie supplémente de jauge U(1). Après quantification, les champs bosoniques correspondants deviennent des variables spinorielles SU(2) pour lequel le champ de jauge de fond est naturellement couplé [3]. La seconde catégorie de modèles de MQS que nous considérons peuvent être obtenus au niveau des composants par la réduction de l'hamiltonien de systèmes discuté précédemment de quatre à trois dimensions. Leur description de superchamp n'est pas trivial et consiste en couplage de supermultiplet de coordonnes $(\mathbf{3},\mathbf{4},\mathbf{1})$ pour un champ de jauge non abélien externe grâce à l'introduction de superchamp auxiliaire $(\mathbf{4},\mathbf{4},\mathbf{0})$. La supersymétrie off-shell $\mathcal{N} = 4$ restreint le champ de jauge externe à une représentation comme une version "statique"

[2] La première formulation de superchamp des MQS généraux $(\mathbf{3},\mathbf{4},\mathbf{1})$ (Sans les couplage de champ de jauge de fond) a été donné dans [15].

[3] L'utilisation des variables bosoniques auxiliaire pour la mise en place de couplage d'une particule à des champs de Yang-Mills peut être retracée à [18]. Dans le contexte de MQS $\mathcal{N} = 4$, elles étaient utilisées dans[19, 20] et [8, 21].

de l'ansatz 't Hooft pour les champs de jauge SU(2) (anti)auto-duaux en quatre dimensions, c'est à dire à une solution particulière des équations générales de monopôle de Bogomolny[22] [4]. Une nouvelle propriété de cas en trois dimensions est l'apparition du terme de potentiel "induite" dans l'action à la suite de l'élimination du champ auxiliaire du supermultiplet des coordonnées $(\mathbf{3},\mathbf{4},\mathbf{1})$. Ce terme est bilinéaire dans les générateurs du groupe de jauge SU(2). En particulier, dans le cas de la construction de la " symétrie sphérique" (avec la symétrie exacte R SU(2)) on récupére la mécanique $\mathcal{N}=4$ avec le monopôle de Wu-Yang [24] (récemment pris en compte dans [21] avec un traitement essentiellement différent des variables spinorielles).

Dans le chapitre 1.2, nous discutons la supersymétrie en mécanique quantique. Les formalismes ordinaires de superespace et de superespace harmoniques sont présentés. En particulier, la structure des supermultiplets $(\mathbf{4},\mathbf{4},\mathbf{0})$ et $(\mathbf{3},\mathbf{4},\mathbf{1})$ est expliquée. De plus, nous introduisons les notations nécessaires qui seront utilisées dans le chapitre 1.3.

Le chapitre 1.3 présente les résultats originaux de cette étude. Nous donnons la description en composant et la description de superchamp pour les modèles en quatre et trois dimensions discutées précédemment. En particulier, les hamiltonien et les supercharges correspondants sont écrites.

[4] Certains champs de monopôle BPS dans le cadre de MQS $\mathcal{N}=2$ ont été considérés, par exemple, dans [23].

1.2 Supersymétrie et superespace harmonique en mécanique quantique

Ce chapitre est consacré à l'approche de superespace harmonique en mécanique quantique supersymétrique. Les définitions essentielles et les notations sont introduites. Elles seront largement utilisées dans le chapitre suivant. En particulier, les supermultiplets $(\mathbf{4},\mathbf{4},\mathbf{0})$ et $(\mathbf{3},\mathbf{4},\mathbf{1})$ sont décrits. Le premier est pertinent dans le contexte de la mécanique quantique à quatre dimensions, tandis que le second est utilisé dans la construction de systèmes tridimensionnels.

1.2.1 Supersymétrie en mécanique quantique

Un système quantique avec des relations de commutation traditionnelles entre les coordonnées et les impulsions et avec un espace de Hilbert des états traditionnel est décrit par sa fonction hamiltonienne H. Nous introduisons un ensemble des opérateurs complexes Q_i avec leurs opérateurs hermitiens conjugués,

$$\bar{Q}^i = (Q_i)^\dagger \,. \tag{1.2.1}$$

Le système avec l'hamiltonien H et *les supercharges* Q_i, $\bar{Q}^j$ est supersymétrique, par définition, si

$$\left\{Q_i, \bar{Q}^j\right\} = 2\delta_i^j H \tag{1.2.2}$$

$$\left\{Q_i, Q_j\right\} = \left\{\bar{Q}^i, \bar{Q}^j\right\} = 0, \tag{1.2.3}$$

où, comme d'habitude, les crochets désignent l'anticommutateur. En particulier, notons la propriété importante $Q_i^2 = \left(\bar{Q}^i\right)^2 = 0$ pour tout i. Une autre conséquence importante est que la supercharge commute avec le hamiltonien :

$$\left[H,\, Q_i\right] = \left[H,\, \bar{Q}^i\right] = 0, \tag{1.2.4}$$

ce qui peut être prouvé par un calcul direct. De cette façon, les supercharges Q_i et $\bar{Q}^j$ sont considérées comme des opérateurs spinoriels conservé dans le système.

Plusieurs commentaires sont pertinents ici. Les indices latin i, j dénotent les nombres de supercharges et varient dans la région suivante :

$$i,\, j = 1,\, 2,\, \ldots,\, \mathcal{N}/2 \tag{1.2.5}$$

avec $\mathcal{N}$ un entier pair, voir ci-dessous.

On distingue la position des indices pour la supercharge Q_i et son conjugué hermitien $\bar{Q}^i$. Ceci est fait parce que le sous-groupe de $\mathrm{SU}(\frac{\mathcal{N}}{2})$ du groupe de symétrie R (dans le cas $\mathcal{N} \geq 4$) agit différemment sur ces indices. Ce sous-groupe sera d'une importance particulière pour nous plus tard, quand nous nous limiterons à $\mathcal{N} = 4$ et développerons, l'approche de superespace harmonique. Les produits spinoriels et de leurs conjugués complexes sont facilement écrits avec de telles notations. Par exemple, pour deux champs $\bar{\psi}^i$ et ξ_j *anticommutantes*

$$\bar{\psi}\,\xi \equiv \bar{\psi}^i\,\xi_i, \qquad \text{tandis que} \qquad \left(\bar{\psi}\,\xi\right)^* = \bar{\xi}\,\psi \equiv \bar{\xi}^i\,\psi_i, \tag{1.2.6}$$

où $\bar{\psi}^i = (\psi_i)^*$ et $\bar{\xi}^i = (\xi_i)^*$.

Enfin, on peut toujours passer à une base *réele* dans l'espace vectoriel des supercharges, par exemple, en utilisant les définitions

$$S_A = \begin{cases} Q_A + \bar{Q}^A, & \text{pour} \quad A = 1,\, 2,\, \ldots,\, \mathcal{N}/2, \\ i\left(Q_A - \bar{Q}^A\right), & \text{pour} \quad A = \mathcal{N}/2+1,\, \ldots,\, \mathcal{N} \end{cases} \tag{1.2.7}$$

qui donnent les relations de commutation

$$\{S_A,\, S_B\} = 4\,\delta_{AB}\, H. \tag{1.2.8}$$

Les nouveaux indices A, B varier de 1 à $\mathcal{N}$. Ainsi, $\mathcal{N}$ compte le nombre de supercharges linéairement indépendantes dans le système présent. Par exemple, $\mathcal{N} = 2$ correspond à une mécanique quantique supersymétrique ordinaire, tandis qu'un système avec $\mathcal{N} = 4$ est doté d'une supersymétrie étendue.

Superespace et superchamps

La supersymétrie peut être réalisée comme une symétrie géométrique qui agit sur les coordonnées dans un certain espace étendu. En mécanique quantique, cet espace comprend la coordonnée temporelle. Les autres "dimensions" de l'espace sont le nature Grassmannienne, c'est à dire que les coordonnées correspondantes *anticommutent* entre elles.

La technique de superchamp est très générale. Elle permet de construire de véritables systèmes supersymétriques. Elle sert aussi comme un instrument efficace pour la généralisation des systèmes supersymétriques déjà connus.

L'espace étendu en mécanique quantique – *superespace* – est décrit par les coordonnées suivantes :

$$\left\{t,\, \theta_i,\, \bar{\theta}^j\right\}, \qquad i,\, j = 1,\, 2,\, \ldots,\, \mathcal{N}/2 \tag{1.2.9}$$

avec l'identification $\bar{\theta}^i = (\theta_i)^*$: les coordonnées de Grassmann θ_i et $\bar{\theta}^j$ sont complexes. Elles anticommutent les unes avec les autres :

$$\left\{\theta_i,\, \theta_j\right\} = \left\{\theta_i,\, \bar{\theta}^j\right\} = \left\{\bar{\theta}^i,\, \bar{\theta}^j\right\} = 0. \tag{1.2.10}$$

Rappelons que l'indice supérieur dans la variable de Grassmann conjuguée reflète la façon dont elle se transforme sous l'action du groupe de symétrie R.

Dans la définition du superespace, on introduit des *superchamps* sur le superespace. La propriété $\theta^2 = 0$ pour toute variable de Grassmann θ limite le nombre de termes dans le développement d'un superchamp général $\Phi(t,\, \theta_i,\, \bar{\theta}^j)$ sous forme de séries de variables de Grassmann. Prenons, par exemple, le cas $\mathcal{N} = 2$. Oubliant les indices sur θ_1 et $\bar{\theta}^1$, on obtient

$$\Phi(t,\, \theta,\, \bar{\theta}) = \phi(t) + \bar{\psi}(t)\theta + \xi(t)\bar{\theta} + D(t)\theta\bar{\theta}. \tag{1.2.11}$$

Deux champs complexes $\phi(t)$, $D(t)$ et deux champs complexes de Grassmann $\bar{\psi}(t)$, $\xi(t)$ ne dépendent que de la variable temps et peuvent représenter les degrés physiques de liberté en mécanique quantique supersymétrique. Cependant, dans beaucoup de cas, le nombre de composantes dans le développement d'un superchamp est excessif, c'est à dire qu'il est possible de réduire le nombre de champs par des contraintes additionnelles sur le superchamp Φ. Ces contraintes doivent être covariantes par rapport aux transformations de supersymétrie. Par exemple, une telle contrainte peut être la condition de réalité pour le superchamp Φ.

Transformations de supersymétrie et opérateurs différentiels sur superespace

Les transformations infinitésimales de supersymétrie en mécanique quantique sont réalisées comme des déplacements dans le superespace (1.2.9),

$$\begin{aligned} t &\to t + i\left(\epsilon_i\bar{\theta}^i + \bar{\epsilon}^i\theta_i\right), \\ \theta_i &\to \theta_i + \epsilon_i, \\ \bar{\theta}^i &\to \bar{\theta}^i + \bar{\epsilon}^i \end{aligned} \tag{1.2.12}$$

avec le paramètre de Grassmann ϵ_i, $\bar{\epsilon}^i = (\epsilon_i)^*$. Ces transformations sont générées par l'opérateur $i\left(\bar{\epsilon}^i Q_i - \epsilon_i\bar{Q}^i\right)$, où les supercharges

$$Q_i = -i\frac{\partial}{\partial\bar{\theta}^i} + \theta_i\frac{\partial}{\partial t}, \qquad \bar{Q}^i = i\frac{\partial}{\partial\theta_i} - \bar{\theta}^i\frac{\partial}{\partial t} \tag{1.2.13}$$

satisfont les relations

$$\left\{Q_i,\, Q_j\right\} \;=\; \left\{\bar{Q}^i,\, \bar{Q}^j\right\} \;=\; 0, \tag{1.2.14}$$

$$\left\{Q_i,\, \bar{Q}^j\right\} \;=\; 2\delta_i^j\, i\partial_t \tag{1.2.15}$$

avec

$$\partial_t = \frac{\partial}{\partial t}. \tag{1.2.16}$$

La supercharge (1.2.13) avec l'opérateur $H = i\partial_t$ réalise une représentation particulière de l'algèbre de supersymétrie (1.2.2), (1.2.3) sur le superespace (1.2.9).

Introduirons également les *superdérivées covariantes* D^i et $\bar{D}_i$ définies par

$$D^i = \frac{\partial}{\partial\theta_i} - i\bar{\theta}^i\frac{\partial}{\partial t}, \qquad \bar{D}_i = \frac{\partial}{\partial\bar{\theta}^i} - i\theta_i\frac{\partial}{\partial t}. \tag{1.2.17}$$

Ces opérateurs sont d'un intérêt particulier, parce qu'ils anticommutent avec les supercharges définies précédemment,

$$\left\{D^i,\, Q_j\right\} = \left\{D^i,\, \bar{Q}^j\right\} = \left\{\bar{D}_i,\, Q_j\right\} = \left\{\bar{D}_i,\, \bar{Q}^j\right\} = 0, \tag{1.2.18}$$

ce qui signifie qu'ils sont covariants par rapport aux supertransformations (1.2.12). Ainsi, ils peuvent être utilisés dans les contraintes covariantes sur les superchamps pour réduire le nombre de leur composantes indépendantes. Les relations d'anticommutation pour les superdérivées sont

$$\left\{D^i,\, D^j\right\} = \left\{\bar{D}_i,\, \bar{D}_j\right\} = 0, \qquad \left\{D^i,\, \bar{D}_j\right\} = -2\delta_j^i\, i\partial_t. \tag{1.2.19}$$

Remarquons que, par commodité, nous utilisons une convention différente des multiplicateurs pour les superdérivées (1.2.17) par rapport aux supercharges (1.2.13).

1.2.2 Approche de superespace harmonique

Soulignons que l'existence du sous-espace (anti)chiral qui conduit à la réduction du développement en composantes de tout superchamp (anti)chiral est spécifique au cas $\mathcal{N} = 2$. Le superespace (1.2.9) en mécanique quantique $\mathcal{N} \geq 4$ n'a pas cette propriété. On peut, cependant, encore utiliser des contraintes covariantes (par exemple des conditions de réalité, ou des équations impliquant des superdérivées) pour réduire le nombre de composantes dans les superchamps. Pourtant, ce n'est parfois pas suffisant de réduire le nombre au minimum prévu. Par ailleurs, les contraintes peuvent être assez compliquées. Des exemples de manipulation des superchamps et de leurs contraintes covariantes sont décrits dans la Réf. [42].

Il s'avère que le cas $\mathcal{N} = 4$ est spécial : il admet aussi un superespace qui a deux sous-espaces invariants et permet de réduire le nombre de variables de Grassmann des superchamps par un facteur 2. C'est l'approche de superespace harmonique (HSS) [5] inventé par Galperin, Ivanov, Ogievetsky et Sokatchev. L'idée clé est que le superespace standard (1.2.9) doit être complété par des coordonnées supplémentaires de nature bosonique.

Limitons-nous au cas $\mathcal{N} = 4$ de la mécanique quantique supersymétrique.

Le superespace harmonique peut être considéré comme l'un des superespaces sur lequel le groupe de supersymétrie agit. Le superespace (1.2.9) est l'un des choix possibles. Tous les espaces imaginables sur lesquels agit la supersymétrie peuvent être décrits comme un quotient

$$\text{superespace} = \frac{\text{groupe de supersymétrie}}{\text{l'un de ses certains sous-groupes}}. \tag{1.2.20}$$

Nous rappelons que l'algèbre de supersymétrie $\mathcal{N} = 4$ en mécanique quantique est invariante par rapport au groupe de symétrie R de SU(2). Jusqu'à maintenant, ce dernier n'a pas été pris en considération. Cependant, tout en recherchant des superespaces possibles, l'espace, du groupe de symétrie R, peut être ajouté au superespace (1.2.9). Il appairait que les superchamps sur le superespace étendu qui offrent un contenu en champ avec un minimum de composantes seront des fonctions sur une 2-sphère $\mathrm{S}^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ – un quotient du groupe de symétrie R par rapport à l'un de ses sous-groupes U(1).

Notations

L'approche de superespace harmonique (HSS) pour la mécanique quantique a été développée en Réf. [6]. Les conventions dans ce manuscrit suivant les conventions de la Réf. [2] et diffèrent des conventions de la Réf. [6] par le changement de direction du temps $t \to -t$. Avec cela, on reproduit le signe correct dans le terme cinétique pour le champ de spineurs dans Éq. (1.3.22).

A partir de maintenant, en supersymétrie $\mathcal{N} = 4$ nous utiliserons une notation différente pour les indices spinoriels : les indices de l'alphabet grec

$$\alpha,\, \beta = 1, 2 \tag{1.2.21}$$

sont utilisés au lieu des indices i, j. Par exemple, superespace $\mathcal{N} = 4$ le ordinaire est

$$\left\{t,\, \theta_\alpha,\, \bar{\theta}^\beta\right\}, \qquad \bar{\theta}^\beta = (\theta_\beta)^*. \tag{1.2.22}$$

Pour une utilisation ultérieure, rappelons ici l'expression des supercharges de l'équation (1.2.13),

$$Q_\alpha = -i\frac{\partial}{\partial\bar{\theta}^\alpha} + \theta_\alpha\frac{\partial}{\partial t}, \qquad \bar{Q}^\alpha = i\frac{\partial}{\partial\theta_\alpha} - \bar{\theta}^\alpha\frac{\partial}{\partial t}, \tag{1.2.23}$$

et des superdérivées de l'équation (1.2.17),

$$D^\alpha = \frac{\partial}{\partial\theta_\alpha} - i\bar{\theta}^\alpha\frac{\partial}{\partial t}, \qquad \bar{D}_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} - i\theta_\alpha\frac{\partial}{\partial t}. \tag{1.2.24}$$

L'algèbre supersymétrique correspondant aux équations (1.2.2) et (1.2.3) est donnée par

$$\begin{aligned} \left\{Q_\alpha, \bar{Q}^\beta\right\} &= 2\delta_\alpha^\beta H \\ \left\{Q_\alpha, Q_\beta\right\} &= \left\{\bar{Q}^\alpha, \bar{Q}^\beta\right\} = 0, \end{aligned} \tag{1.2.25}$$

Montée et descente des indices spinoriels

Le groupe de symétrie R SU(2) admet la possibilité de monter et descendre les indices spinoriels avec les tenseurs antisymétriques invariants de Levi-Civita $\varepsilon_{\alpha\beta}$ et $\varepsilon^{\alpha\beta}$. Par définition,

$$\begin{aligned} \varepsilon_{\alpha\beta} &= -\varepsilon_{\beta\alpha}, \qquad \varepsilon_{12} = 1, \\ \varepsilon^{\alpha\beta} &= -\varepsilon^{\beta\alpha}, \qquad \varepsilon^{12} = -1, \end{aligned} \tag{1.2.26}$$

de telle sorte que, par exemple, on a pour un spineur v_α :

$$v^\alpha = \varepsilon^{\alpha\beta}v_\beta, \qquad v_\alpha = \varepsilon_{\alpha\beta}v^\beta. \tag{1.2.27}$$

En raison de l'invariance des tenseurs de Levi-Civita par rapport à l'action du groupe de symétrie R SU(2), les équations les contenant sont également invariantes.

Nous allons introduire par la suite le groupe SU(2) de Pauli-Gürsey [5] et les indices "pairtés". De façon analogue, on peut introduire les tenseurs de Levi-Civita avec des indices pairtés comme suit :

$$\begin{aligned} \varepsilon_{\dot\alpha\dot\beta} &= -\varepsilon_{\dot\beta\dot\alpha}, \qquad \varepsilon_{\dot1\dot2} = 1, \\ \varepsilon^{\dot\alpha\dot\beta} &= -\varepsilon^{\dot\beta\dot\alpha}, \qquad \varepsilon^{\dot1\dot2} = -1. \end{aligned} \tag{1.2.28}$$

Travailler avec la sphère S^2

Avant de discuter du superespace harmonique dans sa globalité, il est instructif d'étudier les coordonnées dans l'espace SU(2)/U(1) qui est une deux-sphère. On pourrait choisir, par exemple, les coordonnées polaires ou stéréographique sur S^2. Cependant, il s'avère plus pratique de travailler avec les coordonnées homogènes dans le groupe l'espace du groupe SU(2) et contraindre les fonctions dans cet espace à vivre dans l'espace S^2.

Pour plus de détails sur ce point, on introduit les coordonnées complexes homogènes $u^\pm_\alpha$, $\alpha = 1, 2$ qui portent le nom d'*harmoniques* dans la trois-sphère SU(2). Ils satisfont par définition les relations suivantes :

$$u^{+\alpha}u^-_\alpha = 1, \qquad u^-_\alpha = (u^{+\alpha})^*. \tag{1.2.29}$$

(Nous avons utilisé les indices élevés dans ce qui précède (cf. le paragraphe précédent). Par exemple, $u^{+\alpha} = \varepsilon^{\alpha\beta}u^{+}_{\beta}$.) A noter également l'identité importante :

$$u^+_\alpha u^-_\beta - u^-_\alpha u^+_\beta = \varepsilon_{\alpha\beta}. \tag{1.2.30}$$

Nous sommes intéressés par des fonctions considérées dans l'espace SU(2)/U(1). Considérons des fonctions dans la trois-sphère qui a une charge déterminée U(1). À titre illustratif, on prend une fonction $f^+(u)$ de charge 1. Elle peut être écrite comme un développement en série d'harmoniques $u^\pm_\alpha$:

$$f^+(u) = f^\alpha u^+_\alpha + f^{\alpha\beta\gamma}u^+_\alpha u^+_\beta u^-_\gamma + \ldots, \tag{1.2.31}$$

où les constantes f^α, $f^{\alpha\beta\gamma}$, ... peuvent toujours être prises symétriques dans leurs indices. En effet, en utilisant la relation (1.2.30), on peut transformer n'importe quel produit d'harmoniques en combinaisons symétriques plus des produits d'harmoniques d'ordre plus petit. Par exemple,

$$\begin{aligned} u^+_\alpha u^+_\beta u^-_\gamma &= \frac{1}{3}\left(u^+_\alpha u^+_\beta u^-_\gamma + u^-_\alpha u^+_\beta u^+_\gamma + u^+_\alpha u^-_\beta u^+_\gamma\right) + \\ &\quad + \frac{1}{3}\left(u^+_\alpha u^+_\beta u^-_\gamma - u^-_\alpha u^+_\beta u^+_\gamma\right) + \\ &\quad + \frac{1}{3}\left(u^+_\alpha u^+_\beta u^-_\gamma - u^+_\alpha u^-_\beta u^+_\gamma\right) = \\ &= \frac{1}{3}\left(u^+_\alpha u^+_\beta u^-_\gamma + u^-_\alpha u^+_\beta u^+_\gamma + u^+_\alpha u^-_\beta u^+_\gamma\right) + \frac{1}{3}\varepsilon_{\alpha\gamma}u^+_\beta + \frac{1}{3}\varepsilon_{\beta\gamma}u^+_\alpha. \end{aligned} \tag{1.2.32}$$

Avec l'action du groupe de symétrie R, la fonction $f^+(u)$ subit des transformations de phase U(1) homogènes (selon la charge globale) et donc est bien définie sur une deux-sphère SU(2)/U(1). La même chose est également vraie pour toute fonction d'harmoniques avec une charge U(1) fixée.

En fait, les harmoniques $u^\pm_\alpha$ sont les harmoniques sphériques fondamentales de spin 1/2 de la mécanique quantique. C'est pourquoi elles sont appelées variables harmoniques.

Les harmoniques peuvent être utilisées pour la projection d'indices de spineurs sur l'espace des harmoniques. Par exemple, $f^+ = u^+_\alpha f^\alpha$ et $f^- = u^-_\alpha f^\alpha$. Le spineur original peut être restauré en utilisant l'équation (1.2.30) :

$$f^\alpha = u^{+\alpha}f^- - u^{-\alpha}f^+. \tag{1.2.33}$$

Opérateurs différentiels sur le groupe $\mathrm{SU}(2)_\mathrm{R}$

$$D^{++} = u^+_\alpha\frac{\partial}{\partial u^-_\alpha}, \qquad D^{--} = u^-_\alpha\frac{\partial}{\partial u^+_\alpha}\ , \qquad D^0 = u^+_\alpha\frac{\partial}{\partial u^+_\alpha} - u^-_\alpha\frac{\partial}{\partial u^-_\alpha} \tag{1.2.34}$$

sont appelés *dérivés harmoniques*. L'opérateur D^0 joue un rôle d'opérateur de charge dans U(1). On a ainsi pour une fonction $f^{+q}(u)$ avec une charge U(1) de valeur $+q$:

$$D^0 f^{+q}(u) = q\, f^{+q}(u). \tag{1.2.35}$$

Les coordonnées u^+_α ont une charge 1, tandis que les coordonnées u^-_α ont une charge -1.

Le superespace harmonique $\mathcal{N} = 4$

Le formalisme du superespace harmonique $\mathcal{N} = 4$ en mécanique quantique a été développé dans la Réf. [6]. Dans ce formalisme, les superchamps dépendent du temps t et des harmoniques $u^{\pm\alpha}$ qui paramétrisent le groupe de symétrie R SU(2) de la superalgèbre $\mathcal{N} = 4$. es superchamps dépendent également des variables de Grassmann θ_α, $\bar{\theta}^\beta$. Le superespace est donné par

$$\left\{t,\, \theta_\alpha,\, \bar{\theta}^\beta,\, u^\pm_\gamma\right\}, \qquad \bar{\theta}^\beta = (\theta_\beta)^*. \tag{1.2.36}$$

C'est ce que l'on appelle *la base standard* dans le superespace harmonique.

Habituellement, au lieu des spineurs θ_α et $\bar{\theta}^\beta$, il est préférable d'utiliser les projections harmoniques suivantes :

$$\theta^\pm = u^\pm_\alpha \theta^\alpha, \qquad \bar{\theta}^\pm = u^\pm_\alpha \bar{\theta}^\alpha. \tag{1.2.37}$$

On peut également définir des projections harmoniques des superdérivées, $D^\pm = u^\pm_\alpha D^\alpha$, $\bar{D}^\pm = u^\pm_\alpha \bar{D}^\alpha$. On peut vérifier dans la base standard (1.2.36) que

$$D^+ = \frac{\partial}{\partial\theta^-} - i\bar{\theta}^+\frac{\partial}{\partial t}, \qquad D^- = -\frac{\partial}{\partial\theta^+} - i\bar{\theta}^-\frac{\partial}{\partial t}, \tag{1.2.38}$$

$$\bar{D}^+ = -\frac{\partial}{\partial\bar{\theta}^-} - i\theta^+\frac{\partial}{\partial t}, \qquad \bar{D}^- = \frac{\partial}{\partial\bar{\theta}^+} - i\theta^-\frac{\partial}{\partial t}, \tag{1.2.39}$$

Base analytique dans le superespace harmonique

La caractéristique la plus frappante du superespace harmonique est la présence en son sein d'un *sous-espace analytique*

$$\left\{t_{\mathrm{A}},\, \theta^+,\, \bar{\theta}^+,\, u^{\pm\alpha}\right\} \tag{1.2.40}$$

(un analogue du superespace chiral $\mathcal{N} = 2$) impliquant le temps “analytique”

$$t_A = t + i(\theta^+\bar{\theta}^- + \theta^-\bar{\theta}^+) \tag{1.2.41}$$

et contenant deux fois moins de coordonnées fermioniques.

On va maintenant donner plus de détails sur ce point. Il est commode de passer à *la base analytique* dans le superespace harmonique,

$$\left\{t_{\mathrm{A}},\, \theta^\pm,\, \bar{\theta}^\pm,\, u^\pm_\alpha\right\}. \tag{1.2.42}$$

Dans cette base, les dérivées spinorielles covariantes D^+, $\bar{D}^+$ sont simplement données par

$$D^+ = \frac{\partial}{\partial\theta^-}, \qquad \bar{D}^+ = -\frac{\partial}{\partial\bar{\theta}^-}. \tag{1.2.43}$$

C'est ceci qui nous permet de transformer les contraintes de superchamp $D^+f = \bar{D}^+f$ pour un superchamp f donné afin d'offert une indépendance en θ^- et $\bar{\theta}^-$: $f = f(t_A,\, \theta^+,\, \bar{\theta}^+,\, u^\pm_\alpha)$.

On peut vérifier directement que le sous-espace (1.2.40) est invariant par rapport aux transformations supersymétriques $\mathcal{N} = 4$. En effet, en utilisant l'équation (1.2.12), on obtient

$$\begin{aligned} &t_A \to t_A + 2i\left(\epsilon^- \bar{\theta}^+ - \bar{\epsilon}^- \theta^+\right), \\ &\theta^\pm \to \theta^\pm + \epsilon^\pm, \\ &\bar{\theta}^\pm \to \bar{\theta}^\pm + \bar{\epsilon}^\pm, \\ &u^\pm_\alpha \to u^\pm_\alpha, \end{aligned} \tag{1.2.44}$$

où

$$\bar{\epsilon}^\alpha = (\epsilon_\alpha)^*, \qquad \epsilon^\pm = u^\pm_\alpha \epsilon^\alpha, \qquad \bar{\epsilon}^\pm = u^\pm_\alpha \bar{\epsilon}^\alpha. \tag{1.2.45}$$

Enfin, on peut écrire la forme des dérivées harmoniques D^{++} et D^{--} de (1.2.34) dans la base analytique par :

$$D^{++} = u^+_\alpha \frac{\partial}{\partial u^-_\alpha} + \theta^+ \frac{\partial}{\partial \theta^-} + \bar{\theta}^+ \frac{\partial}{\partial \bar{\theta}^-} + 2i\theta^+ \bar{\theta}^+ \frac{\partial}{\partial t_{\mathrm{A}}}, \tag{1.2.46}$$

$$D^{--} = u^-_\alpha \frac{\partial}{\partial u^+_\alpha} + \theta^- \frac{\partial}{\partial \theta^+} + \bar{\theta}^- \frac{\partial}{\partial \bar{\theta}^+} + 2i\theta^- \bar{\theta}^- \frac{\partial}{\partial t_{\mathrm{A}}}. \tag{1.2.47}$$

On note que dans le sous-espace (1.2.40), le deuxième et le troisième termes disparaissent.

La symétrie d'involution

Le superespace (1.2.36) admet *la symétrie d'involution* qui commute avec les transformations supersymétriques [6, 5]. On dénote l'involution avec le signe "$\sim$", par exemple son action est $f \to \widetilde{f}$ pour un superchamp donné $f(t_A, \theta^\pm, \bar{\theta}^\pm, u^\pm_\alpha)$.

Par définition, la transformation d'involution agit exactement comme la conjugaison complexe ordinaire *sauf* lors de son action sur les harmoniques $u^\pm_\alpha$ pour laquelle on a

$$\widetilde{u^\pm_\alpha} = u^{\pm\alpha}, \qquad \widetilde{u^{\pm\alpha}} = -u^\pm_\alpha. \tag{1.2.48}$$

Cela donne

$$\widetilde{t_{\mathrm{A}}} = t_{\mathrm{A}}, \qquad \widetilde{\theta^\pm} = \bar{\theta}^\pm, \qquad \widetilde{\bar{\theta}^\pm} = -\theta^\pm. \tag{1.2.49}$$

L'action de la transformation d'involution sur les harmoniques peut être vue comme une composition de la conjugaison complexe et de l'inversion par rapport à un point sur la sphère S^2. En général, la symétrie d'involution est très similaire à l'opération de conjugaison complexe, mais elle ne change pas les charges U(1) des superchamps. Elle permet de mettre des contraintes supplémentaires sur les superchamps (par exemple condition de réalité) et elle est utilisée dans la construction de lagrangiens supersymétriques.

Supermultiplets de différentes dimensions

Nous allons maintenant discuter des supermultiplets minimaux possibles qui sont souvent impliqués dans la description des superchamps. Tous ces supermultiplets $\mathcal{N} = 4$ sont généralement dénotés par trois nombres, $(\mathbf{b}, \mathbf{f}, \mathbf{a})$, où

- $\mathbf{b}$ est le nombre de degrés de liberté bosoniques physiques ;

- **f** est le nombre de degrés de liberté fermioniques physiques ;
- **a** est le nombre de degrés de liberté bosoniques non dynamique auxiliaires, qui sont intégrés à des Lagrangiens finaux.

Les supermultiplets très largement utilisés sont $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ et $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ [43, 6, 42, 15], qui décrivent habituellement les dynamiques à quatre et trois dimensions respectivement. Nous discuterons de ces supermultiplets en détail ci-dessous. En outre, la manière commune d'obtenir la supersymétrie $\mathcal{N} = 8$ ou des théories à dimensions supérieures avec la supersymétrie $\mathcal{N} = 4$ est de prendre plusieurs superchamps de ces types.

Parmi les autres supermultiplets, on trouve $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ et $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ [44, 45] qui sont habituellement utilisés pour construire des systèmes à nombreuses particules à une ou deux dimensions. Ces multiplets ne sont pas abordés dans ce manuscrit.

En général, le nombre de fermions physiques dans tous ces multiplets est quatre, alors que la somme des degrés de liberté bosonique physiques et auxiliaires est aussi quatre. Il est même possible d'introduire le supermultiplet $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ [6].

Le supermultiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$

Les opérateurs différentiels D^+, $\bar{D}^+$, D^{++} (anti)commutent entre eux et avec les supercharges. Pour cette raison, il est possible d'envisager un superchamp q^+ avec une charge U(1) du valeur $+1$ satisfaisant

$$D^+ q^+ = 0, \qquad \bar{D}^+ q^+ = 0, \qquad D^{++} q^+ = 0. \tag{1.2.50}$$

Dans les coordonnées de superespace analytique, la première et la seconde équation signifient que q^+ ne dépend que de θ^+ et $\bar{\theta}^+$, mais pas de θ^- et $\bar{\theta}^-$, voir Éq. (1.2.43). De cette façon, la première et la seconde équation forment ce qu'on appelle *les conditions d'analyticité des superchamps*.

Lors du développement du superchamp $q^+(t_A, \theta^+, \bar{\theta}^+, u^\pm_\alpha)$ dans les coordonnées spinorielles et harmoniques, on obtient un ensemble infini de champs physiques. Cependant, en imposant également la condition $D^{++} q^+ = 0$, cela réduit drastiquement le nombre de ces champs, en rendant leur nombre fini. Dans la base analytique (1.2.42), la solution des contraintes (1.2.50) se traduit par

$$q^+ = x^\alpha(t_\mathrm{A}) u^+_\alpha - 2\theta^+ \chi(t_\mathrm{A}) - 2\bar{\theta}^+ \bar{\chi}'(t_\mathrm{A}) - 2i\theta^+ \bar{\theta}^+ \partial_\mathrm{A} x^\alpha(t_\mathrm{A}) u^-_\alpha \tag{1.2.51}$$

avec

$$\partial_\mathrm{A} \equiv \frac{\partial}{\partial t_A} \tag{1.2.52}$$

et les facteurs -2 sont introduit par commodité. Ainsi, le superchamp $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ q^+ comprend deux coordonnées bosoniques complexes x^α et deux fermions complexes – χ, $\bar{\chi}'$.

Les contraintes $D^+ q^+ = \bar{D}^+ q^+ = 0$ sont semblables aux contraintes de chiralité dans les théories des champs supersymétriques $\mathcal{N} = 1$ à quatre dimensions. Ces contraintes apparaissent naturellement dans le formalisme HSS et sont communes aussi dans les théories des champs à quatre dimensions. Une possibilité d'imposer la contrainte supplémentaire $D^{++} q^+ = 0$ est spécifique à la mécanique quantique uniquement, où elle a une nature cinématique pure. Dans les théories des champs supersymétriques $\mathcal{N} = 2$, la relation

$D^{++}q^{+} = 0$ n'est pas une contrainte cinématique, c'est l'équation du mouvement pour l'*hypermultiplet libre* dérivé de l'action $S = \int d^4x\, du\, d^4\theta^+\, \widetilde{q^+} D^{++} q^+$ [5].

Les contraintes (1.2.50) admettent une symétrie d'involution $q^+ \to \widetilde{q^+}$ qui commute avec les transformations supersymétriques [6, 5] :

$$\widetilde{q^+} = [x_\alpha(t_{\mathrm{A}})]^* u^+_\alpha - 2\theta^+ \bar{\chi}'^*(t_{\mathrm{A}}) + 2\bar{\theta}^+ \chi^*(t_{\mathrm{A}}) - 2i\theta^+\bar{\theta}^+ \partial_{\mathrm{A}} [x_\alpha(t_{\mathrm{A}})]^* u^-_\alpha. \tag{1.2.53}$$

Il est facile de voir que le champ $\widetilde{q^+}$ satisfait les mêmes contraintes (1.2.50) que le champ q^+.

Comme nous allons utiliser le supermultiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ pour construire un système supersymétrique en mécanique quantique à quatre dimensions d'espace, il sera plus commode pour nous d'utiliser le supermultiplet q^+ sous une forme différente. Plus précisément, nous introduisons le supermultiplet

$$q^{+\dot{\alpha}} = \left\{ q^+, \widetilde{q^+} \right\}, \qquad \dot{\alpha} = 1, 2. \tag{1.2.54}$$

La symétrie d'involution peut être utilisée pour imposer la condition de pseudoréalité sur le champ $q^{+\dot{\alpha}}$,

$$q^{+\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}} \widetilde{\left(q^{+\dot{\beta}}\right)}, \tag{1.2.55}$$

qui est en fait équivalente à l'équation (1.2.54). En termes de composantes,

$$q^{+\dot{\alpha}} = x^{\alpha\dot{\alpha}}(t_{\mathrm{A}}) u^+_\alpha - 2\theta^+ \chi^{\dot{\alpha}}(t_{\mathrm{A}}) - 2\bar{\theta}^+ \bar{\chi}^{\dot{\alpha}}(t_{\mathrm{A}}) - 2i\theta^+\bar{\theta}^+ \partial_{\mathrm{A}} x^{\alpha\dot{\alpha}}(t_{\mathrm{A}}) u^-_\alpha. \tag{1.2.56}$$

La contrainte (1.2.55) implique

$$x^{\alpha\dot{\alpha}} = -\left(x_{\alpha\dot{\alpha}}\right)^*, \qquad \bar{\chi}^{\dot{\alpha}} = \left(\chi_{\dot{\alpha}}\right)^*. \tag{1.2.57}$$

La forme du supermultiplet $q^{+\dot{\alpha}}$ suggère que l'on peut associer le groupe SU(2) relié à l'indice pointé $\dot{\alpha}$. Ce groupe de Pauli-Gürsey [5] est également réalisé avec le supermultiplet q^+, mais pas manifestement.

Par conséquent, le système de mécanique quantique qui sera discuté dans le chapitre suivant, hérite du groupe $\mathrm{SO}(4) = \mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{SU}(2)_{\mathrm{PG}}$ composé du groupe de symétrie R et du groupe de Pauli-Gürsey. En général, ce groupe de rotation SO(4) est complètement brisé par la présence d'un champ de jauge à quatre dimensions (cf. prochain chapitre).

Le supermultiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$

Au lieu de la coordonnée du superchamp $q^{+\dot{\alpha}}$, on peut traiter le superchamp analytique L^{++} de charge +2 qui englobe le supermultiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ et est soumis aux contraintes

$$\begin{gathered} D^+ L^{++} = \bar{D}^+ L^{++} = 0, \\ D^{++} L^{++} = 0, \qquad \widetilde{(L^{++})} = -L^{++}, \end{gathered} \tag{1.2.58}$$

Elles restreignent le superchamp analytique L^{++} d'avoir la composante off-shell du champ approprié, à savoir $(\mathbf{3}, \mathbf{4}, \mathbf{1})$:

$$L^{++} = \ell^{\alpha\beta} u^+_\alpha u^+_\beta + 2i\theta^+ \chi^\alpha u^+_\alpha + 2i\bar{\theta}^+ \bar{\chi}^\alpha u^+_\alpha + \theta^+\bar{\theta}^+ [F - 2i\dot{\ell}^{\alpha\beta} u^+_\alpha u^-_\beta] \tag{1.2.59}$$

avec

$$(\ell_{\alpha\beta})^* = -\ell^{\alpha\beta}, \qquad (\chi^\alpha)^* = \bar{\chi}_\alpha. \tag{1.2.60}$$

Le multiplet L^{++} implique l'espace cible des coordonnées à 3 dimensions $\ell^{\alpha\beta} = \ell^{\beta\alpha}$, leur partenaires fermioniques χ^α, $\bar{\chi}^\alpha$ et un champ auxiliaire réel F.

Intégrales harmoniques

Les actions invariantes impliquent l'intégrale harmonique $\int du$. Pour trouver une telle intégral d'une fonction $f(u^\pm_\alpha)$, on doit écrire le développement de f en série harmonique de Taylor et, pour chaque terme, on effectue les intégrales en utilisant les règles

$$\int du\, 1 = 1, \qquad \int du\, u^+_{\{\alpha_1} \ldots u^+_{\alpha_k} u^-_{\alpha_{k+1}} \ldots u^-_{\alpha_{k+\ell}\}} = 0\ , \tag{1.2.61}$$

où l'intégrant dans l'équation de droite est symétrisé sur tous les indices. Les valeurs des intégrales de tous les autres monômes harmoniques (par exemple, $\int du\, u^+_\alpha u^-_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}$) découlent de (1.2.61) et des définitions (1.2.29), (1.2.30).

Notations en mécanique à quatre dimensions

Des indices vectoriels de l'espace Euclidien à quatre dimensions sont

$$\mu, \nu = 0, 1, 2, 3. \tag{1.2.62}$$

Nous utilisons les notations SO(4) suivantes pour les matrices sigma Euclidiennes à quatre dimensions :

$$(\sigma_\mu)_{\alpha\dot{\alpha}} = \{i, \vec{\sigma}\}_{\alpha\dot{\alpha}}\,, \qquad \left(\sigma^\dagger_\mu\right)^{\dot{\alpha}\alpha} = \{-i, \vec{\sigma}\}^{\dot{\alpha}\alpha}\,, \tag{1.2.63}$$

où $\vec{\sigma}$ sont les matrices de Pauli ordinaires. (Ce sont plus au moins les conventions de [33] tourné à l'espace euclidien.) La matrice $\sigma^\dagger_\mu$ est obtenue à partir de la matrice σ_μ par l'opération de levée des indices :

$$\left(\sigma^\dagger_\mu\right)^{\dot{\alpha}\alpha} = -\varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\alpha\gamma}(\sigma_\mu)_{\gamma\dot{\gamma}}\,. \tag{1.2.64}$$

Les matrices σ_μ, $\sigma^\dagger_\mu$ satisfont les identités

$$\begin{aligned} \sigma_\mu\sigma^\dagger_\nu + \sigma_\nu\sigma^\dagger_\mu = \sigma^\dagger_\mu\sigma_\nu + \sigma^\dagger_\nu\sigma_\mu = 2\delta_{\mu\nu}, \\ \sigma^\dagger_\mu\sigma_\nu - \sigma^\dagger_\nu\sigma_\mu = 2i\,\eta^a_{\mu\nu}\sigma_a, \\ \sigma_\mu\sigma^\dagger_\nu - \sigma_\nu\sigma^\dagger_\mu = 2i\,\bar{\eta}^a_{\mu\nu}\sigma_a, \end{aligned} \tag{1.2.65}$$

où $\eta^a_{\mu\nu}$, $\bar{\eta}^a_{\mu\nu}$ sont les symboles de 't Hooft,

$$\eta^a_{ij} = \bar{\eta}^a_{ij} = \varepsilon_{aij}, \quad \eta^a_{i0} = -\eta^a_{0i} = \bar{\eta}^a_{0i} = -\bar{\eta}^a_{i0} = \delta_{ai} \tag{1.2.66}$$

(σ_a – matrices de Pauli, les indices a, i, j ont varient de 1 à 3). Ils sont auto-duaux et anti-auto-duaux respectivement,

$$\eta^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}\eta^a_{\rho\lambda}, \qquad \bar{\eta}^a_{\mu\nu} = -\frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}\bar{\eta}^a_{\rho\lambda}, \tag{1.2.67}$$

avec la convention

$$\varepsilon_{0123} = -1. \tag{1.2.68}$$

Une autre identité utile est

$$\sigma_2 \sigma_\mu^T \sigma_2 = -\sigma_\mu^\dagger. \tag{1.2.69}$$

Le supermultiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ (1.2.56) implique le champ bosonique $x^{\alpha\dot{\alpha}}(t)$. En fait, ce champ bosonique est équivalent à un quadrivecteur réel dans l'espace Euclidien. Nous allons décrire la transformation entre la notation spinorielle $v^{\alpha\dot{\alpha}}$ et la notation vectorielle correspondante v^μ pour un champ arbitraire v :

$$\begin{aligned} v_{\alpha\dot{\alpha}} &= v_\mu (\sigma_\mu)_{\alpha\dot{\alpha}}, \\ v_\mu &= \tfrac{1}{2} v_{\alpha\dot{\alpha}} (\sigma_\mu^\dagger)^{\dot{\alpha}\alpha} \ = \ -\tfrac{1}{2} v^{\alpha\dot{\alpha}} (\sigma_\mu)_{\alpha\dot{\alpha}}, \\ v^{\alpha\dot{\alpha}} &= \varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}} v_{\beta\dot{\beta}} \ = \ -v_\mu (\sigma_\mu^\dagger)^{\dot{\alpha}\alpha}. \end{aligned} \tag{1.2.70}$$

En particulier, il est facile de vérifier que pour le champ $x^{\alpha\dot{\alpha}}$ avec la contrainte (1.2.57) le champ vecteur correspondant x^μ est réel.

1.3 Nouveaux modèles supersymétrique en mécanique quantique

C'est le chapitre central du manuscrit décrivant certains nouveaux modèles de MQS discutés et étudiés dans les articles [2, 3, 4]. La forme explicite des actions de superchamp et des actions en composantes, ainsi que des hamiltoniens quantiques et opérateurs de supercharges est donnée. Voici un bref résumé des résultats obtenus.

Il est démontré que l'hamiltonien $H = \not{\mathcal{D}}^2$ avec $\not{\mathcal{D}}$ l'opérateur plat de Dirac en dimension quatre dans un champ de jauge de fond *auto-dual* externe, abélien ou non abélien, est supersymétrique avec la supersymétrie $\mathcal{N} = 4$.

Une généralisation de ce hamiltonien existe dans le cas d'une variété courbée, conformément plate à quatre dimensions. Pour un champ de jauge de fond *abélien* auto-dual, le lagrangien correspondant peut être obtenue à partir de certaines expressions de superespace harmonique.

Si l'hamiltonien implique un champ de jauge auto-dual *non abélien*, on va construire sa formulation lagrangienne par l'introduction des variables auxiliaires bosoniques avec une action de type Wess-Zumino. Pour une classe spéciale de ces lagrangiens, il est possible de donner une description de superchamp utilisant le formalisme de superespace harmonique. En particulier, pour les lagrangiens dont le groupe de jauge est SU(2) et le champ de jauge est exprimé sous la forme de l'ansatz de 't Hooft. Comme autre exemple explicite, la mécanique quantique $\mathcal{N} = 4$ avec *monopôle de Yang* dans $\mathbb{R}^5$ (ce qui correspond à un instanton sur S^4) est ensuite considérée.

Enfin, en de façon indépendante, il est montré que un système avec la supersymétrie $\mathcal{N} = 4$ en *trois dimensions* admet également la description en langage de superchamps. Bien que le système en trois dimensions implique différents superchamps, ses composantes lagrangien et hamiltonien apparaissent comme la réduction du système mentionné de quatre dimensions à trois dimensions. La supersymétrie off-shell $\mathcal{N} = 4$ nécessite que le champ de jauge soit une forme statique de l'ansatz de 't Hooft pour les champs de jauge auto-duaux SU(2) en dimension quatre. C'est une solution particulière des équations de Bogomolny pour les *monopôles BPS*.

1.3.1 Fermions en quatre dimensions d'auto-dual de bruit de fond

Description Matrice

Considérons l'opérateur de Dirac dans l'espace Euclidien plat en dimension quatre

$$\not{\mathcal{D}} = \sum_{\mu=0,1,2,3} \mathcal{D}_\mu \gamma_\mu \,, \tag{1.3.1}$$

où $\mathcal{D}_\mu = \partial_\mu - i\mathcal{A}_\mu$ avec $\mathcal{A}_\mu$ le potentiel de jauge et γ_μ les matrices gamma euclidiennes, anti-hermitiennes,

$$\gamma_\mu = \begin{pmatrix} 0 & -\sigma_\mu^\dagger \\ \sigma_\mu & 0 \end{pmatrix}, \qquad \{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}. \tag{1.3.2}$$

Les matrices σ_μ et $\sigma_\mu^\dagger$ ont été introduits dans l'équation (1.2.63). Les hamiltoniens que nous allons construire satisfont la covariance $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$. Par la suite l'indice spinoriel nous pointé se réfère au premier groupe SU(2), tandis que celui pointe – correspondante au seconde groupe.

Considérons l'opérateur

$$H = \frac{1}{2}\not{\mathcal{D}}^2 \;=\; -\frac{1}{2}\mathcal{D}^2 - \frac{i}{4}\mathcal{F}_{\mu\nu}\gamma_\mu\gamma_\nu \;, \tag{1.3.3}$$

où $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i\,[\mathcal{A}_\mu, \mathcal{A}_\nu]$ est le champ de jauge dérivé du potentiel $\mathcal{A}_\mu$. Il est bien connu que les valeurs propres non nulles de l'opérateur de Dirac euclidien forment des couples $(-\lambda, \lambda)$ et donc que le spectre du hamiltonien H est deux fois dégénérés pour tous les états excités. Cela signifie que, pour tout champ extérieur $\mathcal{A}_\mu$, ce hamiltonien est supersymétrique [9] et qu'il admet deux supercharges différentes anticommutantes réels : $\not{\mathcal{D}}$ et $i\not{\mathcal{D}}\gamma_5$ $(\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3)$.

Supposons maintenant que le champ de fond est auto-dual,

$$\mathcal{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\delta}\mathcal{F}_{\rho\delta} \quad\longleftrightarrow\quad \mathcal{F}_{\mu\nu} = \eta^a_{\mu\nu}B_a \;, \tag{1.3.4}$$

où $\eta^a_{\mu\nu}$ sont les symboles de 't Hooft définis dans l'équation (1.2.65). On peut facilement être convaincu que dans ce cas, l'hamiltonien admet q̂uatre racines différentes hermitiennes S_A qui satisfont l'algèbre de supersymétrie étendue (1.2.8) ; on répète ces relations ici :

$$\{S_A, S_B\} = 4\delta_{AB}H. \tag{1.3.5}$$

L'un des choix est

$$\begin{aligned}
S_1 =& \not{\mathcal{D}} = \gamma_0\mathcal{D}_0 + \gamma_1\mathcal{D}_1 + \gamma_2\mathcal{D}_2 + \gamma_3\mathcal{D}_3,\\
S_2 =& \gamma_0\mathcal{D}_3 + \gamma_1\mathcal{D}_2 - \gamma_2\mathcal{D}_1 - \gamma_3\mathcal{D}_0,\\
S_3 =& \gamma_0\mathcal{D}_2 - \gamma_1\mathcal{D}_3 - \gamma_2\mathcal{D}_0 + \gamma_3\mathcal{D}_1,\\
S_4 =& \gamma_0\mathcal{D}_1 - \gamma_1\mathcal{D}_0 + \gamma_2\mathcal{D}_3 - \gamma_3\mathcal{D}_2.
\end{aligned} \tag{1.3.6}$$

Avec les supercharges complexes

$$\begin{aligned}
Q_1 &= (S_1 - iS_2)/2, \qquad & Q_2 &= (S_3 - iS_4)/2,\\
\bar{Q}^1 &= (S_1 + iS_2)/2, \qquad & \bar{Q}^2 &= (S_3 + iS_4)/2,
\end{aligned} \tag{1.3.7}$$

nous obtenons l'algèbre standard de la supersymétrie $\mathcal{N} = 4$ (1.2.25). En conséquence, le spectre excité de H est quatre fois dégénérés, alors que le spectre de $\not{\mathcal{D}}$ se compose des quatuors impliquant deux valeurs propres dégénérés positives et deux dégénérés négatives. Remarquons que, contrairement à $\not{\mathcal{D}}$, l'opérateur $\not{\mathcal{D}}\gamma_5$ n'e exprime pas comme une combinaison linéaire de S_A. En d'autres termes, l'algèbre de supersymétrie $\mathcal{N} = 2$ avec les opérateurs $\not{\mathcal{D}}(1 \pm \gamma_5)$ n'est pas une sous-algèbre de l'algèbre $\mathcal{N} = 4$ (1.2.25).

L'algèbre (1.3.5) avec les supercharges (1.3.6) est la bonne pour tout les champs auto-duaux, indépendamment du fait qu'il soit abélien ou non abélien. Ainsi, la dégénérescence additionnelle double du spectre de l'opérateur de Dirac mentionné précédemment doit être aller que pour un champ générique auto-dual. Un exemple particulier d'un champ non-abélien auto-dual est la solution d'instanton, où cette dégénérescence est observée dans [10] (voir les équations (4.15)). La généralisation de l'opérateur de Dirac et des conditions de (anti)auto-dualité aux variétés de plusieurs dimensions a été prise en compte dans la Réf. [14].

Description covariante

Pour prendre contact avec la description du lagrangien (et, surtout, superchamp), il est commode d'introduire des variables fermiones holomorphes $\psi_{\dot\alpha}$ et $\bar\psi^{\dot\alpha}$, qui répondent eux relations standards d'anticommutation

$$\{\psi_{\dot\alpha}, \psi_{\dot\beta}\} \;=\; \{\bar\psi^{\dot\alpha}, \bar\psi^{\dot\beta}\} \;=\; 0, \qquad \{\bar\psi^{\dot\alpha}, \psi_{\dot\beta}\} = \delta^{\dot\alpha}_{\dot\beta} \tag{1.3.8}$$

et sont réalisés comme des matrices de la façon suivante :

$$\begin{aligned} \psi_{\dot 1} &= \frac{-\gamma_0 + i\gamma_3}{2}, \qquad & \bar\psi^{\dot 1} &= \frac{\gamma_0 + i\gamma_3}{2}, \\ \psi_{\dot 2} &= \frac{\gamma_2 + i\gamma_1}{2}, \qquad & \bar\psi^{\dot 2} &= \frac{-\gamma_2 + i\gamma_1}{2}. \end{aligned} \tag{1.3.9}$$

Puis deux supercharges complexes (1.3.7) sont exprimés d'une façon très simple, à savoir

$$\begin{aligned} Q_\alpha &= \left(\sigma_\mu \bar\psi\right)_\alpha (\hat p_\mu - \mathcal{A}_\mu)\,, \\ \bar Q^\alpha &= \left(\psi \sigma^\dagger_\mu\right)^\alpha (\hat p_\mu - \mathcal{A}_\mu)\,, \end{aligned} \tag{1.3.10}$$

avec $\hat p_\mu = -i\partial_\mu$. L'hamiltonien (1.3.3) est exprimée en ces termes

$$H = \frac{1}{2}\,(\hat p_\mu - \mathcal{A}_\mu)^2 + \frac{i}{4}\,\mathcal{F}_{\mu\nu}\,\psi\sigma^\dagger_\mu\sigma_\nu\bar\psi\ . \tag{1.3.11}$$

Il est clair maintenant pourquoi les indices spinoriels dans les équations (1.2.25) sont sans le point, tandis que dans l'équation (1.3.9), ils sont avec le point : les supercharges sont en rotation par la première groupe SU(2) et les variables $\psi_{\dot\alpha}$ par le seconde [5]. Une distinction soigneuse entre deux différents rapports SU(2) permet mieux comprendre la raison pourquoi la supercharge $\{Q_\alpha, \bar Q^\beta\}$ satisfait l'algèbre simple (1.2.25) dans un champ de font auto-dual. La densité du champ auto-dual $\mathcal{F}$ porte dans la notation spinorielle seulement les indices avec les points. Par conséquent, toute expression impliquant $\mathcal{F}, \psi, \bar\psi$ sont les scalaires sans les points SU(2). Les seules scalaires qui peuvent apparaître à droite de l'anticommutators des supercharges $\{Q_\alpha, \bar Q^\beta\}$ est la structure proportionnelle à δ^β_α, c'est à dire à l'hamiltonien. Aucun autre opérateur n'est pas autorisé.

Dans le cas *abélienne*, les supercharges (1.3.10) et l'hamiltonien (1.3.11) sont des opérateurs scalaires et ne portant plus des indices matricielles. Cela permet de dériver le lagrangien,

$$L = \frac{1}{2}\,\dot x_\mu \dot x_\mu + \mathcal{A}_\mu(x)\dot x_\mu + i\bar\psi^{\dot\alpha}\dot\psi_{\dot\alpha} - \frac{i}{4}\mathcal{F}_{\mu\nu}\,\psi\sigma^\dagger_\mu\sigma_\nu\bar\psi\ . \tag{1.3.12}$$

Dans le cas non abélien, les expressions (1.3.10) et (1.3.11) gardent encore leur structure de couleur matricielle, et on ne peut pas dériver le lagrangien d'une manière aussi simple. Un des moyens pour gérer la structure de la matrice est d'introduire un ensemble de variables fermions couleur (par exemple, dans la représentation fondamentale du groupe) et d'imposer la contrainte supplémentaire ne considérant que le secteur avec unité de charge fermion [9]. Une alternative de la construction non-abélienne du lagrangien est présentée dans la Section 1.3.3. Dans cette section, nous nous limitons seulement aux lagrangiens pour les champs abéliens.

[5] Remarquons que la conjugaison complexe laisse les spineurs dans la même représentation, le groupe de symétrie est ici SO(4) plutôt que SO(1, 3).

La généralisation à conforme métrique plate

Comme il sera démontré explicitement dans la Section 1.3.2, le lagrangien en composantes (1.3.12) découle de l'action de superchamp présenté par Ivanov et Lechtenfeld dans le cadre du superespace harmonique approché de [6].. Nous allons voir que l'on peut naturellement déduire de cette façon un type de σ-modèle généralisation de lagrangien (1.3.12) décrivant le mouvement sur le variété conformément plate métriques $ds^2 = \{f(x)\}^{-2}\, dx_\mu dx_\mu$. Il est écrit comme suit :

$$L = \frac{1}{2} f^{-2}\, \dot{x}_\mu \dot{x}_\mu + i\bar{\psi}^{\dot{\alpha}} \dot{\psi}_{\dot{\alpha}} + \frac{1}{4}\left\{3\,(\partial_\mu f)^2 - f\partial^2 f\right\}\psi^4 + \frac{i}{2} f^{-1}\partial_\mu f\, \dot{x}_\nu\, \psi\sigma^\dagger_{[\mu}\sigma_{\nu]}\bar{\psi}$$
$$+ \mathcal{A}_\mu(x)\dot{x}^\mu - \frac{i}{4} f^2 \mathcal{F}_{\mu\nu}\, \psi\sigma^\dagger_\mu\sigma_\nu\bar{\psi}, \quad (1.3.13)$$

Les supercharges de Nöether quantiques correspondants et l'hamiltonien sont

$$\begin{aligned} Q_\alpha &= f\left(\sigma_\mu\bar{\psi}\right)_\alpha (\hat{p}_\mu - \mathcal{A}_\mu) - \psi_{\dot{\gamma}}\bar{\psi}^{\dot{\gamma}}\left(\sigma_\mu\bar{\psi}\right)_\alpha i\partial_\mu f, \\ \bar{Q}^\alpha &= \left(\psi\sigma^\dagger_\mu\right)^\alpha (\hat{p}_\mu - \mathcal{A}_\mu)\, f + i\partial_\mu f\left(\psi\sigma^\dagger_\mu\right)^\alpha \cdot \psi_{\dot{\gamma}}\bar{\psi}^{\dot{\gamma}}, \end{aligned} \quad (1.3.14)$$

$$H = \frac{1}{2} f\,(\hat{p}_\mu - \mathcal{A}_\mu)^2\, f + \frac{i}{4} f^2\, \mathcal{F}_{\mu\nu}\, \psi\sigma^\dagger_\mu\sigma_\nu\bar{\psi}$$
$$- \frac{1}{2} f i\partial_\mu f\,(\hat{p}_\nu - \mathcal{A}_\nu)\,\psi\sigma^\dagger_{[\mu}\sigma_{\nu]}\bar{\psi} + f\partial^2 f\left\{\psi_{\dot{\gamma}}\bar{\psi}^{\dot{\gamma}} - \frac{1}{2}\left(\psi_{\dot{\gamma}}\bar{\psi}^{\dot{\gamma}}\right)^2\right\}. \quad (1.3.15)$$

D'autre part, on peut calculer explicitement l'anticommutators des supercharges (1.3.14) pour tout champs auto-duaux [6] $\mathcal{A}_\mu(x)$ abélien ou non-abélien et vérifier que l'algèbre (1.2.25) détient. En faisant cela, c'est pratique d'utiliser l'identité de Fierz suivante

$$\left(\bar{\psi}\sigma^\dagger_\mu\right)^\beta \left(\sigma_\nu\psi\right)_\alpha - \left(\sigma_\mu\bar{\psi}\right)_\alpha \left(\psi\sigma^\dagger_\nu\right)^\beta = \delta^\beta_\alpha\, \bar{\psi}\sigma^\dagger_\mu\sigma_\nu\psi\ , \quad (1.3.16)$$

qui peut être prouvé en utilisant (1.2.69).

Notez que, avec un rapport non trivial $f(x)$, l'opérateur de Dirac $\not{\mathcal{D}}$ dans un champ de fond conformément plate peut être exprimé comme une combinaison linéaire de Q_α et $\bar{Q}^\alpha$ *que si* l'on ajoute également une certain *torsion* proportionnelle aux dérivés de $f(x)$ [46]. L'hamiltonien (1.3.15) serait également coïncident avec $\not{\mathcal{D}}^2/2$ dans ce cas. En fait, la généralisation du système avec l'hamiltonien (1.3.11) dans le cas conformément plate, ce qui préserve la supersymétrie $\mathcal{N} = 4$, comme dans l'équation (1.3.15) implique toujours le champ de torsion. L'hamiltonien (1.3.11) admet aussi un autre type de l'extension $\mathcal{N} = 4$ à un espace courbée dotée avec métriques de Hyper-Kähler et sans une torsion [14]. Dans sa forme la plus générale, l'hamiltonien implique la géométrie HKT faible – la généralisation de la technologie de la métrique de Hyper-Kähler obtenue par l'introduction de rapport conformel et de torsion [46].

[6] Anti-auto-dualité conditions sont obtenues lorsque l'on échangeurs σ_μ et $\sigma^\dagger_\mu$ dans toutes les formules. Ceci est équivalent à l'échange de deux représentations spinorielles de SO(4).

Le modèle (1.3.13), (1.3.14), (1.3.15) est un parent proche du modèle construit en Réf. [12] (Voir les équations (30) et (31)), qui décrit le mouvement sur la variété *trois-dimensionnel* conformelement plate dans un champ magnétique externe et d'un potentiel scalaire. En fait, le dernier modèle peuvent être obtenus auprès de l'ancien, en supposant que la métrique et le vecteur potentiel de $\mathcal{A}_\mu \equiv (\Phi, \vec{\mathcal{A}})$ ne dépendent que de trois coordonnées spatiales x_i. Si supposer en outre que la métrique est plate, on est emmene à l'hamiltonien [11]

$$H = \frac{1}{2}\left(\hat{\vec{p}} - \vec{\mathcal{A}}\right)^2 + \frac{1}{2}U^2 + \vec{\nabla}U \, \psi\vec{\sigma}\bar{\psi}, \tag{1.3.17}$$

qui est supersymétriques en condition $\mathcal{F}_{ij} = \varepsilon_{ijk}\partial_k U$ (la réduction de 3 dimensions de la condition d'auto-dualité quadri-dimensionnel). Il a été remarqué dans Réf. [12] que l'hamiltonien effectif de l'électrodynamique chiralle supersymétrique dans le volume spatial fini appartient à cette classe avec $U \propto 1/|\vec{A}|$. Le vecteur de potentiel $\vec{\mathcal{A}}(\vec{A})$ décrit dans ce cas un champ magnétique de Dirac Monopole de type de la phase de Berry apparaît. Les trois variables dynamiques $\vec{A}$ (ne confondrez pas avec le $\vec{\mathcal{A}}$ bouclée !) ont dans ce cas le sens de la nulle harmoniques de transformation de Fourier du vecteur potentiel dans la théorie du champ originale. Dans le premier ordre, la métrique est plate. Lorsque les corrections des boucles supérieures sont incluses, une métrique (conformément plate !) sur l'espace des modules $\{\vec{A}\}$ apparaît.

En exécutant la réduction de hamiltonien de l'équation (1.3.15) avec $\mathcal{A}_\mu$ non-abéliens, la généralisation non-abélienne de l'équation (1.3.17) peut facilement être dérivée. Elle garde la structure de la jauge de l'équation (1.3.17) avec la matrice d'une valeur $\vec{\mathcal{A}}$ et U satisfaisant la condition $\mathcal{F}_{ij} = \varepsilon_{ijk}\mathcal{D}_k U$. Notez que ces hamiltonien *ne* coïncide pas avec l'hamiltonien non-abélien dérivé en 3 dimensions dans Réf. [21].

1.3.2 Description superespace harmonique dans le cas abélien

Dans cette section, nous dérivons l'hamiltonien (1.3.15) de l'approche en langage de superespace harmonique. L'action pertinents de superchamp a été écrit dans [6]. On montre ici que la composante correspondante de Lagrange coïncide avec (1.3.13). La supercharge correspondant (1.3.14) et l'hamiltonien (1.3.15) impliquent un champ de jauge abélien auto-dual $\mathcal{A}_\mu(x)$. Le cas de champ non abélien est discuté plus tard dans ce chapitre.

Contenu du superchamp

Ici on présente un doublet de superchamps $q^{+\dot\alpha}$ avec la charge +1 ($D^0 q^{+\dot\alpha} = q^{+\dot\alpha}$) satisfaisant les contraintes (1.2.50), (1.2.55). L'indice $\dot\alpha$ est l'indice d'une représentation fondamentale du groupe (de Pauli-Gürsey) additionnelle externe SU(2). La solution à ces contraintes dans la base analytique a été écrit dans les équations (1.2.56), (1.2.57). Elle peut être présentée dans la base centrale (1.2.36) comme $q^{+\dot\alpha} = u^+_\alpha q^{\alpha\dot\alpha}$, où $q^{\alpha\dot\alpha}$ ne dépend pas de $u^\pm_\alpha$ (ce dernier découle de la contrainte $D^{++}q^{+\dot\alpha} = 0$ et la définition $D^{++} = u^+_\alpha \frac{\partial}{\partial u^-_\alpha}$). Il est commode de passer à la notation vectorielle à quatre dimensions (1.2.70), en introduisant

$$q_\mu = -\frac{1}{2}\left(\sigma_\mu\right)_{\alpha\dot\alpha} q^{\alpha\dot\alpha}, \qquad q^{+\dot\alpha} = -q_\mu \left(\sigma^\dagger_\mu\right)^{\dot\alpha\alpha} u^+_\alpha. \tag{1.3.18}$$

Maintenant, q_μ est un vecteur par rapport au groupe $\mathrm{SO}(4) = \mathrm{SU}_{\mathrm{R}}(2) \times \mathrm{SU}_{\mathrm{PG}}(2)$, avec le premier rapport représentant la symétrie de groupe R $\mathcal{N} = 4$ et le second étant le groupe globale de Pauli-Gürsey SU(2) qui tourne les indices du “saveurs” avec les pointes.

La condition de pseudorealité (1.2.55) implique que le superchamp q_μ est réel. Ce dernier est exprimé en composantes comme suit :

$$q_\mu = x_\mu + \theta\sigma_\mu\chi + \bar\theta\sigma_\mu\bar\chi - \frac{i}{2}\dot x_\nu\,\bar\theta\sigma_{[\mu}\sigma^\dagger_{\nu]}\theta + \frac{i}{2}\bar\theta\sigma_\mu\dot\chi\,\theta^2 - \frac{i}{2}\theta\sigma_\mu\dot{\bar\chi}\,\bar\theta^2 - \frac{1}{4}\ddot x_\mu\,\theta^4\ , \qquad (1.3.19)$$

où $\theta^2 \equiv \theta^\alpha\theta_\alpha$, $\bar\theta^2 \equiv \bar\theta^\alpha\bar\theta_\alpha$, $\theta^4 \equiv \theta^2\bar\theta^2$. Par ailleurs, la première égalité dans l'équation (1.2.57) implique que

$$x_\mu = -\frac{1}{2}x^{\alpha\dot\alpha}(\sigma_\mu)_{\alpha\dot\alpha} \qquad (1.3.20)$$

est également réel, et on se retrouve avec quatre variables bosoniques dynamique.

L'action du superchamp

L'action classique supersymétrique $\mathcal{N} = 2$ pour le superchamp q_μ peut maintenant être écrite. Il se compose de deux parties, $S = S_{\mathrm{kin}} + S_{\mathrm{int}}$. La partie cinétique,

$$S_{\mathrm{kin}} = \int dt\, d^4\theta du\, R'_{\mathrm{kin}}(q^{+\dot\alpha}, q^{-\dot\beta}, u^\pm_\gamma) = \int dt\, d^4\theta\, R_{\mathrm{kin}}(q_\mu), \qquad (1.3.21)$$

dépend d'une fonction arbitraire $R_{\mathrm{kin}}(q_\mu)$. Remarquons qu'on peut oublier les coordonnées de superespace harmonique ici et travailler dans un superespace ordinaire. En d'autres termes, la terme cinétique S_{kin} ne nécessite pas les coordonnées supplémentaires $u^\pm_\alpha$ de superespace harmonique. Branchement (1.3.19) dans (1.3.21) et en ajoutant/soustrayant les dérivés totale, on obtient

$$S_{\mathrm{kin}} = \int dt \left\{ \frac{1}{2}g(x)\,\dot x_\mu\dot x_\mu + \frac{i}{2}g(x)\left(\bar\chi^{\dot\alpha}\dot\chi_{\dot\alpha} - \dot{\bar\chi}^{\dot\alpha}\chi_{\dot\alpha}\right) \right.$$
$$\left. + \frac{1}{8}\partial^2 g(x)\,\chi^4 - \frac{i}{4}\partial_\mu g(x)\,\dot x_\nu\,\chi\sigma^\dagger_{[\mu}\sigma_{\nu]}\bar\chi \right\}, \qquad (1.3.22)$$

où $g(x) = \frac{1}{2}\partial_x^2 R_{\mathrm{kin}}(x)$ et $\chi^4 = \chi^{\dot\alpha}\chi_{\dot\alpha}\,\bar\chi^{\dot\beta}\bar\chi_{\dot\beta}$.

Pour coupler x_μ à un champ de jauge externe, il faut ajouter le terme d'interaction S_{int} qui représente une intégrale sur *analytique* superespace dans le superespace harmonique,

$$S_{\mathrm{int}} = \int dt\, du\, d\bar\theta^+ d\theta^+ R^{++}_{\mathrm{int}}\left(q^{+\dot\alpha}(t_{\mathrm{A}}, \theta^+, \bar\theta^+), u^\pm_\gamma\right). \qquad (1.3.23)$$

Nous choisissons R^{++}_{int} (il porte la charge $+2$, $D^0 R^{++}_{\mathrm{int}} = 2R^{++}_{\mathrm{int}}$) satisfaisant la condition $\widetilde{R^{++}_{\mathrm{int}}} = -R^{++}_{\mathrm{int}}$ (l'opération d'involution $\widetilde{X}$ a été définie à la section 1.2.2) de telle sorte que l'action (1.3.23) est réelle. En revanche le terme cinétique, le terme d'interaction implique la dépendance sur les harmoniques $u^\pm_\gamma$ et ne peut donc pas être écrit en termes de superchamps du superespace ordinaires (1.2.22).

Pour ce faire l'intégrale sur θ^+ et $\bar{\theta}^+$, nous substituons l'équation (1.2.56) dans (1.3.23) et d'étendre celle-ci en série de Taylor au cours θ^+, $\bar{\theta}^+$, en gardant ce qui concerne seulement $\sim \theta^+\bar{\theta}^+$:

$$R^{++}_{\text{int}}(q^{+\dot{\alpha}}, u^{\pm}_{\gamma}) = \partial_{+\dot{\alpha}} R^{++}_{\text{int}} \cdot \left(-2i\theta^+\bar{\theta}^+ u^-_{\alpha} \dot{x}^{\alpha\dot{\alpha}}\right) + 2\partial_{+\dot{\alpha}}\partial_{+\dot{\beta}} R^{++}_{\text{int}} \cdot \theta^+\bar{\theta}^+ \left(\chi^{\dot{\alpha}}\bar{\chi}^{\dot{\beta}} + \chi^{\dot{\beta}}\bar{\chi}^{\dot{\alpha}}\right) + \dots \tag{1.3.24}$$

(points de suspension indiquent les termes non proportionnelle à $\theta^+\bar{\theta}^+$) avec

$$\partial_{+\dot{\alpha}} R^{++}_{\text{int}}(x, u) \equiv \frac{\partial R^{++}_{\text{int}}(x^{+\dot{\gamma}}, u^{\pm}_{\gamma})}{\partial x^{+\dot{\alpha}}} \tag{1.3.25}$$

Laissez-nous aussi passer à la notation vectorielle x_μ pour les coordonnées $x^{\alpha\dot{\alpha}}$, voir l'équation (1.3.20). En conséquence,

$$x^{+\dot{\alpha}} \equiv x^{\alpha\dot{\alpha}} u^+_{\alpha} = -x_\mu \left(\sigma^\dagger_\mu\right)^{\dot{\alpha}\alpha} u^+_{\alpha}. \tag{1.3.26}$$

Puis

$$S_{\text{int}} = \int dt\, du \left\{2i \left(\sigma^\dagger_\mu\right)^{\dot{\alpha}\alpha} \partial_{+\dot{\alpha}} R^{++}_{\text{int}}\, u^-_{\alpha} \cdot \dot{x}_\mu - 4\chi^{\dot{\alpha}}\bar{\chi}^{\dot{\beta}}\, \partial_{+\dot{\alpha}}\partial_{+\dot{\beta}} R^{++}_{\text{int}}\right\}. \tag{1.3.27}$$

Maintenant, définir le champ de jauge,

$$\mathcal{A}_\mu(x) \equiv \int du \left\{2i \left(\sigma^\dagger_\mu\right)^{\dot{\alpha}\alpha} \partial_{+\dot{\alpha}} R^{++}_{\text{int}}\, u^-_{\alpha}\right\}. \tag{1.3.28}$$

Comme l'action (1.3.27) est réelle, le domaine $\mathcal{A}_\mu(x)$ est également réel. Il a automatiquement divergence nulle,

$$\partial_\mu \mathcal{A}_\mu = 0. \tag{1.3.29}$$

L'intensité du champ est exprimée en

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu = -2\eta^a_{\mu\nu} \int du\, \partial_{+\dot{\alpha}}\partial_{+\dot{\beta}} R^{++}_{\text{int}}\, \varepsilon^{\dot{\alpha}\dot{\gamma}} \left(\sigma_a\right)^{\dot{\beta}}_{\ \dot{\gamma}} \tag{1.3.30}$$

(les identités (1.2.65) ont été utilisés). Il est évidemment auto-dual, puisque les symboles de 't Hooft sont auto-duaux, voir l'équation (1.2.67). Avec les définitions (1.3.28) et (1.3.30) en main, on peut représenter le terme d'interaction (1.3.27) simplement comme

$$S_{\text{int}} = \int dt \left\{\mathcal{A}_\mu(x)\dot{x}_\mu - \frac{i}{4}\mathcal{F}_{\mu\nu}\, \chi\sigma^\dagger_\mu\sigma_\nu\bar{\chi}\right\}. \tag{1.3.31}$$

Enfin, on peut se débarrasser du rapport $g(x)$ dans le terme cinétique de fermions (1.3.22) en introduisant canoniquement conjugués

$$\psi_{\dot{\alpha}} = f^{-1}(x)\chi_{\dot{\alpha}}, \qquad \bar{\psi}^{\dot{\alpha}} = f^{-1}(x)\bar{\chi}^{\dot{\alpha}} \tag{1.3.32}$$

avec

$$f(x) = g^{-1/2}(x) \equiv \left[\frac{1}{2}\,\partial^2_\mu R_{\text{kin}}(x)\right]^{-1/2}. \tag{1.3.33}$$

Ajoutant du terme cinétique dans (1.3.22) au terme d'interaction (1.3.31), on peut vérifier explicitement que le lagrangien $L = L_{\text{kin}} + L_{\text{int}}$ coïncide, à une dérivée totale, avec (1.3.13).

Comme cela a été remarqué, le champ A_μ est obtenue naturellement dans le cadre HSS satisfait la contrainte $\partial_\mu \mathcal{A}_\mu = 0$ [6]. Ce n'est pas vraiment impose une restriction, cependant, parce que les transformations de jauge de A_μ qu'il décalage en le gradient d'un arbitraire le montant de fonction à l'ajout d'une dérivée totale dans le lagrangien (1.3.31).

1.3.3 Le lagrangien composante dans le cas non abélien

Pour un champ auto-dual non abélien en valeurs de matrices $\mathcal{A}_\mu$, le de lagrangien (scalaire) ne peut pas être directement dérivée de l'hamiltonien (1.3.15) par une transformation de Legendre comme il a été fait dans le cas abélien, Éq. (1.3.13). Néanmoins, cela peut être fait dans le cas de SU(N) groupe de jauge en introduisant les champs "semi-dynamique" additionnels φ_i dans la représentation fondamentale de SU(N) et le champ de jauge auxiliaire U(1) $B(t)$. La deuxième ligne de (1.3.13) est alors généralisée à

$$L_{\rm int}^{{\rm SU}(N)} = i\bar{\varphi}^i\left(\dot{\varphi}_i + iB\varphi_i\right) + kB + \mathcal{A}^a_\mu T^a\, \dot{x}_\mu - \frac{i}{4} f^2 \mathcal{F}^a_{\mu\nu} T^a\, \psi\sigma^\dagger_\mu\sigma_\nu\bar{\psi} \tag{1.3.34}$$

avec entier k et

$$T^a = \bar{\varphi}^i\left(t^a\right)_i^{\ j}\varphi_j, \tag{1.3.35}$$

t^a étant la norme SU(N) générateurs de l'algèbre. L'interaction de Lagrange (1.3.34) possède la supersymétrie $\mathcal{N} = 4$. Ce n'est pas difficile de vérifier qu'elle est invariante par rapport aux transformations de jauge non abéliennes de l'espace de cible :

$$\begin{aligned} &\mathcal{A}^a_\mu\, t^a \to U^\dagger \mathcal{A}^a_\mu\, t^a U + iU^\dagger\partial_\mu U \\ &\varphi_i \to \left(U^\dagger\varphi\right)_i, \quad \bar{\varphi}^i \to \left(\bar{\varphi}U\right)^i, \end{aligned} \tag{1.3.36}$$

où $U(x) \in {\rm SU}(N)$. En addition, l'expression (1.3.34) est aussi invariant par rapport à la transformation des champs de jauge auxiliaires $B(t)$ et φ_i suivants :

$$B(t) \to\ B(t) + \frac{d\alpha(t)}{dt}, \qquad \varphi_i(t) \to e^{-i\alpha(t)}\varphi_i(t). \tag{1.3.37}$$

Il n'est pas immédiatement évident comment étendre le description de superchamp abélien en cas général non abélien, c'est à dire en groupe de jauge SU(N) . Nous avons réussi à construire une telle description pour le cas particulier de SU(2) champs de jauge auto-dual ou anti-auto-dual exprimée sous la forme

$$\mathcal{A}^a_\mu \ =\ -\bar{\eta}^a_{\mu\nu}\partial_\nu \ln h(x) \quad \text{or} \quad \mathcal{A}^a_\mu \ =\ -\eta^a_{\mu\nu}\partial_\nu \ln h(x) \tag{1.3.38}$$

respectivement, avec la fonction harmonique $h(x)$,

$$\partial^2_\mu h(x) = \text{la sum finie de delta fonctions.} \tag{1.3.39}$$

C'est ce qu'on appelle l'ansatz de 't Hooft pour une solution de multi-instanton SU(2) [16] avec les symboles de 't Hooft $\eta^a_{\mu\nu}$ définis précédemment dans l'équation (1.2.65). Si on prend la fonction $h(x)$ qui disparaissent à $|x| \to \infty$, alors cette fonction peut être présentée comme la somme suivante sur instantons :

$$h(x) = 1 + \sum_I \frac{c_I}{(x^\mu - b^\mu_I)^2}. \tag{1.3.40}$$

Il s'agit notamment des positions instanton b^μ_I ainsi que les numéros c_I associés à chaque instanton.

Quantification des champs auxiliaires dans le cas du le groupe de jauge SU(2)

Entendons-nous comment l'interaction de Lagrange (1.3.34) donne lieu à l'hamiltonien de matrice (1.3.15). Ceci est réalisé lors de la quantification des variables auxiliaires φ_α. Nous considérons que le cas particulier de SU(2) groupe de jauge lorsque les indices i, j pour les variables auxiliaires ne prendre que deux valeurs, 1 et 2, et sont notés α, β. Voir la section 1.3.4 pour une discussion générale de SU(N) groupe de jauge.

On observe que les variables φ_α saisir le lagrangien avec une seule dérivée temporelle. Ainsi, ils ne sont pas à part entière des variables dynamiques (comme x_μ) et non plus des champs auxiliaires (comme le champ $B(t)$ ou les champs $\omega_{1,2}$ dans l'équation (1.3.50), voir ci-dessous). Ils ont une espèce de la nature intermédiaire. Dans le contexte des modèles de MQS $\mathcal{N}=4$, des variables telles (ensemble avec leurs transporteurs superchamp analytique $v^+, \widetilde{v^+}$, voir les équations (1.3.50), (1.3.51) ci-dessous) ont été introduites dans [19, 20]. Voir aussi [21] pour une application récente. Pour mieux comprendre la nature des champs auxiliaires, effectuez la quantification.

Les relations de commutation canoniques suivantes de l'action (1.3.34) grâce à la prescription standard de Dirac sont

$$[\varphi_\alpha, \bar\varphi^\beta] = \delta_\alpha^\beta\ , \qquad\qquad [\varphi_\alpha, \varphi_\beta] = [\bar\varphi^\alpha, \bar\varphi^\beta] = 0. \tag{1.3.41}$$

Le fait que k doit être entier emmene aux représentations finies de l'algèbre de l'opérateur φ_α, $\bar\varphi^\alpha$. En effet, considérons la contrainte

$$\bar\varphi^\alpha \varphi_\alpha = k\,, \tag{1.3.42}$$

qui suit de 1.3.34 en faisant varier par rapport à B. Toutes les valeurs réelles positives de k sont classiquement autorisée. Dans la théorie quantique, k doit être entier, mais pas nécessaire positif.

Prenons le cas de valeurs positives de l'entier k. En théorie quantique, on peut choisir $\varphi_\alpha \equiv \partial/\partial\bar\varphi^\alpha$ et d'imposer 1.3.42 sur les fonctions d'onde :

$$\bar\varphi^\alpha\varphi_\alpha\Psi = \bar\varphi^\alpha \frac{\partial}{\partial\bar\varphi^\alpha}\Psi = k\Psi\ . \tag{1.3.43}$$

En d'autres termes, les fonctions d'onde représentent des polynômes homogènes de $\bar\varphi^\alpha$ de (un entier) degré k. Dans le cas $k<0$ l'algèbre (1.3.41) est le même, mais il faut choisir $\bar\varphi^\alpha = -\partial/\partial\varphi_\alpha$ et d'envisager des polynômes de φ_α de degré $|k|$. Le nombre de ces (linéairement indépendants) polynômes est $|k|+1$. Par ailleurs, il est également facile de voir que les opérateurs (1.3.35) (qui entrent dans l'interaction de Lagrange (1.3.34)) satisfont l'algèbre suivantes :

$$[T^a, T^b] \;=\; i\varepsilon^{abc}T^c. \tag{1.3.44}$$

En outre, en supposant que $k>0$ et en prenant (1.3.43), on tire

$$T^aT^a \;=\; \frac{1}{4}\left[(\bar\varphi^\alpha\varphi_\alpha)^2 + 2(\bar\varphi^\alpha\varphi_\alpha)\right] \;=\; \frac{k}{2}\left(\frac{k}{2}+1\right). \tag{1.3.45}$$

En d'autres termes, T^a peuvent être traités comme les générateurs de SU(2) dans la représentation spinorielle $k/2$.

Cette façon de quantification des variables semi-dynamique $\varphi_\alpha, \bar{\varphi}^\alpha$ a été employé dans Réf. [20]. Alternativement, on pourrait interpréter $\varphi_\alpha, \bar{\varphi}^\alpha$ avec la contrainte 1.3.42 comme une sorte des variables harmoniques de la cible représentant une sphère S^2, de résoudre des 1.3.42 en termes de projection stéréographique coordonnées et de quantification du système (voir, par exemple, Réf. [48]).

Une propriété intéressante est que cette jauge SU(2) groupe est en fait le groupe de symétrie R de algèbre de supersymétrie $\mathcal{N} = 4$.

Le rôle crucial de la contrainte 1.3.43 est de restreindre l'espace des états quantiques du modèle considéré à l'ensemble *finie* des irréductibles SU(2) multiplets de spin fixe (par exemple, du spin $k/2$ dans le sectre bosonique). C'est une différence essentielle de cette approche de l'approche implique, par exemple, dans [18] (et plus tard dans [21, 17]) où aucune des contraintes analogiques 1.3.42 et 1.3.43 a été imposée, permettant ainsi l'espace des états d'impliquer une *infinite* nombre de SU(2) multiplets de tous les spins. Le schéma de quantification que nous suivons ici, a été précédemment utilisé dans le cadre MQS dans [19, 20] et on peut faire remonter à l'ouvre [49].

1.3.4 Lagrangien en superespace harmonique avec des champs de jauge non abéliens

Contenu de superchamp

Pour construire l'action impliquant des champs de jauge non abéliennes, introduire, comme précédemment, une doublet de superchamps $q^{+\dot{\alpha}}$ avec la charge +1 satisfaisant les contraintes (1.2.50), (1.2.55). *En plus de cela*, nous introduisons une analytique de calibre superchamp V^{++} de charge +2 en satisfaisant les contraintes

$$D^+V^{++} = \bar{D}^+V^{++} = 0\ , \qquad V^{++} = \widetilde{V^{++}} \tag{1.3.46}$$

et “la question” de superchamp v^+ de la charge +1. Les contraintes aux quelles elle répond,

$$D^+v^+ = 0, \qquad \bar{D}^+v^+ = 0, \qquad (D^{++} + iV^{++})v^+ = 0\,, \tag{1.3.47}$$

diffèrent de (1.2.50) par la présence de la dérivée covariante harmoniques $\mathcal{D}^{++} = D^{++} + iV^{++}$ [5]. La contrainte $\mathcal{D}^{++}v^+ = 0$ est covariant par rapport à mesurer les transformations

$$V^{++} \ \rightarrow\ V^{++} + D^{++}\Lambda, \qquad v^+ \ \rightarrow\ e^{-i\Lambda}v^+, \qquad D^+\Lambda = \bar{D}^+\Lambda = 0\ . \tag{1.3.48}$$

Nous pouvons utiliser cette liberté de jauge afin d'éliminer presque toutes les composantes de V^{++} et de présenter comme

$$V^{++} \ = \ 2i\,\theta^+\bar{\theta}^+B, \tag{1.3.49}$$

où le champ de jauge $B(t)$ est bien réel. Ceci est une contrepartie à une dimension du familier de Wess-Zumino jauge en quatre dimensions théories. Observons aussi que l'équation (1.3.37) est un vestige de transformations de jauge (1.3.48), qui survit dans la jauge de Wess-Zumino (1.3.49).

Ensuite, le superchamp v^+ est exprimée dans la base analytique que

$$v^+ = \phi^\alpha u^+_\alpha - 2\theta^+\omega_1 - 2\bar\theta^+\bar\omega_2 - 2i\theta^+\bar\theta^+(\dot\phi^\alpha + iB\phi^\alpha)u^-_\alpha \,, \tag{1.3.50}$$

d'où il s'ensuit que

$$\widetilde{v^+} = \bar\phi^\alpha u^+_\alpha - 2\theta^+\omega_2 + 2\bar\theta^+\bar\omega_1 - 2i\theta^+\bar\theta^+(\dot{\bar\phi}^\alpha - iB\bar\phi^\alpha)u^-_\alpha \tag{1.3.51}$$

avec $\bar\phi^\alpha = (\phi_\alpha)^*$. Ainsi, les champs ϕ_α et $\bar\phi^\alpha$ non nulle mener face U(1) les charges associés à l'auxiliaire de champ de jauge B.

L'action de superchamp

Le $\mathcal{N} = 4$ l'action supersymétrie-invariante se compose de trois parties, $S = S_{\rm kin} + S_{\rm int} + S_{\rm FI}$. La partie cinétique est plus commode d'exprimer dans la base centrale $\{t,\, \theta_\alpha,\, \bar\theta^\beta\}$. Il a la même forme que dans l'équation (1.3.21) et son expansion composante coïncide avec la première ligne dans l'équation (1.3.13), où le même changement de variables (1.3.32) et (1.3.33) est effectuée comme dans le cas abélien.

La partie d'interaction est considérée comme

$$S_{\rm int} = -\frac{1}{2}\int dt\, du\, d\bar\theta^+ d\theta^+\, K\left(q^{+\dot\alpha}, u^\pm_\beta\right) v^+\widetilde{v^+}, \tag{1.3.52}$$

où la condition $\widetilde{K} = K$ est imposée pour garantir l'action pour être vrai. Enfin, nous ajoutons le terme de Fayet-Iliopoulos

$$S_{\rm FI} \;=\; -\frac{ik}{2}\int dt\, du\, d\bar\theta^+ d\theta^+\, V^{++} \;=\; k\int dt\, B, \tag{1.3.53}$$

qui est invariant sous les transformations de jauge (1.3.48).

Concentrons-nous sur la partie d'interaction. Il est commode d'introduire de nouvelles variables

$$\varphi_\alpha \;=\; \phi_\alpha \sqrt{h(x)}\;, \tag{1.3.54}$$

où

$$h(x) = \int du\, K(x^{+\dot\alpha}, u^\pm_\beta) \qquad \left(x^{+\dot\alpha} = x^{\alpha\dot\alpha}u^+_\alpha\right), \tag{1.3.55}$$

est une fonction harmonique [7]. En effet,

$$\partial^2_\mu h(x) = 4\,\varepsilon^{\dot\alpha\dot\beta}\int du\, \partial_{+\dot\alpha}\partial_{-\dot\beta}K(x^{+\dot\gamma}, u^\pm_\beta) = 0\,. \tag{1.3.56}$$

En substituant (1.2.56), (1.3.50) et (1.3.51) dans (1.3.52) et en éliminant les degrés auxiliaires fermioniques de la liberté de $\omega_{1,2}$, $\bar\omega_{1,2}$ par leurs équations algébriques du mouvement, nous tirons au bout de quelques algèbres

$$L_{\rm int} \;=\; i\bar\varphi^\alpha(\dot\varphi_\alpha + iB\varphi_\alpha) - \frac{1}{2}\bar\varphi^\beta\varphi_\gamma\left(\mathcal{A}_{\alpha\dot\alpha}\right)_\beta{}^\gamma\,\dot x^{\alpha\dot\alpha} - \frac{i}{4}\left(\mathcal{F}_{\dot\alpha\dot\beta}\right)_\beta{}^\gamma\,\chi^{\dot\alpha}\bar\chi^{\dot\beta}\bar\varphi^\beta\varphi_\gamma\,. \tag{1.3.57}$$

[7] Nous supposons ici que $h(x) > 0$. Le cas $h(x) < 0$ est traité de même, si l'on redéfinit $h(x) \to -h(x)$.

Voici

$$(\mathcal{A}_{\alpha\dot\alpha})_\beta^{\ \gamma} = -\frac{2i}{\int du\, K}\int du\,\partial_{+\dot\alpha}K\left(u^{+\gamma}\varepsilon_{\alpha\beta} - \frac{1}{2}u^+_\alpha\delta^\gamma_\beta\right) = \frac{i}{h}\left(\varepsilon_{\alpha\beta}\,\partial^\gamma{}_{\dot\alpha}h - \frac{1}{2}\delta^\gamma_\beta\,\partial_{\alpha\dot\alpha}h\right) \quad (1.3.58)$$

($\partial_{\alpha\dot\alpha} \equiv (\sigma_\mu)_{\alpha\dot\alpha}\,\partial_\mu = -2\partial/\partial x^{\alpha\dot\alpha}$) est une matrice hermitienne sans trace – le champ de jauge, et

$$\left(\mathcal{F}_{\dot\alpha\dot\beta}\right)_\beta^{\ \gamma} = (\mathcal{F}_{\mu\nu})_\beta^{\ \gamma}\left(\sigma^\dagger_\mu\sigma_\nu\right)_{\dot\alpha\dot\beta} = \partial_{\delta\dot\alpha}(\mathcal{A}^\delta{}_{\dot\beta})_\beta^{\ \gamma} - i(\mathcal{A}_{\delta\dot\alpha})_\beta^{\ \lambda}(\mathcal{A}^\delta{}_{\dot\beta})_\lambda^{\ \gamma} + (\dot\alpha \leftrightarrow \dot\beta) \quad (1.3.59)$$

est son auto-dual partie. Il est facile de vérifier de façon explicite, que la partie anti-auto-duale du champ de jauge $\mathcal{A}_\mu$ s'annule,

$$(\mathcal{F}_{\alpha\beta})_\gamma^{\ \delta} = (\mathcal{F}_{\mu\nu})_\gamma^{\ \delta}\left(\sigma_\mu\sigma^\dagger_\nu\right)_{\alpha\beta} = -\partial_{\alpha\dot\alpha}(\mathcal{A}_\beta{}^{\dot\alpha})_\gamma^{\ \delta} + i(\mathcal{A}_{\alpha\dot\alpha})_\gamma^{\ \lambda}(\mathcal{A}_\beta{}^{\dot\alpha})_\lambda^{\ \delta} + (\alpha \leftrightarrow \beta) = 0. \quad (1.3.60)$$

Ainsi, l'intensité du champ $\mathcal{F}^a_{\mu\nu}$ est auto-dual et appartient à la représentation $(0,1)$ de $\mathrm{SO}(4) = \mathrm{SU}(2)\times\mathrm{SU}(2)$. Passant à $\mathcal{A}^a_\mu$ tel que $(\mathcal{A}_\mu)_\beta^{\ \gamma} = \mathcal{A}^a_\mu\,(\sigma_a)_\beta^{\ \gamma}/2$, nous constatons que la représentation 1.3.58 sommes précisément à l'auto-dual l'ansatz de 't Hooft (1.3.38), l'équation gauche. L'expression anti-auto-duale de l'équation de droit naît si l'on en pointillés les échangeurs et les indices sans pointés, à savoir les échanges de manière efficace σ_μ et $\sigma^\dagger_\mu$. Cela implique aussi de passer à des harmoniques $u^\pm_{\dot\alpha}$ et en fait à un autre $\mathcal{N}=4$ la supersymétrie, avec le seconde SU(2) (agissant sur les indices en pointillés) comme le groupe de symétrie R.

Enfin, en substituant $\bar\varphi^\beta\varphi_\gamma = T^a\,(\sigma_a)_\gamma^{\ \beta}$ et $\chi_{\dot\alpha} = f\psi_{\dot\alpha}$ dans (1.3.57), où T^a est défini dans (1.3.35) avec $t^a = \frac{1}{2}\sigma_a$, une convainc lui-même que le terme d'interaction avec le terme de FI 1.3.53 rendements juste (1.3.34) pour le SU(2) cas de groupe de jauge, et l'hamiltonien quantique dérivée du lagrangien $L_{\mathrm{kin}} + L_{\mathrm{int}} + L_{\mathrm{FI}}$ a la forme (1.3.15) avec $\mathcal{A}_\mu \equiv \mathcal{A}^a_\mu T^a$ et $\mathcal{F}_{\mu\nu} \equiv \mathcal{F}^a_{\mu\nu}T^a$.

Supersymétrie $\mathcal{N}=4$ avec monopôle de Yang

Nous avons construit l'action pour le superchamp $\mathcal{N}=4$ la mécanique quantique supersymétrique correspondant à l'hamiltonien (1.3.15) avec un non-abéliens SU(2) champ de jauge $\mathcal{A}_\mu$ ce qui vit sur une 4-variété conformément plate et est représentable sous la forme 't Hooft ansatz (1.3.38).

Comme exemple d'un tel champ, citons la solution instanton sur S^4. De manière générique, elle dépend du rayon R de la sphère et la taille instanton ρ. Les configurations de taille maximale, $\rho = R$, présentent un intérêt particulier. Dans les coordonnées stéréographiques sur S^4,

$$ds^2 \;=\; \frac{4R^4dx^2_\mu}{(x^2+R^2)^2}\,, \quad (1.3.61)$$

ils sont exprimés par les mêmes formules que l'instantons à plat dans la jauge de singulier,

$$\mathcal{A}^a_\mu \;=\; \frac{2R^2\bar\eta^a_{\mu\nu}x_\nu}{x^2(x^2+R^2)} \quad \text{or} \quad (\mathcal{A}_{\alpha\dot\alpha})_\beta^{\ \gamma} = -\frac{2i\,R^2}{x^2(x^2+R^2)}\left(\varepsilon_{\alpha\beta}x^\gamma_{\dot\alpha} - \frac{1}{2}\,\delta^\gamma_\beta\,x_{\alpha\dot\alpha}\right), \quad (1.3.62)$$

et

$$(\mathcal{F}_{\dot\alpha\dot\beta})_\beta{}^\gamma = \frac{8i\,R^2}{x^2(x^2+R^2)^2}\left(x^\gamma_{\dot\beta}x_{\beta\dot\alpha} + x^\gamma_{\dot\alpha}x_{\beta\dot\beta}\right). \qquad (1.3.63)$$

Les fonctions correspondantes dans l'équation (1.3.55) sont prises sous la forme

$$K(x^{+\dot\alpha}, u^\pm_\beta) = 1 + \frac{1}{\left(c^-_{\dot\alpha}x^{+\dot\alpha}\right)^2}\,, \qquad h(x) \equiv \int du\,K(x^{+\dot\alpha}, u^\pm_\beta) = 1 + \frac{R^2}{x^2_\mu}, \qquad (1.3.64)$$

où $c^-_{\dot\alpha} = c^\alpha{}_{\dot\alpha}u^-_\alpha$, $c^{\alpha\dot\alpha}$ – vecteur constant et $R^2 = 1/c^2_\mu$. L'intégrale sur le côté droit de l'équation (1.3.64) peut être calculée comme la série entière en $c^-_{\dot\alpha}c^{+\dot\alpha} = -c^2_\mu$ ou directement après avoir constaté que la forme de cette intégrale est SO(4) invariant et mettre $c_\mu = (c, 0, 0, 0)$, $x_\mu = (x_1, x_2, 0, 0)$.

Le champ $\mathcal{A}^a_\mu$ peut être apporté à la jauge non singulière

$$\mathcal{A}^a_\mu = \frac{2\eta^a_{\mu\nu}x_\nu}{x^2+R^2}\,, \qquad \mathcal{F}^a_{\mu\nu} = -\frac{4R^2\eta^a_{\mu\nu}}{(x^2+R^2)^2}\,, \qquad (1.3.65)$$

par les transformations de jauge (1.3.36) avec $U(x) = -i\sigma_\mu x_\mu/\sqrt{x^2}$ (cette forme particulière $U(x)$ est demandé par le formulaire du champ la force 1.3.63). La densité de l'action $\sim \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ est la même dans ce cas à tous les points de S^4. Il est à noter que la transformation de jauge singulière convertit le indices de groupe de jauge sans pointés dans les pointillés : l'auto-dual potentiel de jauge et de l'intensité du champ dans la notation spinorielle devenu

$$(\mathcal{A}_{\alpha\dot\alpha})^{\dot\gamma}{}_{\dot\beta} = \frac{2i}{x^2+R^2}\left(\varepsilon_{\dot\alpha\dot\beta}x^{\dot\gamma}_\alpha - \frac{1}{2}\,\delta^{\dot\gamma}_{\dot\beta}\,x_{\alpha\dot\alpha}\right), \quad (\mathcal{F}_{\dot\alpha\dot\beta})^{\dot\gamma}{}_{\dot\delta} = -\frac{8i\,R^2}{(x^2+R^2)^2}\left(\varepsilon_{\dot\alpha\dot\delta}\delta^{\dot\gamma}_{\dot\beta} + \varepsilon_{\dot\beta\dot\delta}\delta^{\dot\gamma}_{\dot\alpha}\right) \qquad (1.3.66)$$

et, aussi, $\varphi_\alpha \to \varphi^{\dot\alpha} = -i\varphi_\alpha\,x^{\alpha\dot\alpha}/\sqrt{x^2}$, $\bar\varphi^\alpha \to \bar\varphi_{\dot\alpha} = -i\bar\varphi^\alpha\,x_{\alpha\dot\alpha}/\sqrt{x^2}\,$.

Notez que, sur le terrain (1.3.62), (1.3.65) décrit la vie monopôle Yang dans $\mathbb{R}^5$ [51]. Le potentiel 1.3.65 non singulière a un groupe-théorique belle signification comme l'un des deux SU(2) connexions sur le collecteur coset SO(5)/[SU(2) × SU(2)] $\sim \mathrm{S}^4$ (voir par exemple [52]). Elle coïncide avec le plat auto-dual instanton seulement dans la paramétrisation conformément plate de S^4 comme dans 1.3.61. Lorsque couplé à la ligne à travers le monde nos semi-dynamique des variables $\varphi_\alpha, \bar\varphi^\alpha$, le monopole à 5 dimensions Yang est réduite à cette SU(2) connexion définie sur S^4.

Laissez-nous élaborer sur ce point plus en détail, en choisissant, sans perte de généralité, $R = 1$ dans les formules ci-dessus. Considérez ce qui suit lagrangien $d = 1$ bosonique avec l'espace $\mathbb{R}^5$ cible et un couplage supplémentaire pour Yang monopôle

$$L_{\mathbb{R}^5} = \frac{1}{2}\,(\dot y_5\dot y_5 + \dot y_\mu\dot y_\mu) + \mathcal{B}^a_\mu(y)T^a\,\dot y_\mu\,. \qquad (1.3.67)$$

Ici, $\mathcal{B}^a_\mu$ est la forme standard du monopole dans le Yang coordonnées $\mathbb{R}^5$,

$$\mathcal{B}^a_\mu = \frac{\eta^a_{\mu\nu}y_\nu}{r(r+y_5)}\,, \quad r = \sqrt{y^2_5 + y^2_\mu}\,, \qquad (1.3.68)$$

T^a sont définis comme dans 1.3.35 avec $t^a = \frac{1}{2}\sigma^a$, et l'action pour les demi-dynamique variables $\varphi_\alpha, \bar{\varphi}^\alpha$ est omis. Maintenant on passe à la décomposition polaire de $\mathbb{R}^5$ dans un rayon r et la partie angulaire S^4 , $(y_5, y_\mu) \;\rightarrow\; (r, \tilde{y}_5, \tilde{y}_\mu)$, $\tilde{y}_5 = \sqrt{1-\tilde{y}_\mu^2}$, et réécrit 1.3.67 comme

$$L_{\mathbb{R}^5} = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\left(\dot{\tilde{y}}_5\dot{\tilde{y}}_5 + \dot{\tilde{y}}_\mu\dot{\tilde{y}}_\mu\right) + \frac{\eta^a_{\mu\nu}\tilde{y}_\nu\dot{\tilde{y}}_\mu\,T^a}{1+\sqrt{1-\tilde{y}_\mu^2}}\,. \tag{1.3.69}$$

Les coordonnées $\tilde{y}_\mu$ donner une paramétrisation particulière de S^4. Passant aux coordonnées stéréographiques est accompli par la redéfinition

$$\tilde{y}_\mu = 2\,\frac{x_\mu}{1+x^2}\,,$$

ce qui jette 1.3.69 dans le formulaire

$$L_{\mathbb{R}^5} = \frac{1}{2}\left\{\dot{r}^2 + 4r^2\frac{\dot{x}_\mu\dot{x}_\mu}{(1+x^2)^2}\right\} + \frac{2\eta^a_{\mu\nu}x_\nu\dot{x}_\mu\,T^a}{1+x^2}\,. \tag{1.3.70}$$

On voit que les S^4 métriques 1.3.61 (avec $R=1$) et les instantons vecteur potentiel 1.3.65 apparaissent.

Ainsi, l'approche actuelle, comme un sous-produit, fournit une solution au problème de longue date de la construction $\mathcal{N}=4$ supersymétriques mécanique quantique avec des monopôles Yang (voir par exemple [8] et références dedans). Évidemment, la composante lagrangienne (1.3.13) (avec la fonction correspondante $f(x)$) est juste S^4 le cadre de la lagrangienne 1.3.70 avec le radiale variable “gelés” $r=1$. Vraisemblablement, on peut restaurer la complète à 5 dimensions en partie cinétique 1.3.70 en ajoutant un couplage scalaire convenablement contraints $\mathcal{N}=4$ de charge nulle superchamp $X(t,\theta,\bar{\theta})$ qui décrit un multiplet off-shell $(\mathbf{1},\mathbf{4},\mathbf{3})$ avec un champ bosonique physiques [44], telle que $X|_{\theta=\bar{\theta}=0} = r$.

Quelques remarques dans le cas non abélien

Le problème de trouver une formulation superchamp pour un générique $\mathrm{SU}(N)$ field auto-dual est plus complexe et est encore une question ouverte. Toutefois, en introduisant des variables supplémentaires φ_i, il est toujours possible d'écrire un lagrangien *en composante* (1.3.34) (avec la première ligne (1.3.13)) correspondant à l'hamiltonien de la matrice (1.3.15).

Cette observation a effectivement rien à voir avec la supersymétrie. Son point d'ébullition jusqu'à la suivante. Considérons le problème aux valeurs propres pour une matrice hermitienne habituelle H_{jk}. Elle peut être traitée comme un Schrödinger problème $\hat{H}\Psi(\varphi_j) = \lambda\Psi(\varphi_j)$ with the constraint $\hat{G}\Psi = 0$, où

$$\hat{H} \;=\; \varphi_j H_{jk}\frac{\partial}{\partial\varphi_k}\,, \qquad \hat{G} = \varphi_j\frac{\partial}{\partial\varphi_j} - 1. \tag{1.3.71}$$

Le lagrangien correspondant est

$$L \;=\; i\bar{\varphi}_j\dot{\varphi}_j - B(\bar{\varphi}_j\varphi_j - 1) - \bar{\varphi}_j H_{jk}\varphi_k, \tag{1.3.72}$$

où $\bar{\varphi}_j = (\varphi_j)^*$. Ceci généralise facilement au cas où H est un opérateur en fonction d'un ensemble de variables conjuguées canoniquement $\{p_\mu, x_\mu\}$. La seule différence est que $-H_{jk}$ est maintenant remplacée par la matrice L_{jk} obtenu à partir de H_{jk} par le cas échéant transformation de Legendre [8].

L'objectif initial était de trouver une représentation de Lagrange pour l'hamiltonien (1.3.15) avec la matrice à valeurs $\mathcal{A}_\mu$, $\mathcal{F}_{\mu\nu}$. La construction décrite, avec φ_i dans la représentation fondamentale de SU(N), conduit à matrice hamiltonienne $N \times N$. Le lagrangien (1.3.72) coïncide dans ce cas avec le lagrangien (1.3.34) avec le choix $k = 1$, à laquelle la première ligne de Éq. (1.3.13) est également ajouté.

Évidemment, on peut décrire les Hamiltoniens en plus des représentations de SU(N) de la même façon, en choisissant le nombre de composants φ_i égale à la dimension de la représentation. Nous l'avons vu, cependant, que dans le cas SU(2) on peut être plus économique, en introduisant seulement un couple de variables dynamiques φ_α en multipliant le terme proportionnel à B dans le lagrangien par un arbitraire entier k. Cela conduit à l'hamiltonien dans la représentation de spin $|k|/2$. Certaines représentations SU(N) (les produits symétriques de $|k|$ fondamentale ou $|k|$ représentations antifundamental) peut également être atteint de cette manière.

On peut aussi construire de cette façon un lagrangien supersymétrique $\mathcal{N} = 2$ de l'hamiltonien (1.3.15) avec $\mathcal{A}_\mu$ générique (pas nécessairement auto-dual). Une construction similaire (mais avec des variables supplémentaires fermioniques plutôt que bosonique) a été en fait discuté dans Réf. [9]. Une beauté de l'approche superespace harmonique exploré ici est, toutefois, que ces variables supplémentaires et la contrainte (1.3.43) ne sont pas mis en place par la main, mais se posent naturellement des manifestement off-shell actions superchamps supersymétriques.

1.3.5 MQS avec des champs de jauge non abélien de monopôle en trois dimensions

L'hamiltonien (1.3.15) présente la généralisation de l'hamiltonien (1.3.11) à métrique conformément plate en *quatre dimensions*. Nous avons réussi dans la construction de cette généralisation en utilisant le formalisme des superchamps. Dans cette section, nous employons de construction similaire et généraliser le système en *trois dimensions* décrit par l'hamiltonien (1.3.17) pour le cas conformément plate. Bien que l'hamiltonien résultante, les supercharges et le lagrangien composante être simplement la réduction en trois dimensions de la système à quatre dimensions, le formalisme des superchamps dans le cas de trois dimensions implique une superchamp spécifique pour les coordonnées d'espace x^i et est donc mis en œuvre différemment.

Composant superchamp et action superchamp

Dans cette section, au lieu du superchamp des coordonnées $q^{+\dot\alpha}$ nous travaillons avec le superchamp analytique L^{++} qui englobent le multiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ et est soumis à des

[8] Cette observation élémentaire devrait être bien connu, par exemple, dans les modèles de matrices. Étonnamment, il n'est pas trouvé dans une telle forme "chimiquement pur", à l' littérature, mais des constructions similaires ont été discutées, par exemple, dans les références [49, 53].

contraintes (1.2.58). Le superchamp V^{++} et les superchamps auxiliaires v^+ et $\widetilde{v^+}$ sont définis par les équations mêmes (1.3.46), (1.3.47). Le superchamp V^{++} dans la jauge de Wess-Zumino (1.3.49) est exprimé avec une composante indépendante $B(t)$. Nous rappelons que $B(t)$ est un "champ de jauge" réel unidimensionnele qui se transforme comme $B \to B + \dot{\lambda}$, avec $\lambda(t)$ étant le paramètre du jauge résiduelle symétrie U(1).

Les expressions explicites pour les superchamps $q^{+\dot{\alpha}}$, v^+, $\widetilde{v^+}$ et V^{++} sont écrits dans Éqs. (1.2.56), (1.3.50), (1.3.51) et (1.3.49) respectivement. L'expansion composante du superchamp analytique L^{++} peut être trouvé dans Éqs. (1.2.59), (1.2.60). Le multiplet L^{++} implique les coordonnées cible espace $\ell^{\alpha\beta} = \ell^{\beta\alpha}$ en trois dimensions, leurs partenaires fermioniques χ^α, $\bar{\chi}^\alpha$ et un champ auxiliaire réel F. Remarquons que le cas de trois dimensions implique seulement un seule groupe symétrie R SU(2) et donc pas d'indices en pointillés présents dans la description.

Notez également que la contrainte $\bar{\chi}_\alpha = (\chi^\alpha)^*$ implique la position différente des indices spinoriels par rapport à Éq. (1.2.57) dans le cas en quatre dimensions (voir la Section 1.3.5 ci-dessous). Le passage de la notation spinorielle $\ell^{\alpha\beta}$ à la notation vectorielle ℓ^i,

$$\ell_\alpha^\beta = \ell_i \, (\sigma_i)_\alpha^{\ \beta}, \qquad \ell_i = \frac{1}{2} \ell_\beta^\alpha \, (\sigma_i)_\alpha^{\ \beta}, \qquad i = 1, 2, 3 \tag{1.3.73}$$

(σ_i est matrices de Pauli et, comme d'habitude, indices spinoriels sont soulevées et abaissées avec les tenseurs antisymétriques de Levi-Civita $\varepsilon_{\alpha\beta}$ et $\varepsilon^{\alpha\beta}$), pour les coordonnées en trois dimensions est considéré dans la Section 1.3.5. La condition (1.2.60) veille à ce que les coordonnées ℓ_i sont réels.

La lagrangien pleine $\mathcal{L}$ entrant dans le action invariante off-shell $\mathcal{N} = 4$ $S = \int dt \mathcal{L}$ se compose des trois pièces [9]

$$\begin{aligned} \mathcal{L} = \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm int} + \mathcal{L}_{\rm FI} = & \int du \, d^4\theta \, R_{\rm kin}(L^{++}, L^{+-}, L^{--}, u) \\ & - \frac{1}{2} \int du \, d\bar{\theta}^+ d\theta^+ \, K(L^{++}, u) v^+ \widetilde{v^+} - \frac{ik}{2} \int du \, d\bar{\theta}^+ d\theta^+ \, V^{++}, \end{aligned} \tag{1.3.74}$$

où $L^{+-} = \frac{1}{2} D^{--} L^{++}$ et $L^{--} = D^{--} L^{+-}$. Les fonctions de superchamps $R_{\rm kin}$ et K supporter une dépendance arbitraire sur leurs arguments. Le sens de trois termes en 1.3.74 est expliqué ci-dessous.

De superespace harmonique à des composants

Le premier terme de type sigma-modèle dans Éq. 1.3.74, après intégration sur les variables de Grassmann et variables harmoniques, on obtient les termes cinétiques généralisées pour $\ell^{\alpha\beta}, \chi^\alpha, \bar{\chi}_\alpha$:

$$\begin{aligned} \mathcal{L}_{\rm kin} = & \frac{1}{8} f^{-2} \left(-2 \dot{\ell}_{\alpha\beta} \dot{\ell}^{\alpha\beta} + F^2 \right) + \frac{i}{2} f^{-2} \left(\bar{\chi}_\alpha \dot{\chi}^\alpha - \dot{\bar{\chi}}_\alpha \chi^\alpha \right) + \frac{1}{4} \left(\partial_{\alpha\beta} \partial^{\alpha\beta} f^{-2} \right) \chi^4 \\ & + \frac{i}{f^3} \dot{\ell}^{\alpha\beta} \Big\{ \partial_{\alpha\gamma} f \chi_\beta \bar{\chi}^\gamma + \partial_{\beta\gamma} f \chi^\gamma \bar{\chi}_\alpha \Big\} - \frac{1}{f^3} F \chi^\alpha \bar{\chi}^\beta \partial_{\alpha\beta} f, \end{aligned} \tag{1.3.75}$$

[9] La formulation superchamp premier de MQS général $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ sans interaction avec champ de jauge a été donnée dans [15].

où $\chi^4 = \chi^\alpha \chi_\alpha \bar{\chi}^\beta \bar{\chi}_\beta$, $\partial_{\alpha\beta} \equiv \frac{\partial}{\partial \ell^{\alpha\beta}}$ et $f(\ell)$ est un facteur conformel. Les calculs sont plus facilement réalisée dans la base centrale, où $L^{++} = u^+_\alpha u^+_\beta L^{\alpha\beta}\left(t, \theta_\gamma, \bar{\theta}^\delta\right)$. Then

$$f^{-2}(\ell) = -\partial_{\alpha\beta}\partial^{\alpha\beta} \int R_{\mathrm{kin}}\left(\ell^{\alpha\beta}u^+_\alpha u^+_\beta, \ell^{\alpha\beta}u^+_\alpha u^-_\beta, \ell^{\alpha\beta}u^-_\alpha u^-_\beta\right) du.$$

Le terme fermionique cinétique peut être portée à la forme canonique par le changement de variables

$$\chi^\alpha = f\psi^\alpha, \qquad \bar{\chi}_\alpha = f\bar{\psi}_\alpha. \tag{1.3.76}$$

Il est intéressant de souligner que la groupe symétrie R SU(2) donne au groupe de rotation SO(3) dans l'espace cible $\mathbb{R}^3$ paramétrée par ℓ^i de l'équation (1.3.73). Le facteur conformel $f(\ell)$ peut avoir une dépendance arbitraire sur $\ell^{\alpha\beta}$, donc cette groupe SO(3) peut être totalement rompu dans le lagrangien 1.3.75.

La deuxième pièce dans Éq. (1.3.74) décrit le couplage à un champ de jauge non abélien externe. Exécution de l'intégration sur θ^+, $\bar{\theta}^+$ et $u^\pm_\alpha$, éliminer les champs auxiliaires fermioniques $\omega_{1,2}$ et, finalement, mise à l'échelle des variables bosoniques doublet que $\varphi_\alpha = \phi_\alpha\sqrt{h(\ell)}$, où

$$h(\ell) = \int du\, K\left(\ell^{\alpha\beta}u^+_\alpha u^+_\beta, u^\pm_\gamma\right), \tag{1.3.77}$$

après un peu d'algèbre, nous obtenons

$$\mathcal{L}_{\mathrm{int}} = i\bar{\varphi}^\alpha\left(\dot{\varphi}_\alpha + iB\varphi_\alpha\right) + \bar{\varphi}^\gamma\varphi^\delta\frac{1}{2}\left(\mathcal{A}_{\alpha\beta}\right)_{\gamma\delta}\dot{\ell}^{\alpha\beta} - \frac{1}{2}F\,\bar{\varphi}^\gamma\varphi^\delta\,U_{\gamma\delta} + \chi^\alpha\bar{\chi}^\beta\bar{\varphi}^\gamma\varphi^\delta\nabla_{\alpha\beta}U_{\gamma\delta}. \tag{1.3.78}$$

Ici, le champ de jauge non abélien et le potentiel scalaire (matrice) sont entièrement spécifié par la fonction h :

$$\left(\mathcal{A}_{\alpha\beta}\right)_{\gamma\delta} = \frac{i}{2h}\Big\{\varepsilon_{\gamma\beta}\partial_{\alpha\delta}h + \varepsilon_{\gamma\alpha}\partial_{\beta\delta}h + \varepsilon_{\delta\beta}\partial_{\alpha\gamma}h + \varepsilon_{\delta\alpha}\partial_{\beta\gamma}h\Big\}, \qquad U_{\gamma\delta} = \frac{1}{h}\partial_{\gamma\delta}h\,. \tag{1.3.79}$$

Par sa définition, la fonction h obéit à l'équation de Laplace en trois dimensions,

$$\partial^{\alpha\beta}\partial_{\alpha\beta}\,h = 0\,. \tag{1.3.80}$$

En utilisant les expressions explicites 1.3.79, il est facile de vérifier la relation

$$\left(\mathcal{F}_{\alpha\beta}\right)_{\gamma\delta} = 2i\nabla_{\alpha\beta}U_{\gamma\delta}, \tag{1.3.81}$$

où

$$\left(\mathcal{F}_{\alpha\beta}\right)_{\gamma\delta} = -2\partial^{\ \lambda}_\alpha\left(\mathcal{A}_{\lambda\beta}\right)_{\gamma\delta} + i\left(\mathcal{A}_\alpha^{\ \lambda}\right)_{\gamma\sigma}\left(\mathcal{A}_{\lambda\beta}\right)_\delta^{\ \sigma} + (\alpha \leftrightarrow \beta)\,, \tag{1.3.82}$$

$$\nabla_{\alpha\beta}U_{\gamma\delta} = -2\partial_{\alpha\beta}U_{\gamma\delta} + i\left(\mathcal{A}_{\alpha\beta}\right)_{\gamma\lambda}U_\delta^{\ \lambda} + i\left(\mathcal{A}_{\alpha\beta}\right)_{\delta\lambda}U_\gamma^{\ \lambda}, \tag{1.3.83}$$

et $\left(\mathcal{F}_{\alpha\beta}\right)_{\gamma\delta}$ est liée à l'intensité standard du champ de jauge dans la notation vectorielle, voir ci-dessous. Comme nous le verrons bientôt, la condition 1.3.81 n'est autre que la forme statique de la condition générale auto-dual pour le champ SU(2) de Yang-Mills dans $\mathbb{R}^4$ (voir Éq. (1.3.90)), à savoir les équations Bogomolny pour BPS monopôles [22], tandis que 1.3.79 fournit une solution particulière à ces équations, étant une forme statique de 'ansatz de 't Hooft [16].

Notez que la relation 1.3.81 est covariante et le lagrangien 1.3.78 est la forme invariante sous la transformations jauge SU(2) de "espace cible" écrite dans Éq. (1.3.36). Ce n'est pas une symétrie véritable ; plutôt, il s'agit d'une reparamétrisation du lagrangien qui permet de jeter les potentiels de fond 1.3.79 dans certaines formes différentes équivalents. Il faut noter que les indices de groupe de jauge coïncident avec ceux du groupe symétrie R, comme dans le cas en quatre dimensions. Néanmoins, reparamétrisations "jauge" 1.3.36 n'affectent pas les indices doublet des coordonnées d'espace cible $\ell^{\alpha\beta}$ et leurs superpartenaires qui présents dans le superchamp L^{++}. Ils agissent uniquement sur les variables de spin demi-dynamiques $\varphi_\alpha, \bar{\varphi}^\alpha$ sur les potentiels jauge et scalaires 1.3.79.

Finalement, le terme dernier de Éq. (1.3.74) donne le terme de Fayet-Iliopoulos,

$$\mathcal{L}_{\text{FI}} = kB\,. \tag{1.3.84}$$

Dans le cas quantique, le coefficient k est quantifiée, $k \in \mathbb{Z}$, sur les mêmes motifs que dans le cas de quatre dimensions.

Notations vectorielles en trois dimensions

Il est instructif de réécrire les relations et les expressions ci-dessus, y compris la lagrangien pleine (1.3.74), dans les notations vectorielles. Pour ce faire, nous passer de $\ell^{\alpha\beta}$ à ℓ_i comme dans Éq. (1.3.73) et associer le vecteur $\mathcal{A}_i$ aver le champ de jauge $\mathcal{A}_{\alpha\beta}$ (avec des indices de matrice supplémentaires qui ont été omis ici pour simplifier) par la règle

$$\mathcal{A}_\alpha^\beta = \mathcal{A}_i \left(\sigma_i\right)_\alpha^{\ \beta}, \qquad \mathcal{A}_i = \frac{1}{2}\mathcal{A}_\beta^\alpha \left(\sigma_i\right)_\alpha^{\ \beta}, \qquad i = 1, 2, 3. \tag{1.3.85}$$

On peut vérifier que les coordonnées ℓ_i sont réels tandis que la matrice $\left(\mathcal{A}_i\right)_\gamma^{\ \delta}$ est hermitienne. Notez également la relation entre les dérivées partielles $\partial_{\alpha\beta} = \partial/\partial\ell^{\alpha\beta}$ et $\partial_i = \partial/\partial\ell_i$:

$$\partial_{\alpha\beta} = -\frac{1}{2}\left(\sigma_i\right)_{\alpha\beta}\partial_i, \qquad \partial_i = -\left(\sigma_i\right)_\alpha^{\ \beta}\partial_\beta^\alpha\,. \tag{1.3.86}$$

Nous faisons également une conversion similaire des indices des groupes de jauge,

$$M_\gamma^{\ \delta} = \frac{1}{2}M^a \left(\sigma_a\right)_\gamma^{\ \delta}, \qquad M^a = M_\delta^{\ \gamma}\left(\sigma_a\right)_\gamma^{\ \delta}, \qquad a = 1, 2, 3\,, \tag{1.3.87}$$

pour toute matrice M hermitienne 2×2 de trace nulle.

Dans les nouvelles notations, le lagrangien total 1.3.74 prend la forme suivante :

$$\begin{aligned}\mathcal{L} = \frac{1}{2}f^{-2}\dot{\ell}_i^2 + \mathcal{A}_i^a T^a \dot{\ell}_i + i\bar{\varphi}^\alpha\left(\dot{\varphi}_\alpha + iB\varphi_\alpha\right) + kB + i\bar{\psi}_\alpha\dot{\psi}^\alpha + f^2\nabla_i U^a T^a\,\psi\sigma_i\bar{\psi} \\ + \frac{1}{4}\left\{f\partial_i^2 f - 3\left(\partial_i f\right)^2\right\}\psi^4 + 2f^{-1}\varepsilon_{ijk}\partial_i f\,\dot{\ell}_j\,\psi\sigma_k\bar{\psi} \\ + \frac{1}{8}f^{-2}F^2 + \frac{1}{2}F\left(U^a T^a - f^{-1}\partial_i f\,\psi\sigma_i\bar{\psi}\right).\end{aligned} \tag{1.3.88}$$

où T^a définie dans Éq. (1.3.35). Ici

$$\nabla_i U^a = \partial_i U^a + \varepsilon^{abc}\mathcal{A}_i^b U^c \tag{1.3.89}$$

et les équations de Bogomolny 1.3.81 qui se rapportent $\mathcal{A}_i^a$ et U^a sont de manière équivalente réécrite sous la forme plus familière,

$$\mathcal{F}_{ij}^a = \varepsilon_{ijk} \nabla_k U^a \,, \tag{1.3.90}$$

où $\mathcal{F}_{ij}^a = \partial_i \mathcal{A}_j^a - \partial_j \mathcal{A}_i^a + \varepsilon^{abc} \mathcal{A}_i^b \mathcal{A}_j^c$. Finalement, le champ de jauge et le potentiel matrice définie dans 1.3.79 sont réécrits comme

$$\mathcal{A}_i^a = -\varepsilon_{ija} \partial_j \ln h, \qquad U^a = -\partial_a \ln h \,, \qquad \Delta\, h = 0 \,. \tag{1.3.91}$$

Après avoir éliminé le champ auxiliaire F par son équation du mouvement,

$$F = 2 f^2 \left(f^{-1} \partial_i f \, \psi \sigma_i \bar{\psi} - U^a T^a \right) \,, \tag{1.3.92}$$

le lagrangien 1.3.88 prend la forme

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} f^{-2} \dot{\ell}_i^2 + \mathcal{A}_i^a T^a \dot{\ell}_i + i \bar{\varphi}^\alpha \left(\dot{\varphi}_\alpha + i B \varphi_\alpha \right) + k B + i \bar{\psi}_\alpha \dot{\psi}^\alpha + f^2 \psi \sigma_i \bar{\psi} \left(\nabla_i + f^{-1} \partial_i f \right) U^a T^a \\ & + \frac{1}{4} \left\{ f \partial_i^2 f - 4 \left(\partial_i f \right)^2 \right\} \psi^4 + 2 f^{-1} \varepsilon_{ijk} \partial_i f \, \dot{\ell}_j \, \psi \sigma_k \bar{\psi} - \frac{1}{2} f^2 (U^a T^a)^2 \,. \end{aligned} \tag{1.3.93}$$

Nous voyons que ce lagrangien implique trois champs physiques bosoniques ℓ_i et quatre champs physiques fermioniques ψ_α . Il est entièrement spécifié par deux fonctions indépendantes : le facteur métrique conformel $f(\ell)$ qui peut supporter une dépendance arbitraire sur ℓ_i et la fonction $h(\ell)$ qui satisfait l'équation de Laplace en trois dimensions et détermine les potentiels jauge non abélien et scalaire. La représentation 1.3.77 pour h en termes de la fonction analytique $K(\ell^{++}, u)$ donne en fait une solution générale de l'équation de Laplace en trois dimensions [54]. Si on prend la fonction $h(\ell)$ à disparaître au $|\vec{\ell}| \to \infty$, cette fonction peut être présentée comme la somme sur monopôles suivante :

$$h(\ell) = 1 + \sum_M \frac{c_M}{\left| \vec{\ell} - \vec{b}_M \right|} . \tag{1.3.94}$$

Il s'agit des positions particulières de monopôles $\vec{b}_M$ ainsi que les numéros c_M associé à chaque monopôle.

Le lagrangien 1.3.93 contient également des variables de spin "semi-dynamiques" $\varphi_\alpha, \bar{\varphi}^\alpha$, dont le rôle est le même que dans le cas en quatre dimensions : après quantification ils assurent que T^a défini par (1.3.35) devenir la matrice générateurs SU(2) correspondant à la représentation de spin $|k|/2$.

Hamiltonien et supercharges

Le lagrangien 1.3.93 est le point de départ pour la mise en place de la formulation hamiltonienne du modèle à l'étude et la quantification de la dernière. Après substitution de générateurs spin-$k/2$ SU(2) au lieu de T^a, l'hamiltonien quantique de ce système prend la forme

$$\begin{aligned} H = & \frac{1}{2} f \left(\hat{p}_i - \mathcal{A}_i \right)^2 f + \frac{1}{2} f^2 U^2 - f^2 \nabla_i U \psi \sigma_i \bar{\psi} \\ & + \left\{ \varepsilon_{ijk} f \partial_i f \left(\hat{p}_j - \mathcal{A}_j \right) - f \partial_k f U \right\} \psi \sigma_k \bar{\psi} + f \partial^2 f \left\{ \psi^\gamma \bar{\psi}_\gamma - \frac{1}{2} \left(\psi^\gamma \bar{\psi}_\gamma \right)^2 \right\} , \end{aligned} \tag{1.3.95}$$

qui est juste un réduction statique en trois dimensions de l'hamiltonien en quatre dimensions donné par Éq. (1.3.15). Dans cette expression, le champ de jauge $\mathcal{A}_i = \mathcal{A}_i^a T^a$ et le potentiel scalaire $U = U^a T^a$ sont matrices SU(2) soumis à la contrainte (1.3.90). Il est également facile de trouver les supercharges $Q_\alpha, \bar{Q}^\beta$:

$$\begin{aligned} Q_\alpha &= f\left(\sigma_i \bar{\psi}\right)_\alpha (\hat{p}_i - \mathcal{A}_i) - \psi^\gamma \bar{\psi}_\gamma \left(\sigma_i \bar{\psi}\right)_\alpha i\partial_i f - ifU\bar{\psi}_\alpha, \\ \bar{Q}^\alpha &= (\psi\sigma_i)^\alpha (\hat{p}_i - \mathcal{A}_i) f + i\partial_i f\,(\psi\sigma_i)^\alpha \psi^\gamma \bar{\psi}_\gamma + ifU\psi^\alpha, \end{aligned} \tag{1.3.96}$$

L'ambiguïté de commande résultant dans le cas du facteur conformel générale $f(\ell)$ peut être fixé, comme d'habitude, par la procédure de commande de Weyl [41].

Laissez-nous mettre l'accent que la seule condition requise dans les champs de la matrice $\mathcal{A}_i$ et U pour les générateurs Q_α et $\bar{Q}^\beta$ pour former le superalgebra $\mathcal{N} = 4$ 1.2.25 que ces champs satisfont aux équations de Bogomolny 1.3.90. Ainsi, les expressions 1.3.95 et 1.3.96 définissent le modèle MQS $\mathcal{N} = 4$ dans le champ *arbitraire* de BPS monopôle, pas nécessairement limitée à la ansatz 1.3.91. En outre, on peut étendre le groupe de jauge SU(2) à SU(N) dans 1.3.95 et 1.3.96.

Notons que l'hamiltonien (1.3.95) en trois dimensions et les supercharges (1.3.96) ont été considérés pour la première fois en Réf. [12] (dans le cas abélien).

Supersymétrie $\mathcal{N} = 4$ avec monopôle de Wu-Yang

finalement, comme un exemple simple de champ de monopôle compatible avec supersymétrie $\mathcal{N} = 4$ off-shell et on-shell, laissez-nous envisager un particulier le cas à symétrie sphérique en trois dimensions. Il correspond à la solution SO(3) invariante plus générale de l'équation de Laplace de la fonction h,

$$h_{\mathrm{so}(3)}(\ell) = c_0 + c_1 \frac{1}{\sqrt{\ell^2}} \,. \tag{1.3.97}$$

Les potentiels correspondants calculés selon les équations 1.3.91 donnent

$$\mathcal{A}_i^a = \varepsilon_{ija} \frac{\ell_j}{\ell^2} \frac{c_1}{c_1 + c_0\sqrt{\ell^2}}, \qquad U^a = \frac{\ell_a}{\ell^2} \frac{c_1}{c_1 + c_0\sqrt{\ell^2}}. \tag{1.3.98}$$

Cette configuration devient le monopôle de Wu-Yang [24] pour le choix $c_0 = 0$. Il est facile de trouver la fonction analytique $K(\ell^{++}, u)$ qui génère la solution 1.3.97 (voir [6]) :

$$\begin{gathered} h_{\mathrm{so}(3)}(\ell) = \int du\, K_{\mathrm{so}(3)}(\ell^{++}, u)\,, \quad K_{so(3)}(\ell^{++}, u) = c_0 + c_1 \left(1 + a^{--}\hat{\ell}^{++}\right)^{-\frac{3}{2}}\,, \\ \ell^{++} \equiv \hat{\ell}^{++} + a^{++}\,, \quad a^{\pm\pm} = a^{\alpha\beta} u_\alpha^\pm u_\beta^\pm\,, \quad a^\alpha_\beta a^\beta_\alpha = 2\,. \end{gathered} \tag{1.3.99}$$

On pourrait également choisir comme $h(\ell)$, par exemple, le bien-connu solution multicentre de l'équation de Laplace, avec SO(3) cassé. Notez que la mécanique $\mathcal{N} = 4$ avec couplage à monopôle de Wu-Yang a été récemment construit dans [21], de procéder à partir d'une approche différente, avec l'invariance SO(3) intégré et le traitement des variables de spin dans l'esprit de la référence [18]. Notre examen général montre, en particulier, que la demande de symétrie SO(3) n'est pas nécessaire pour l'existence de modèles MQS $\mathcal{N} = 4$ avec champs de jauge de monopôle non abélien.

Relation avec le système MQS $\mathcal{N}=4$ en quatre dimensions

Il est instructif de montrer que 1.3.91 peut en effet être considéré comme une réduction 3-dimensionnelle de l'ansatz de 't Hooft pour la solution de l'équation général auto-dual en $\mathbb{R}^4$ pour le grouppe de jauge SU(2), avec l'identification $U^a = \mathcal{A}_0^a$, tandis que la condition 1.3.90 est la réduction de cette équation en trois dimensions.

Pour établir cette relation, nous utilisons les règles suivantes de la correspondance entre le formalisme spinoriel SO(4) = SU(2) × SU(2) et son réduction SU(2) :

$$\begin{aligned} &(\sigma_\mu)_{\alpha\dot\beta} \to \left\{ i\delta_\alpha^\beta, (\sigma_i)_\alpha{}^\beta \right\}, \\ &\varepsilon^{\dot\alpha\dot\beta} \to -\varepsilon_{\alpha\beta}, \qquad \varepsilon_{\dot\alpha\dot\beta} \to -\varepsilon^{\alpha\beta}, \\ &x_{\alpha\dot\beta} \to \ell_\alpha^\beta, \qquad x^{\alpha\dot\beta} \to -\ell_\beta^\alpha, \\ &\psi_{\dot\alpha} \to \psi^\alpha. \end{aligned} \tag{1.3.100}$$

Cela reflète le fait que le symétrie R SU(2) dans les modèles $(\mathbf{3},\mathbf{4},\mathbf{1})$ peut être traité comme un sous-groupe diagonale dans la groupe de symétrie SO(4) = SU(2) × SU(2) de modèles $(\mathbf{4},\mathbf{4},\mathbf{0})$, avec les facteurs SU(2) agissant, respectivement, sur les indices non-pointillés et pointillés.

Le champ de jauge SU(2) auto-dual en $\mathbb{R}^4$ dans l'ansatz de 't Hooft est écrit dans la notation spinorielle dans Éq. (1.3.58). Puis, en utilisant les règles 1.3.100, on effectue la réduction $\mathbb{R}^4 \to \mathbb{R}^3$ comme

$$\begin{aligned} &(\mathcal{A}_{\alpha\dot\beta})_\gamma{}^\delta \;\to\; iU_\gamma{}^\delta\delta_\alpha^\beta + (\mathcal{A}_\alpha^\beta)_\gamma{}^\delta, \qquad (\mathcal{A}_\alpha^\alpha)_\gamma{}^\delta = 0, \\ &h(x) \;\to\; h(\ell), \qquad \partial_\beta^\alpha\partial_\alpha^\beta\, h = 0. \end{aligned} \tag{1.3.101}$$

Sur cette réduction, l'ansatz 1.3.58 en quatre dimensions donne précisément 1.3.79, tandis que le condition général auto-dual (1.3.60) devient les équations de Bogomolny 1.3.81. Bien sûr, la même réduction peut être effectuée dans la notation vectorielle, avec $\mathcal{F}_{\mu\nu} \to \{\mathcal{F}_{ij}, \mathcal{F}_{0k} = \nabla_k U\}$, et Éqs. 1.3.90, 1.3.91 comme le résultat.

Ainsi, le champ de jauge général prescrite par supersymétrie off-shell $\mathcal{N}=4$ dans ce système $(\mathbf{3},\mathbf{4},\mathbf{1})$ est une forme statique de l'ansatz de 't Hooft pour un champ de jauge SU(2) auto-dual en $\mathbb{R}^4$. Ceci suggère que la réduction de l'espace cible bosonique dessusa sa contrepartie superchamp concernant le système en quatre dimensions décrites dans la Section 1.3.4 à celui qui est envisagé ici.

Effectivement, l'action de superchamp $(\mathbf{3},\mathbf{4},\mathbf{1})$ 1.3.74 peut être obtenue à partir de l'action de multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ composé les Éqs. (1.3.21), (1.3.52), (1.3.53) par “la dualité automorphe” [55] en considérant un classe restreint des actions $(\mathbf{4},\mathbf{4},\mathbf{0})$ avec U(1) isométrie et effectuer une jaugeage superchamp de cette isométrie par un superchamp de jauge supplémentaire $V^{++\prime}$ selon ligne générale de Réf. [56]. En fait, la réduction de l'espace cible bosonique qui nous venons de décrire correspond à l'isométrie décalage de superchamp analytique $q^{+\dot\alpha}$ recevoir le multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$, à savoir, par $q^{+\dot\alpha} \to q^{+\dot\alpha} + \omega u^{+\dot\alpha}$. Il est la projection invariante $q^{+\dot\alpha}u^+_{\dot\alpha}$ qui va devenir le superchamp $(\mathbf{3},\mathbf{4},\mathbf{1})$ L^{++} sur jauger cette isométrie et par le choix de la jauge appropriée que est manifestement supersymétrique $\mathcal{N}=4$.

Un impact important de cette réduction superchamp sur la structure de l'action composante est l'apparition du nouveau potentiel induit qui est bilinéaire dans les générateurs de groupe de jauge $\sim U^2 = U^a U^b T^a T^b$. Il sort à la suite de l'élimination du champ auxiliaire F en multiplet off-shell $(\mathbf{3}, \mathbf{4}, \mathbf{1})$, et ainsi est nécessairement prescrit par la supersymétrie $\mathcal{N} = 4$. Il est intéressant que les termes potentiels analogues ont été introduits dans [57] au niveau bosonique de veiller à l'existence de certains symétries cachée dans les modèles de particule en trois dimensions dans un contexte de monopôle non abélien.

La même réduction $\mathbb{R}^4 \to \mathbb{R}^3$ peut être effectuée au niveau de hamiltonien et supercharges. En particulier, la réduction de l'hamiltonien du système en quatre dimensions de Éq. (1.3.15) donne l'hamiltonien en trois dimensions (1.3.95).

1.4 Conclusion

Nous avons étudié un couplage assez générale off-shell supersymétrique $\mathcal{N} = 4$ de $d = 1$ de supermultiplets des coordonnés $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ et $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ avec une champ de jauge externe abélien auto-dual (ou anti-auto-dual) et nous avons discuté le cas de $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ en détails. L'approche superespace harmonique est le cadre principal de l'etude.

L'utilisation d'un multiplet analytique "semi-dynamiques" $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ avec le type d'action de Wess-Zumino nous a permis de faire le couplage des multiplets de coordonner $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ et $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ avec une champ de jauge SU(2) externe. Ce multiplet auxiliaire intègre le doublet de variables spinorielles bosoniques SU(2) qui sont cruciaux pour l'organisation de couplage aux champs de jauge non abéliens. Dans le cas de quatre dimensions, la supersymétrie off-shell $\mathcal{N} = 4$ restreint le champ de jauge non abélien d'être auto-dual (ou anti-auto-dual) et dans une forme de l'ansatz de 't Hooft pour le champ de jauge SU(2). Dans le cas de trois dimensions, le champ de jauge non-abélien est une réduction en trois dimensions de cette ansatz de 't Hooft, c'est à dire une solution particulière des équations de monopôle de Bogomolny. De plus, en trois dimensions, au niveau des composants, le couplage d'un champ de jauge est nécessairement accompagné par un potentiel induit qui est bilinéaire en générateurs SU(2) et survient à la suite de l'élimination du champ auxiliaire du multiplet des coordonnés $(\mathbf{3}, \mathbf{4}, \mathbf{1})$.

La forme explicite des hamiltoniens et des supercharges a été présentés. Les expressions correspondantes satisfont la supersymétrie *on-shell* $\mathcal{N} = 4$ pour tout champ de jauge auto-dual ou anti-auto-dual, abélien ou non abélien en quatre dimensions, pas nécessairement dans l'ansatz de forme de 't Hooft. En trois dimensions, un champ de fond arbitraires de monopôle BPS peut être utilisé dans le cas non abélien.

Il est intéressant de noter que des contraintes similaires (équations Bogomolny) sur le champ de jauge extérieur non abélien en trois dimensions ont été trouves dans [58], tout en considérant l'extension $\mathcal{N} = 4$ de phase de Berry en mécanique quantique. Néanmoins, aucune action invariante et/ou les expressions explicites pour l'hamiltonien et supercharges $\mathcal{N} = 4$ n'ont pas été présentés dans cette référence.

La contrepartie non linéaire de multiplet $q^{+\dot{\alpha}}$ est discuté dans [46]. Dans ce cas, la géométrie de cible bosonique est plus général par rapport à la géométrie conforme plate associée à multiplet linéaire $(\mathbf{4}, \mathbf{4}, \mathbf{0})$.

Parmi les orientations possibles de la suite d'étude plus loin, nous mentionnons la construction de modèles SQM avec $\mathcal{N}$ supérieurs avec des champs de jauges non-abéliens, par exemple $\mathcal{N} = 8$, ainsi que l'étude des réductions différentes supersymétriques de ces modèles à variétés de dimensions basses de cibles bosoniques par la procédure de calibrage [56]. En fait, la méthode de multiplet auxiliaire semi-dynamique $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ avec l'action de type de Wess-Zumino, qui a été appliquée avec succès dans notre construction, pourrait être utilisé avec l'efficacité égale pour construire une description lagrangienne d'autres problèmes supersymétriques de la mécanique quantique, qui implique le couplage d'un champ de jauge non abélien externe. Hors des exemples évidents de l'effet Hall quantique (ou un problème de Landau) dans les dimensions supérieures (voir par exemple la discussion dans [8]), laissez-nous aussi mentionner la boucle de Wilson fonctionnelle supersymétrique qui peut être interprétée en termes d'une version non abélienne de Chern-

Simons en mécanique (super)quantique [59], avec le paramètre le long de la boucle comme un paramètre d'évolution. Nous espérons que des variables semi-dynamiques quantifiées pourrait fournir un nouvel outil efficace pour étudier cette classe des problèmes.

L'ansatz de type de 't Hooft 1.3.38 et le choix de SU(2) comme le groupe de jauge sont nécessaires pour l'existence de formulation superchamp *off-shell* des systèmes de SQM discuté. On ne sait pas si le système le plus général peut être dérivé d'un formalisme superchamp off-shell, avec des horizons général instanton/monopôle obtenus à partir de la construction ADHM [60] ou de se réduction en trois dimensions. De plus, il ne reste q'un problème de l'extension des modèles d'un groupe de jauge SU(N) générique. Peut-être, les questions ci-dessus sont liés à la généralisation du terme d'interaction (1.3.52) comme

$$S_{\text{int}} = \int dt\, du\, d\bar{\theta}^{+} d\theta^{+}\, K^{++}\left(q^{+\dot{\alpha}},\, u^{\pm}_{\beta},\, v^{+}\widetilde{v^{+}}\right).$$

Il serait également intéressant d'étudier les modèles de SQM avec le multiplet semi-dynamique non linéaire $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ [56].

Chapitre 2 Introduction

Supersymmetric quantum mechanics (SQM) provides a proper venue for exploring and modeling salient features of supersymmetric field theories in diverse dimensions [1]. Some SQM models represent one-dimensional reductions of higher-dimensional supersymmetric theories. At the same time, many interesting models of this kind can be constructed directly in (0+1) dimensions, without any reference to the dimensional reduction procedure. They exhibit some surprising properties related to peculiarities of one-dimensional supersymmetry. For any SQM model (like for any supersymmetric field theory), it is desirable, besides the component Hamiltonian and Lagrangian description, to have the appropriate superfield Lagrangian formulation. The latter makes supersymmetry manifest, prompts possible generalizations of the model and allows one to reveal relationships with other cognate theories.

This study is devoted to the Hamiltonian and the Lagrangian as well as the superfield Lagrangian formulation for a certain class of $\mathcal{N} = 4$ SQM models [1] with self-dual or anti-self-dual Abelian or non-Abelian gauge field backgrounds [2, 3, 4]. Surprisingly, such systems did not attracted much attention so far. A natural framework for this formulation proves to be the harmonic superspace (HSS) approach [5] adapted to the one-dimensional case [6].

The models of supersymmetric quantum mechanics with background gauge fields are of obvious interest for several reasons. One of them is a close relation of these systems to the Landau problem (motion of a charged particle in an external magnetic field) and its generalizations (see e.g. [7]). The Landau-type models constitute a basis of the theoretical description of quantum Hall effect (QHE), and it is natural to expect that their supersymmetric extensions, with extra fermionic variables added, may be relevant to spin versions of QHE. Also, these systems can provide quantum-mechanical realizations of various Hopf maps closely related to higher-dimensional QHE (see e.g. [8] and references therein).

The first type of SQM models considered in this work represents a subclass of well-known systems which describe the motion of a fermion on an even-dimensional manifold with an arbitrary gauge background. It was observed many years ago that one can treat these systems as supersymmetric ones such that, e.g., the Atiyah-Singer index of the massless Dirac operator $\not{\mathcal{D}}$ can be interpreted as the Witten index of a certain supersymmetric Hamiltonian [9]. The corresponding supercharges and the Hamiltonian are

$$Q = \not{\mathcal{D}}(1+\gamma_5), \qquad \bar{Q} = \not{\mathcal{D}}(1-\gamma_5), \qquad H = \not{\mathcal{D}}^2,$$

[1]Hereafter, in quantum mechanics, $\mathcal{N}$ counts the number of *real* supercharges.

where γ_5 is the appropriate "fifth gamma matrix" obeying $\gamma_5^2 = 1$ and anticommuting with the Dirac operator, $\{\gamma_5, \not{\mathcal{D}}\} = 0$. Indeed, for any eigenstate Ψ of the massless Dirac operator $\not{\mathcal{D}}$ with a nonzero eigenvalue λ, the state $\gamma^5\Psi$ is also an eigenstate of $\not{\mathcal{D}}$ with the eigenvalue $-\lambda$. Thus, all excited states of H are doubly degenerate.

It turns out that for a four-dimensional flat manifold and self-dual or anti-self-dual gauge field, Abelian or non-Abelian, the spectrum of H is 4-fold degenerate implying the extended $\mathcal{N} = 4$ supersymmetry. For a flat Dirac operator in the instanton background, this can be traced back to Ref. [10].

$\mathcal{N} = 4$ SQM models with the background Abelian gauge fields were treated in the pioneer papers [11, 12] and, more recently, e.g. in [13, 6, 14, 2]. In particular, in [6] an off-shell Lagrangian superfield formulation of the general models associated with the multiplets $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ and $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ was given in the $\mathcal{N} = 4$, $d = 1$ harmonic superspace [2]. It was found that $\mathcal{N} = 4$ supersymmetry requires the gauge field to be (anti)self-dual in the four-dimensional $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ case, or to obey a "static" version of the (anti)self-duality condition in the three-dimensional $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ case. In the papers [14, 2], it was observed (in a Hamiltonian approach) that the Abelian $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ $\mathcal{N} = 4$ SQM admits a simple generalization to arbitrary self-dual non-Abelian background. In [3], an off-shell Lagrangian formulation was shown to exist for a particular class of such non-Abelian $\mathcal{N} = 4$ SQM models, with SU(2) gauge group and 't Hooft ansatz [16] for the self-dual SU(2) gauge field (see also [17]). As in the Abelian case, it was the use of $\mathcal{N} = 4$, $d = 1$ harmonic superspace that allowed us to construct such an off-shell formulation. A new non-trivial feature of the construction of [3] is the involvement of an auxiliary "semi-dynamical" $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet with the Wess-Zumino type action possessing an extra gauged U(1) symmetry. After quantization, the corresponding bosonic fields become a sort of spin SU(2) variables to which the background gauge field naturally couples [3].

The second class of SQM models that we consider can be obtained at the component level by the Hamiltonian reduction of the systems discussed above from four to three dimensions. Their superfield description is nontrivial and consists in coupling the coordinate supermultiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ to an external non-Abelian gauge field through the introduction of the auxiliary $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ superfield. The off-shell $\mathcal{N} = 4$ supersymmetry restricts the external gauge field to be represented by a "static" version of the 't Hooft ansatz for four-dimensional (anti)self-dual SU(2) gauge fields, i.e. to a particular solution of the general monopole Bogomolny equations [22] [4]. A new feature of the three-dimensional case is the appearance of "induced" potential term in the action as a result of eliminating the auxiliary field of the coordinate $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ supermultiplet. This term is bilinear in the SU(2) gauge group generators. As a particular "spherically symmetric" case of the construction (with the exact SU(2) R-symmetry) we recover the $\mathcal{N} = 4$ mechanics with Wu-Yang monopole [24] (recently considered in [21] with an essentially different treatment of the spin variables).

The chapter 3 is devoted to the introduction to supersymmetry in four-dimensional

[2]The first superfield formulation of general $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ SQM (without background gauge field couplings) was given in [15].

[3]The use of such auxiliary bosonic variables for setting up coupling of a particle to Yang-Mills fields can be traced back to [18]. In the context of $\mathcal{N} = 4$ SQM, they were employed in [19, 20] and [8, 21].

[4]Some BPS monopole backgrounds in the framework of $\mathcal{N} = 2$ SQM were considered, e.g., in [23].

relativistic field theories. We discuss the motivation and the properties of supersymmetric theories as well as the their practical realization through the superfield approach. As an illustration, we consider the simplest example of the Wess-Zumino model – a complex scalar field coupled to a Weyl spinor field.

In chapter 4, we discuss supersymmetry in quantum mechanics. The ordinary superspace and harmonic superspace formalisms are given. In particular, the structure of the supermultiplets $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ and $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ is explained. Additionally, we introduce necessary notations which will be used in the chapter 5.

The chapter 5 presents the original results of this study. We give the component and the superfield description of the four-dimensional and the three-dimensional models discussed above. In particular, the Hamiltonians and the corresponding supercharges are written.

Finally, to facilitate the reading of the manuscript, the contents of the references [2, 3, 4] are given in the appendices A, B and C respectively.

Chapitre 3

Supersymmetric extension of Poincaré symmetry

This chapter is purely introductory and is devoted to supersymmetric field theories in four-dimensional Minkowski space.

We explain *what* is supersymmetry and why it is the only possible non-trivial extension of the Poincaré symmetry. We discuss main properties of any supersymmetric field theory and motivate *why* supersymmetric theories are interesting.

Finally, we show how to work with such theories and explain the superfield formalism. As an illustration, we consider a simplest possible supersymmetric example – the Wess-Zumino model which describes supersymmetric dynamics of a complex scalar field.

In a supersymmetric field theory, interactions between particles are fine-tuned in a special way, so that an additional continuous symmetry – *supersymmetry* – emerges. This symmetry mixes bosons and fermions (particles with different statistics) between each other.

Supersymmetry admits natural resolution of certain inconsistency problems of field theories. For instance, a vacuum in a field theory usually has infinite energy density. In a theory with unbroken supersymmetry, however, the energy of a vacuum is exactly zero. This subject is discussed in details in Section 3.6.4.

The infinite vacuum energy density in a field theory does not produce a problem by itself. Being coupled to gravity, however, such a theory becomes inconsistent. Thus, every known consistent field theory with gravity must be supersymmetric. In a similar manner, supersymmetry is included into every consistent string theory.

Another notable property of supersymmetric field theories is the equality in the number of bosonic and fermionic particles. It is not what we observe experimentally. This does not mean, however, that the idea of supersymmetry is altogether unreasonable. Indeed, supersymmetry may describe particle interactions at very small distances and very high energies and be broken at our energy scale. Assuming that it is the case, the extra predicted particles may acquire large masses, which explains the fact that they have not been observed so far. In addition, supersymmetric gauge theories provide the lightest supersymmetric particle as a natural candidate for dark matter.

All physical phenomena up to the TeV scale are well described by the Standard Model. Despite its success, however, it is conceivable that a new theory has to exist beyond the TeV scale. One reason is that we need a Higgs boson to break the electroweak symmetry. The radiative corrections to the mass of the Higgs boson are quadratically divergent and thus give an unacceptable large contribution if the cutoff scale is not of the TeV scale. This is called the naturalness problem. One way to address it is to consider a supersymmetric extension of the Standard Model. In the presence of supersymmetry, the mass of the Higgs boson is the same as the mass of its fermionic partner, while the fermion mass obtains only a logarithmic divergence due to the fact that an additional chiral symmetry appears in the absence of the mass term.

Of course, supersymmetry should be broken below the TeV scale to be able to describe our non-supersymmetric world. One can introduce supersymmetry breaking terms by hand. They should break supersymmetry softly in the sense that quadratic divergences should be absent. Alternatively, supersymmetry can be broken dynamically, so that the soft terms are generated in the a energy effective theory.

There is another aspect which makes supersymmetry attractive for a theorist. In particle physics, symmetries restrict particle dynamics and allow one to make theoretical predictions on kinematical grounds without actually doing any concrete dynamical calculation. The introduction of the extra symmetry on top of the Poincaré symmetry imposes more constraints on the amplitudes in a theory and makes it more accessible for theoretical studies. In it even believed that in some cases supersymmetry makes a theory exactly solvable. In fact, supersymmetry is a powerful instrument to study strong coupling dynamics and non-perturbative effects analytically.

In addition, theoretically appealing property of supersymmetry is that it offers the only "loophole" to the Coleman-Mandula theorem (see Section 3.4) which prohibits any nontrivial extension of the Poincaré symmetry in a field theory.

It is possible to make a supersymmetric theory from almost any field theory. There exist en effective *superfield* technique for this. It is discussed in details in this chapter.

In a certain way, supersymmetric extension of a theory can be compared with the extension of real numbers $\mathbb{R}$ to complex numbers $\mathbb{C}$. Indeed, analytical functions on the complex plane are much more constrained than functions of real argument. It is well known that an analytical function of a complex argument, given in some region of the complex plane, can be uniquely analytically continued to other regions or even to the whole complex plane. It it also well known that an analytical function can be reconstructed from the knowledge of its zeros and poles, which is not the case for functions of real argument.

These central properties of complex functions are of great importance in supersymmetric theories : the introduction of supersymmetry renders physical observables (e.g. amplitudes) depend on the parameters of the theory analytically. This allows one to calculate these quantities in one region of the theory (for example, in the region of the weak coupling, where the calculations can be carried out perturbatively) and then analytically continue the results to the strong coupling regime. As an example of such analysis, let us mention the paper by Seiberg and Witten [25] who studied the supersymmetric extension of quantum chromodynamics (QCD) and showed analytically that this theory is confining.

Supersymmetry, however, is reacher than just the complex analysis. Whereas the complex extension of real numbers is unique, several supersymmetric extensions of a quantum theory may be possible. A theory may have an ordinary or an extended supersymmetry. A quantum field theory in four space-time dimensions may have as much as four independent supersymmetries. The four-dimensional field theories with gravity may have as much as eight independent supersymmetries. The number of independent supersymmetries is usually counted by the $\mathcal{N}$ symbol, e.g. $\mathcal{N} = 2$ means that a field theory has two independent supersymmetries.

The more is the number of supersymmetries, the more a theory is constrained. This generally makes the theory more amenable for theoretical studies. The best known example is $\mathcal{N} = 4$ super-Yang-Mills theory − a theory similar to QCD, but extended to have four different supersymmetries. It is believed that this theory is exactly solvable. Still, this theory is very different from QCD, having zero β-function, no dimensional transmutation and no confinement. The simplest supersymmetric extension of QCD (with one supersymmetry) is still too complicated to be understood analytically. The $\mathcal{N} = 2$ super-Yang-Mills theory (which was studied by Seiberg and Witten) is intermediate between $\mathcal{N} = 1$ and $\mathcal{N} = 4$ theories.

Being invented as a form of mathematical construction, supersymmetry produced the deepest impact on theoretical physics over the last several decades and became an essential part of modern high-energy physics.

3.1 Basic notations in four-dimensional Minkowski space

We denote the coordinates in four-dimensional Minkowski space as x^μ with the Lorentz indices taken from the middle of the Greek alphabet,

$$\mu,\ \nu,\ \rho,\ \dots = 0, 1, 2, 3. \tag{3.1.1}$$

Thereby, the coordinate x^0 is associated with time. As for the three space components, x^i, we use the indices from the middle of the Latin alphabet,

$$i,\, j,\, k,\, \dots = 1, 2, 3. \tag{3.1.2}$$

Also, it is convenient to use the vectorial notation $\vec{x}$ for the space components of the four-vector x^μ.

The Minkowski space metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \mathrm{diag}\,(1, -1, -1, -1)\,. \tag{3.1.3}$$

As usual, it is used for raising and lowering Lorentz indices. For instance, one has for a tensor $A_{\mu\nu}$ with two Lorentz indices :

$$A^\mu{}_\nu = g^{\mu\rho} A_{\rho\nu}, \qquad A_{\mu\nu} = g_{\mu\rho} A^\rho{}_\nu, \tag{3.1.4}$$

where $g^{\mu\nu}$ is the inverse metric tensor, which is equal to $g_{\mu\nu}$. As usual, summing over the repeated indices is assumed, as in the formulas above.

Supersymmetry involves fermions. Consequently, spinors and spinor notations are extensively exploited in all the chapters. The spinorial indices are denoted with undotted and dotted Greek letters from the beginning of the alphabet :

$$\alpha, \beta = 1, 2 \qquad \text{and} \qquad \dot\alpha, \dot\beta = 1, 2. \tag{3.1.5}$$

Throughout this chapter the following four-dimensional matrices are used :

$$(\sigma^\mu)_{\alpha\dot\alpha} = \{1,\, \vec{\sigma}\}_{\alpha\dot\alpha}\,, \qquad (\bar\sigma^\mu)^{\dot\alpha\alpha} = \{1,\, -\vec{\sigma}\}^{\dot\alpha\alpha}\,, \tag{3.1.6}$$

where $\vec{\sigma}$ are ordinary Pauli matrices. Note that starting from the next chapter, where a quantum-mechanical formalism is involved, the Euclidean version of these matrices will be used, see Eq. (4.5.44).

3.2 Poincaré group and Poincaré algebra

The Poincaré group in Minkowski space parametrized by the coordinates x^μ can be realized by linear transformations

$$x'^\mu = \Lambda^\mu_\nu x^\nu + c^\mu \tag{3.2.1}$$

which preserve the space-time interval

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{3.2.2}$$

The subgroup of homogeneous transformations (i.e. those with parameters Λ^μ_ν) form the Lorentz group $\mathrm{O}(1,3)$. The invariance of ds^2 implies

$$g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma} \tag{3.2.3}$$

and, as a consequence, $\det \Lambda = \pm 1$. Here we skip the consideration of the discrete Poincaré transformations (i.e. space-time reflections) some of which are related to the $\det \Lambda = -1$ branch of solutions of Eq. (3.2.3). Instead, we take the proper subgroup in the Lorentz group with $\det \Lambda = 1$. The infinitesimal (infinitely small) Lorentz transformation can be written as

$$\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\nu}, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}, \tag{3.2.4}$$

where, as usual, $\omega_{\mu\nu} = g_{\mu\rho}\omega^{\rho}_{\nu}$. In this way, the infinitesimal form of the transformations (3.2.1) is given by

$$\delta x^{\mu} = -i\left[c^{\nu}\hat{P}_{\nu} + \frac{1}{2}\omega^{\nu\rho}\hat{M}_{\nu\rho}\right]x^{\mu}, \tag{3.2.5}$$

where the differential operators

$$\hat{P}_{\mu} = i\,\partial_{\mu}, \qquad \hat{M}_{\mu\nu} = -i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right) \tag{3.2.6}$$

are the generators of the Poincaré algebra. The infinitesimal form for the action of the Poincaré group on functions $f(x)$ on the Minkowski space is

$$f'(x) = f(x) - i\left[c^{\nu}\hat{P}_{\nu} + \frac{1}{2}\omega^{\nu\rho}\hat{M}_{\nu\rho}\right]f(x). \tag{3.2.7}$$

The Poincaré algebra generators – four translations P_{μ} and six space-time rotations $M_{\mu\nu}$ – form the Poincaré algebra [1]

$$\begin{aligned}
&[P_{\mu}\,,\,P_{\nu}] = 0,\\
&[M_{\mu\nu}\,,\,P_{\lambda}] = i\left(g_{\mu\lambda}\,P_{\nu} - g_{\nu\lambda}\,P_{\mu}\right),\\
&[M_{\mu\nu}\,,\,M_{\rho\sigma}] = i\left(g_{\mu\rho}\,M_{\nu\sigma} + g_{\nu\sigma}\,M_{\mu\rho} - g_{\nu\rho}\,M_{\mu\sigma} - g_{\mu\sigma}\,M_{\nu\rho}\right).
\end{aligned} \tag{3.2.8}$$

Thus, the Poincaré algebra in Minkowski space in four dimensions has 10 independent generators : four space-time shifts, three space rotations and three boosts (transformations to other inertial reference frames).

3.3 Two-component spinor notation

Supersymmetry unifies bosons and fermions and thus extensively uses the spinorial formalism. Here we recall the basic properties of this formalism in four-dimensional Minkowski space.

Four-dimensional spinors realize irreducible representation of the Lorentz group (which has six generators : three spatial rotations and three Lorentz boosts). There are two types of spinors : left-handed and right-handed, which are marked by undotted and dotted indices, respectively, in the following way :

$$\begin{array}{lll}
\text{left-handed :} & \xi_{\alpha}, & \alpha = 1,2,\\
\text{right-handed :} & \bar{\eta}_{\dot{\alpha}}, & \dot{\alpha} = 1,2.
\end{array} \tag{3.3.1}$$

[1] Note that here and below we omit "hats" on the operators P_{μ} and $M_{\mu\nu}$.

It is possible to lower and raise spinor indices with the invariant Levi-Civita tensor *from the left*. For instance,

$$\chi^\alpha = \varepsilon^{\alpha\beta}\chi_\beta, \qquad \chi_\alpha = \varepsilon_{\alpha\beta}\chi^\beta \tag{3.3.2}$$

and similar for spinors with dotted indices. The two-index antisymmetric Lorentz-invariant Levi-Civita tensors $\varepsilon^{\alpha\beta}$, $\varepsilon^{\dot\alpha\dot\beta}$, $\varepsilon_{\alpha\beta}$, and $\varepsilon_{\dot\alpha\dot\beta}$ are defined as

$$\begin{aligned} &\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}, \quad \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}, \quad \varepsilon_{12} = -\varepsilon^{12} = 1, \\ &\varepsilon^{\dot\alpha\dot\beta} = -\varepsilon^{\dot\beta\dot\alpha}, \quad \varepsilon_{\dot\alpha\dot\beta} = -\varepsilon_{\dot\beta\dot\alpha}, \quad \varepsilon_{\dot1\dot2} = -\varepsilon^{\dot1\dot2} = 1. \end{aligned} \tag{3.3.3}$$

The Lorentz transformation law for the undotted (left) spinors can be written as

$$\xi'_\alpha = U_\alpha^\beta \xi_\beta, \tag{3.3.4}$$

where the matrix U has the form

$$U = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right) \tag{3.3.5}$$

with $\omega_{\mu\nu}$ being the same as in Eqs. (3.2.4), (3.2.7). The matrices $\sigma^{\mu\nu}$ give a particular *matrix* realization of the Lorentz rotations $M^{\mu\nu}$ and satisfy the last line in Eqs. (3.2.8). To be more specific,

$$\sigma^{\mu\nu} = \frac{i}{4}\left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right) \tag{3.3.6}$$

with the matrices σ^μ and $\bar\sigma^\mu$ being introduced in Eq. (3.1.6).

Let us consider a spatial rotation. The matrix from Eq. (3.3.5) takes the following form :

$$U_{\rm rot} = \exp\left(-i\frac{\theta}{2}\vec n\vec\sigma\right), \qquad \theta\, n^i = \frac{1}{2}\varepsilon^{ijk}\omega^{jk}, \tag{3.3.7}$$

where ε^{ijk} is antisymmetric Levi-Civita tensor ($\varepsilon^{123} = 1$), θ is the rotation angle and $\vec n$ – the unit vector denoting the axis of rotation. Analogously for a Lorentz boost, the matrix from Eq. (3.3.5) has the form

$$U_{\rm boost} = \exp\left(\frac{\phi}{2}\vec n'\vec\sigma\right), \qquad \phi\, n^i = \omega^{oi}. \tag{3.3.8}$$

Here $\tanh\phi = v$, where v is the velocity in the units of speed of light of the first inertial reference frame with respect to the second, $\vec n'$ denotes the velocity direction. Note that in the case of the spatial rotation the matrix $U_{\rm rot}$ is unitary, $U^\dagger_{\rm rot}U_{\rm rot} = 1$, whereas in the case of the Lorentz boost the matrix $U_{\rm boost}$ is not. This reflects the fact that the Lorentz group is the noncompact O$(1,3)$ group rather than the compact O(4) group.

Dotted spinors transform as complex conjugates of undotted spinors :

$$\bar\eta_{\dot\alpha} \sim (\eta_\alpha)^*, \tag{3.3.9}$$

where the sign $\sim$ means "is transformed as". Therefore, for dotted spinors the Lorentz transformation goes with the complex conjugated matrix,

$$\bar\eta'_{\dot\alpha} = (U^*)_{\dot\alpha}^{\dot\beta}\,\bar\eta_{\dot\beta}, \tag{3.3.10}$$

where

$$U^* = \exp\left(\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right), \tag{3.3.11}$$

and the matrices

$$\bar{\sigma}^{\mu\nu} = \frac{i}{4}\left(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu}\right) \tag{3.3.12}$$

give another matrix realization of the Lorentz rotations $M^{\mu\nu}$ and satisfy the last equation in Eqs. (3.2.8).

Similarly for dotted spinors, one has for particular cases of spatial rotations and Lorentz boosts :

$$\bar{\eta}'^{\dot{\alpha}} = \begin{cases} (U_{\text{rot}})^{\dot{\alpha}}_{\dot{\beta}}\, \bar{\eta}^{\dot{\beta}}, & \text{for rotations,} \\ \left(U^{-1}_{\text{boost}}\right)^{\dot{\alpha}}_{\dot{\beta}}\, \bar{\eta}^{\dot{\beta}}, & \text{for boosts,} \end{cases} \tag{3.3.13}$$

where for convenience the index for the spinor $\bar{\eta}_{\dot{\alpha}}$ is raised with the antisymmetric Levi-Civita tensor. Note that under spatial rotations the undotted spinor ξ_α and the dotted spinor $\bar{\eta}^{\dot{\alpha}}$ transform under one and the same matrix U_{rot}.

The spinors ξ_α and $\bar{\eta}_{\dot{\alpha}}$ are referred to as *Weyl spinors.* In Minkowski space in four dimensions one undotted and one dotted Weyl spinor comprise one Dirac spinor (see, for example, the textbook [26] for a more detailed description).

In order to be Lorentz-invariant, an equation which involves spinors must have the same number of undotted and dotted indices on each side, otherwise the equation becomes invalid under a change of reference frame. One should also remember, however, that complex conjugation implies the interchange of dotted and undotted indices. For instance, the relation

$$(\xi_{\alpha\beta})^* = \bar{\eta}_{\dot{\alpha}\dot{\beta}} \tag{3.3.14}$$

is Lorentz-invariant.

Lorentz scalars can be built by convolution of either undotted or dotted spinor indices. For example, the products

$$\chi^\alpha \xi_\alpha \qquad \text{and} \qquad \bar{\psi}_{\dot{\beta}}\bar{\eta}^{\dot{\beta}} \tag{3.3.15}$$

are invariant under the Lorentz transformations.

3.4 The Coleman-Mandula (no-go) theorem

The Poincaré algebra (3.2.8) forms basis of geometric symmetries of a relativistic field theory. Other symmetries like flavour symmetry, isospin symmetry, *etc.* commute with the Poincaré group and are *internal* in the sense that they have nothing to do with Minkowski space. A natural question arises : is it possible to extend the Poincaré group with an additional symmetry which affects space-time coordinates ? It was believed for a long time that this is not possible. In 1967 Coleman and Mandula formulated a theorem which states that, in a dynamically nontrivial relativistic quantum field theories of space-time dimension $d \geq 3$ with interactions and with asymptotic states (particles), no geometric extension of the Poincaré group is possible [27]. In other words, besides already known conserved generators carrying Lorentz indices (the energy-momentum operator P_μ and the Lorentz transformations $M_{\mu\nu}$) no such new conserved charges (algebra generators) can appear. According to the theorem, the only allowed additional conserved charges must be Lorentz scalars, such as the electromagnetic charge. However, in 1970 Golfand and Likhtman found a loophole in this theorem [28] which, together with the Coleman-Mandula theorem, singles out supersymmetry as the only possible geometric extension of the Poincaré invariance in a relativistic field theory. A reason for which this statement is not valid in one and in two space-time dimensions will become clear shortly.

The essence of the proof of the Coleman-Mandula theorem is the following. Let us take an interacting field theory and consider two-particle scattering process. The energy and momentum conservation laws present in every Poincaré-invariant field theory. Particularly, in our case

$$p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu, \qquad \mu = 0, 1, 2, 3, \tag{3.4.1}$$

where p_1^μ, p_2^μ are the particle 4-momenta before the interaction while p_3^μ, p_4^μ are the particle 4-momenta after the interaction. The kinematic constraints above leave only one essential free parameter – the scattering angle θ, see Fig. 3.1. This angle cannot be determined on kinematical grounds and is defined by particular dynamics in the theory. Imagine now that there is an additional symmetry generator of space-time with some Lorentz indices. An additional exotic conservation law which have the same Lorentz indices corresponds to this symmetry and involves the particles 4-momenta. The presence of this conservation law would completely fix the scattering angle θ (or at most would leave only a discrete set of possible angles). Since the scattering amplitude is an analytic function of the angle, it then must vanish for all angles. In other words, the theory has trivial S-matrix, i.e. it is non-interacting.

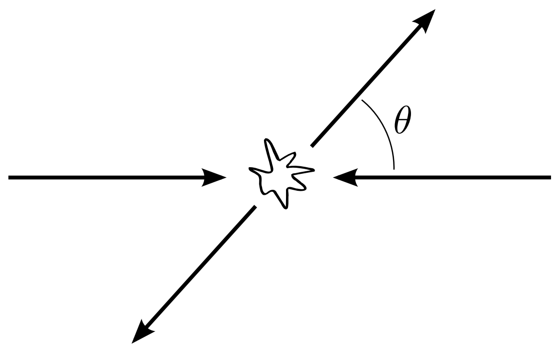

FIG. 3.1: Two-particle scattering. Only the scattering angle θ is undefined from energy and momentum conservation laws.

Consequently, the Coleman-Mandula theorem is not applicable in one and in two space-time dimensions, where there is no scattering angle between the two particles. The details of the proof and also its generalization to non-identical particles, particles with spin, *etc.* can be found in Refs. [27, 29, 30].

Thus, no geometric extension of the Poincaré symmetry is possible on *asymptotic states* in a nontrivial field theory. Saying this differently, either the theory dynamics is trivial or the theory has no asymptotic states (particles), or extra geometric symmetries in the theory are broken on asymptotic states. The latter two statements can be illustrated on an example of a *conformal* field theory. The conformal symmetry adds scale invariance to a theory. For instance, any conformal field theory possesses an extra space-time symmetry

$$x'^\mu = \lambda x^\mu, \tag{3.4.2}$$

where λ is an arbitrary positive number. In its infinitesimal form $\lambda = 1 + \epsilon$, $|\epsilon| \ll 1$ and

$$x'^\mu = x^\mu + i\epsilon\hat{D}x^\mu, \tag{3.4.3}$$

where $\hat{D}$ is the so called dilatational operator,

$$\hat{D} = -ix^\mu \partial_\mu \tag{3.4.4}$$

Together with certain additional conformal operators (special conformal transformations usually denoted as $\hat{K}_\mu$), the Poincaré algebra (3.2.8) extends in a nontrivial way. However, the asymptotic states (particles) in the theory would break conformal symmetry down to the Poincaré symmetry, in full accordance with Coleman-Mandula theorem. The second possibility – the theory has no asymptotic states at all. Such a theory is scale invariant so that the distance between two points in space is undetermined. For instance, $\mathcal{N} = 4$ super-Yang-Mills theory is of this kind.

3.5 Supersymmetric extension of the Poincaré algebra

The Coleman-Mandula theorem assumes that all symmetry generators in a relativistic field theory are operators which possibly have some Lorentz indices (or, equivalently, even number of spinor indices) and thus are bosonic operators, i.e. they form a Lie algebra with certain *commutation relations*. Meanwhile, generators with odd number of spinor indices are of fermionic nature and are not considered in the proof of the theorem since they cannot participate in commutation relations. The loophole in the theorem consists in the possibility to introduce the operators Q_α and $\bar{Q}_{\dot\alpha}$ with *spinor* indices. The algebra which now includes such operators must involve not only commutators, e.g. $[B_1,\, B_2]$ and $[Q_\alpha,\, B_3]$ (with $B_{1,2,3}$ being the bosonic operators), but also anticommutators, e.g. $\{Q_\alpha,\, \bar{Q}_{\dot\alpha}\}$. Hence, the spinor operators Q_α, $\bar{Q}_{\dot\alpha}$, due to their nature, cannot produce additional restrictions on particle momenta in scattering processes so that no new conservation laws appear. Nevertheless, they relate to each other various scattering amplitudes which greatly constrains the S-matrix in a quantum field theory.

The complex operators Q_α and $\bar{Q}_{\dot\alpha}$ are Hermitian conjugated,

$$\bar{Q}_{\dot\alpha} = (Q_\alpha)^\dagger\,, \tag{3.5.1}$$

and transform as ordinary Weyl spinors under the action of the Poincaré algebra, namely they satisfy the following commutation relations with the Poincaré algebra generators :

$$\begin{aligned} &[P_\mu,\, Q_\alpha] = [P_\mu,\, \bar{Q}^{\dot\alpha}] = 0, \\ &[M^{\mu\nu},\, Q_\alpha] = i\,(\sigma^{\mu\nu})_\alpha^{\ \beta}\, Q_\beta, \\ &[M^{\mu\nu},\, \bar{Q}^{\dot\alpha}] = i\,(\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta}\, \bar{Q}^{\dot\beta}, \end{aligned} \tag{3.5.2}$$

where the index for the supercharge $\bar{Q}_{\dot\alpha}$ is raised with the antisymmetric Levi-Civita tensor, $\bar{Q}^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar{Q}_{\dot\beta}$, and the matrices $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$ were introduced in Eqs. (3.3.6) and (3.3.12).

The operators Q_α and $\bar{Q}_{\dot\alpha}$ are also referred to as *supercharges* or *supergenerators*. According to their indices, the minimum number of such supergenerators in four-dimensional Minkowski space is four. To close the algebra (3.2.8), (3.5.2), one needs to specify the anticommutators $\{Q_\alpha, \bar{Q}_{\dot\alpha}\}$, $\{Q_\alpha, Q_\beta\}$ and $\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\}$. The first anticommutator can only be proportional to $P_\mu \left(\sigma^\mu\right)_{\alpha\dot\beta}$ since it is the only operator with the appropriate Lorentz indices. The standard normalization is

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2P_\mu \left(\sigma^\mu\right)_{\alpha\dot\alpha} . \tag{3.5.3}$$

The simplest choice for the other two anticommutators allowed by the Jacobi identities is

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0. \tag{3.5.4}$$

Thereby, Eqs. (3.2.8), (3.5.2), (3.5.3), (3.5.4) form the *super-Poincaré algebra* first obtained by Golfand and Likhtman [28].

This minimal super-Poincaré algebra with four supercharges can be further extended with additional supercharges. As was demonstrated in Ref. [31], one can construct *extended supersymmetries*, with up to sixteen supercharges in four dimensions. The minimal supersymmetry is referred to as $\mathcal{N} = 1$. Correspondingly, one can consider $\mathcal{N} = 2$ (eight supercharges) or $\mathcal{N} = 4$ (sixteen supercharges). The extended supersymmetry in a four-dimensional field theory is discussed in Section 3.11.

As was also demonstrated in Ref. [31], it is possible to modify the super-Poincaré algebra by an introduction of central charges in it. Such superalgebras are referred to as *centrally extended*. A central charge is an element of the superalgebra which commutes with other generators. It acts as a number with numerical value being dependent on a sector of a theory under consideration. The presence of central charges reflect possible existence of conserved topological currents and topological charges [32]. For instance, if a theory under consideration supports topologically stable domain walls, the right-hand side of (3.5.4) can be modified in the following way :

$$\{Q_\alpha, Q_\beta\} = C_{\alpha\beta}, \qquad \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = \left(C_{\alpha\beta}\right)^\dagger . \tag{3.5.5}$$

Here $C_{\alpha\beta} = C_{\beta\alpha}$ are the central charges. Let us remark that they have spinor indices and thus transform under Lorentz rotations. This is why $C_{\alpha\beta}$ are also called tensor central charges to distinguish them from "standard" central charges which commute with all superalgebra generators. (See also the footnote in Section 3.11, where it is shown how the central charges Z^{IJ}, which are Lorentz scalars, can be introduced for the case of extended supersymmetry.)

3.6 Main properties of supersymmetric field theories

For any supersymmetric field theory, the following fundamental properties hold :

- a state in a supersymmetric field theory cannot have negative energy ;
- if supersymmetry is *unbroken*, the vacuum has exactly zero energy ;
- if there is a boson with mass m, there must exist a fermion with exactly the same mass m, and *vice versa* (Bose-Fermi degeneracy) ;
- any supersymmetric field theory has equal number of bosonic and fermionic degrees of freedom in every *supermultiplet*.

Let us discuss these statements in details.

3.6.1 Non-negative energy of an eigenstate and vanishing of the vacuum energy

The first consequence which follows from the super-Poincaré algebra is the fact that a state in a quantum field theory cannot have negative energy. This straightforwardly follows from Eqs. (3.5.1) and (3.5.3) if one takes the sum

$$P^0 = \frac{1}{4}\sum_{\alpha=1}^{2}\left[Q_\alpha\left(Q_\alpha\right)^\dagger + \left(Q_\alpha\right)^\dagger Q_\alpha\right] \tag{3.6.1}$$

and calculates an average of the left and the right hand sides for a normalized eigenstate $|\Psi\rangle$ with the energy E. Indeed,

$$\begin{aligned}\left\langle\Psi\left|P^0\right|\Psi\right\rangle = E = \frac{1}{4}\sum_{\alpha=1}^{2}\left\langle\Psi\left|Q_\alpha\left(Q_\alpha\right)^\dagger + \left(Q_\alpha\right)^\dagger Q_\alpha\right|\Psi\right\rangle \\ = \frac{1}{4}\sum_{\alpha=1}^{2}\left\langle\left(Q_\alpha\right)^\dagger\Psi\middle|\left(Q_\alpha\right)^\dagger\Psi\right\rangle^* + \frac{1}{4}\sum_{\alpha=1}^{2}\left\langle Q_\alpha\Psi\middle|Q_\alpha\Psi\right\rangle^*\end{aligned} \tag{3.6.2}$$

The second line in this equality is always non-negative. Thus, for any quantum eigenstate its energy $E \geq 0$.

The minimum $E = 0$ is achieved on a *vacuum* state $|0\rangle$ which is annihilated by the supercharges,

$$Q_\alpha|0\rangle = \left(Q_\alpha\right)^\dagger|0\rangle = 0. \tag{3.6.3}$$

A field theory may have one or several vacua with zero energy. If a theory have no states with zero energy, i.e. $E_{\text{vac}} > 0$, the supersymmetry is spontaneously broken. In fact, the vanishing of the vacuum energy is the necessary and sufficient condition for supersymmetry to be left unbroken.

3.6.2 Bose-Fermi degeneracy

In a supersymmetric theory, if there is a boson with the mass m, a fermion with the very same mass m must exist too, and *vice versa*.

To elaborate more on this point, let us introduce a bosonic state $|B\rangle$ with the mass m and associate with it one of the following fermionic states : $Q_\alpha|B\rangle$, $\left(Q_\alpha\right)^\dagger|B\rangle$, where $\alpha = 1, 2$. At least one of these four states is nonzero. Indeed, the sum of the norms of these four states in positive :

$$\begin{aligned}\sum_{\alpha=1}^{2}\left\langle\left(Q_\alpha\right)^\dagger B\middle|\left(Q_\alpha\right)^\dagger B\right\rangle + \sum_{\alpha=1}^{2}\left\langle Q_\alpha B\middle|Q_\alpha B\right\rangle \\ = \sum_{\alpha=1}^{2}\left\langle B\left|Q_\alpha\left(Q_\alpha\right)^\dagger + \left(Q_\alpha\right)^\dagger Q_\alpha\right|B\right\rangle^* = 4\left\langle B\left|P^0\right|B\right\rangle^* = 4E_B,\end{aligned} \tag{3.6.4}$$

where $E_B > 0$ is the state $|B\rangle$ energy.

Note also that $P^2 = P_\mu P^\mu$ which is a Casimir operator of the Poincaré algebra (it commutes with all the Poincaré algebra generators) is also a Casimir operator of the super-Poincaré algebra, because

$$[P^2,\, Q_\alpha] = [P^2,\, \bar{Q}_{\dot{\alpha}}] = 0. \tag{3.6.5}$$

Thus, from $P^2\,|B\rangle = m^2\,|B\rangle$ follows that

$$P^2\Big|Q_\alpha B\Big\rangle = m^2\Big|Q_\alpha B\Big\rangle \qquad \text{and} \qquad P^2\,\Big|(Q_\alpha)^\dagger\, B\Big\rangle = m^2\,\Big|(Q_\alpha)^\dagger\, B\Big\rangle\,. \tag{3.6.6}$$

Combining the two observations above, one arrives to the statement of this section. In a similar manner, one can prove the reverse statement : for a fermion with the mass m there exist a boson with the very same mass m.

3.6.3 Supermultiplets

The Poincaré group is not a compact group. That is why all its unitary representations (except for the trivial representation) are infinite-dimensional. This infinite dimensionality reveals itself in a widely known fact that particle states are labeled by the continuous parameters – particle 4-momentum p_μ.

The Poincaré algebra has two Casimir operators : $P^2 = P_\mu P^\mu$ and $W^2 = W_\mu W^\mu$, where W^μ is Pauli–Lubanski vector,

$$W^\mu = \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,P_\nu\,M_{\rho\sigma} \tag{3.6.7}$$

($\varepsilon^{\mu\nu\rho\sigma}$ is antisymmetric Levi-Civita tensor). The eigenvalues of the operator P^2 fix particle mass squared, $p_\mu p^\mu = m^2$, while the eigenvalues of the operator W^2 are responsible for particle spin if the particle mass is not zero.

To understand the latter statement, let us boost to a reference frame where the particle is at rest : $p_\mu = (m,\, 0,\, 0,\, 0)$. One can check that in this reference frame

$$W^2 = W_\mu W^\mu = -m^2 s(s+1), \tag{3.6.8}$$

where s is the particle spin.

For massless particles $P^2 = 0$ and $W^2 = 0$. Then, instead of spin, one must consider particle helicity. One can boost to the reference frame where the particle 4-momentum is $p_\mu = (E,\, 0,\, 0,\, E)$ with E being the particle energy. Then the eigenvalues of the operator M_{12} are $\pm\lambda$ with λ being the helicity.

Hence, besides the particle 4-momentum p_μ, a unitary irreducible representation of the Poincaré algebra is identified by the particle mass m and the particle spin or helicity s, if the particle has zero mass. In contrast with P^2, the operator W^2 does not commute with the supercharges, i.e. $[W^2,\, Q_\alpha] \neq 0$, as follows from Eq. (3.5.2). The same is true for the operator M_{12}. Thus, massive irreducible superalgebra representations must contain particles with different spins, while massless irreducible superalgebra representations must contain particles with different helicities. Due to the property, $Q^2_\alpha = \bar{Q}^2_{\dot{\alpha}} = 0$, the supercharges may change the particle into another particle with different spin/helicity a finite number of times. The corresponding set of particles, all with the same mass m, but with different spins/helicities is called a *supermultiplet*. In the simplest case a supermultiplet consists of two particles with spins s, $s+1/2$ or helicities λ, $\lambda+1/2$. Further details on building the supermultiplets can be found e.g. in [33, 30, 29, 34].

3.6.4 Equal number of bosonic and fermionic degrees of freedom in every supermultiplet

We omit here a formal proof of the equality of the number of bosonic and fermionic states in a supermultiplet. It will be given in the case of supersymmetric quantum mechanics in Section 4.2.4. Instead, let us discuss how this fact follows from the vanishing of the vacuum energy.

Consider a free field theory. It is well known that bosons and fermions contribute to the vacuum energy due to zero-point oscillations. The bosonic contribution is

$$\sum_B \sum_{\vec{p}} \sqrt{m_B^2 + \vec{p}^2}, \tag{3.6.9}$$

where the (divergent) sum runs over all bosonic degrees of freedom and over all spatial momenta. The fermionic contribution is

$$-\sum_F \sum_{\vec{p}} \sqrt{m_F^2 + \vec{p}^2}, \tag{3.6.10}$$

where the sum runs over all fermionic degrees of freedom. The extra minus sign is due to -1 associated with the fermion loop in the corresponding Feynman diagram which describes the vacuum energy density. The vanishing of the vacuum energy density requires the cancellation of two contributions which is possible only if the following equations hold inside each supermultiplet :

$$n_B = n_F \qquad \left[\begin{array}{c}\text{equal number of bosons and fermions}\\ \text{in a supermultiplet}\end{array}\right] \tag{3.6.11}$$

and

$$m_B = m_F \qquad \left[\begin{array}{c}\text{equal masses of bosons and fermions}\\ \text{in a supermultiplet}\end{array}\right]. \tag{3.6.12}$$

Note that the latter property was already proven in Section 3.6.2 from algebraic considerations. Note also that by bosons and fermions we mean physical (positive norm) degrees of freedom. For instance, a photon has two degrees of freedom corresponding to the two transverse polarizations (the two helicities ± 1).

3.7 Superspace and superfields

In a relativistic field theory, fields are functions (probably, with some vector or spinor indices) which locally depend on the space-time point x^μ and transform in a certain way under the action of the Poincaré group. With introduction of supersymmetry which is the geometric extension of the Poincaré symmetry, it is very natural to expand the space-time by an addition of appropriate extra dimensions. By doing so, one expands the concept of space-time to the concept of *superspace*. In the superspace, the supercharges are realized as differential operators which generate *supertranslations* in a way similar to the energy-momentum operator which generates translations in four-dimensional space-time. Due to the anticommuting nature of the supercharges, the extra dimensions in the superspace are described by coordinates of Grassmann (anticommuting) nature. Finally, the concept of fields is extended to the concept of *superfields* which are functions of the coordinates on the superspace. This breakthrough idea was pioneered by Salam and Strathdee [35].

The immediate advantage of this formalism is that it gives simple and explicit description of the action of supersymmetry on *component fields* (see below) and provides a very efficient method for constructing manifestly supersymmetric Lagrangians.

In an ordinary $\mathcal{N}=1$ supersymmetry, the superspace

$$\{x^\mu,\, \theta^\alpha,\, \bar\theta^{\dot\alpha}\}, \qquad \bar\theta^{\dot\alpha} \equiv (\theta^\alpha)^* \tag{3.7.1}$$

includes four complex Grassmann (anticommuting) variables θ^α and $\bar\theta^{\dot\alpha}$ which represent "quantum" or "fermionic" dimensions of the superspace. They are complex conjugated and anticommute between each other,

$$\{\theta^\alpha,\, \theta^\beta\} = \{\bar\theta^{\dot\alpha},\, \bar\theta^{\dot\beta}\} = \{\theta^\alpha,\, \bar\theta^{\dot\beta}\} = 0. \tag{3.7.2}$$

Note also the peculiarity of the Leibniz rule for Grassmann derivatives, e.g.

$$\frac{\partial}{\partial\theta^\alpha}\left(\theta^\beta\theta^\gamma\right) = \left(\frac{\partial}{\partial\theta^\alpha}\theta^\beta\right)\theta^\gamma - \theta^\beta\left(\frac{\partial}{\partial\theta^\alpha}\theta^\gamma\right). \tag{3.7.3}$$

In addition, Hermitian conjugation changes the order of anticommuting numbers :

$$\left(\theta^1\theta^2\right)^\dagger = (\theta^2)^\dagger(\theta^1)^\dagger = \bar\theta^{\dot 2}\bar\theta^{\dot 1}. \tag{3.7.4}$$

A superfield is a function of the coordinates (3.7.1) [35, 36]. One can expand it in power series of the Grassmann variables θ^α and $\bar\theta^{\dot\alpha}$. This expansion has finite number of terms since the square of a given Grassmann parameter vanishes. Thus, the highest term in this expansion is $\theta^2\bar\theta^2$, where $\theta^2 = \theta^\alpha\theta_\alpha$, $\bar\theta^2 = \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}$. The most general superfield with no external indices has the following form :

$$S(x,\theta,\bar\theta) = \phi + \theta^\alpha\psi_\alpha + \bar\theta_{\dot\alpha}\bar\chi^{\dot\alpha} + \theta^2 F + \bar\theta^2 G + \theta^\alpha A_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} + \theta^2(\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}) + \bar\theta^2(\theta^\alpha\rho_\alpha) + \theta^2\bar\theta^2 D, \tag{3.7.5}$$

where ϕ, ψ_α, $\bar\chi^{\dot\alpha}$, $\ldots$, D depend only on x^μ and are referred to as the component fields.

In what follows we will use shorthand notations for contraction of spinor indices :

$$AB = A^\alpha B_\alpha, \qquad \bar A\bar B = \bar A_{\dot\alpha}\bar B^{\dot\alpha}. \tag{3.7.6}$$

In particular, if A and B are *anticommuting* variables,

$$AB = BA, \qquad \bar A\bar B = \bar B\bar A, \qquad (AB)^\dagger = \bar A\bar B. \tag{3.7.7}$$

3.7.1 Supersymmetry transformations and differential operators on superspace

The representation of the supercharges Q_α and $\bar{Q}_{\dot\alpha}$ as differential operators on the superspace (3.7.1) can be derived in the following standard way. Let us associate with each point of the superspace (3.7.1) an element of the group corresponding to the $\mathcal{N}=1$ superalgebra (3.2.8), (3.5.2), (3.5.3), (3.5.4) as

$$G(x^\mu, \theta, \bar\theta) = e^{i\left(-x^\mu P_\mu + \theta^\alpha Q_\alpha + \bar\theta_{\dot\alpha}\bar{Q}^{\dot\alpha}\right)}. \tag{3.7.8}$$

Then the product of two elements $G(0,\epsilon,\bar\epsilon)$ and $G(x^\mu,\theta,\bar\theta)$ is [2]

$$G(0,\epsilon,\bar\epsilon)\ G(x^\mu,\theta,\bar\theta) = G(x^\mu + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha} - i\epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\ ,\ \theta+\epsilon\ ,\ \bar\theta+\bar\epsilon). \tag{3.7.9}$$

This equality can be proven by using the Hausdorff formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\ldots} \tag{3.7.10}$$

(where the ellipsis corresponds to infinite series of multi-commutator terms) and taking into account the fact that the series on the right-hand side terminate at the first commutator for the algebra elements considered here. While doing this calculation, one should also remember that the parameters ϵ, $\bar\epsilon$, θ, $\bar\theta$ as well as the supercharges Q, $\bar{Q}$ all anticommute between each other.

Thereby, the action of the group element $G(0,\epsilon,\bar\epsilon)$ on $G(x^\mu,\theta,\bar\theta)$ induces the following motion in the parameter space (3.7.1) :

$$\begin{aligned} x^\mu &\to x^\mu + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha} - i\epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}, \\ \theta_\alpha &\to \theta_\alpha + \epsilon_\alpha, \\ \bar\theta_{\dot\alpha} &\to \bar\theta_{\dot\alpha} + \bar\epsilon_{\dot\alpha}. \end{aligned} \tag{3.7.11}$$

This motion is generated by the operator $i\left(\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar{Q}^{\dot\alpha}\right)$, where the Hermitian-conjugated supercharges are

$$Q_\alpha = -i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha}\,\partial_\mu, \qquad \bar{Q}_{\dot\alpha} = i\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu. \tag{3.7.12}$$

[2] All the elements (3.7.8) form an invariant space under the action of the Poincaré group : the result of the action of any Poincaré group element on (3.7.8) is of the same type. In fact, this space is invariant under the action of the whole super-Poincaré group. This statement partially follows from the equality (3.7.9).

Let us also remark that the superspace (3.7.1) can be obtained as the factor of the super-Poincaré group over the Lorentz group much in the same way this can be done for the four-dimensional Minkowski space :

$$\frac{\text{Poincaré group}}{\text{Lorentz group}} \qquad \longrightarrow \qquad \frac{\text{super-Poincaré group}}{\text{Lorentz group}}.$$

The points in this factor space are orbits obtained by the action of the Lorentz group on the super-Poincaré group space. If we choose a certain point as the origin, then the superspace can be parametrized by (3.7.8).

They satisfy the anticommutation relations

$$\begin{aligned}\left\{Q_\alpha,\,Q_\beta\right\} &= \left\{\bar{Q}_{\dot\alpha},\,\bar{Q}_{\dot\beta}\right\} = 0,\\ \left\{Q_\alpha,\,\bar{Q}_{\dot\alpha}\right\} &= 2i\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu\end{aligned} \tag{3.7.13}$$

and hence, together with $P_\mu = i\partial_\mu$ and an appropriate expression for $M_{\mu\nu}$ [3] give an explicit realization of the supersymmetry algebra, Eqs. (3.2.8), (3.5.2), (3.5.3), (3.5.4).

One could study right multiplication instead of left multiplication in (3.7.9) and would found that the induced motion is generated by a different operator $\epsilon^\alpha D_\alpha + \bar\epsilon_{\dot\alpha}\bar{D}^{\dot\alpha}$, with the operators D_α and $\bar{D}_{\dot\alpha}$ defined as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha}\,\partial_\mu, \qquad \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu. \tag{3.7.14}$$

with $\bar{D}_{\dot\alpha} = -\left(D_\alpha\right)^\dagger$. Note that we have used a different convention of multipliers in the operators above on purpose : this will be convenient in subsequent sections.

The operators D_α and $\bar{D}_{\dot\alpha}$ are called *superderivatives*. By their very definition, they satisfy the following anticommutation relations :

$$\left\{D_\alpha,\,D_\beta\right\} = \left\{\bar{D}_{\dot\alpha},\,\bar{D}_{\dot\beta}\right\} = 0, \qquad \left\{D_\alpha,\,\bar{D}_{\dot\alpha}\right\} = -2i\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu. \tag{3.7.15}$$

In addition, the superderivatives and the supercharges anticommute :

$$\left\{D_\alpha,\,Q_\beta\right\} = \left\{D_\alpha,\,\bar{Q}_{\dot\beta}\right\} = \left\{\bar{D}_{\dot\alpha},\,Q_\beta\right\} = \left\{\bar{D}_{\dot\alpha},\,\bar{Q}_{\dot\beta}\right\} = 0. \tag{3.7.16}$$

This will allow us to reduce the number of independent components in superfields by imposing covariant (consistent with supersymmetry) constraints on them (see Section 3.8.1).

3.8 Superfields

Superfields form linear representations of superalgebra. In general, however, the representations are highly reducible. Extra components in superfields can be eliminated by imposing covariant constraints which (anti)commute with the supersymmetry algebra. One could say that superfield formalism shifts the problem of finding supersymmetry representations to that of finding appropriate constraints. Note that one should constrain superfields without restricting their x^μ dependence (i.e., for instance, by virtue of differential equations in the x space).

3.8.1 Chiral superfield

Let us remark first that the superspace (3.7.1) has two invariant subspaces in it :

$$\{x^\mu_{\mathrm{L}},\,\theta^\alpha\}, \qquad x^\mu_{\mathrm{L}} = x^\mu + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}, \tag{3.8.1}$$

[3] If $M_{\mu\nu}$ is acting on a scalar, its expression is given in Eq. (3.2.6). One must add certain (matrix) extra terms in this expression, if $M_{\mu\nu}$ is acting on a field with some vector or spinor indices.

and

$$\{x^\mu_{\mathrm{R}},\, \bar\theta^{\dot\alpha}\}, \qquad x^\mu_{\mathrm{R}} = x^\mu - i\theta^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}. \tag{3.8.2}$$

Indeed, the supertransformations (3.7.11) give the following supertransformations in the subspaces :

$$\begin{aligned} x^\mu_{\mathrm{L}} &\to x^\mu_{\mathrm{L}} + 2i\theta^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}, \\ \theta^\alpha &\to \theta^\alpha + \epsilon^\alpha, \end{aligned} \tag{3.8.3}$$

and

$$\begin{aligned} x^\mu_{\mathrm{R}} &\to x^\mu_{\mathrm{R}} - 2i\epsilon^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}, \\ \bar\theta^{\dot\alpha} &\to \bar\theta^{\dot\alpha} + \bar\epsilon^{\dot\alpha}. \end{aligned} \tag{3.8.4}$$

These two subspaces are referred to as *chiral* (or left) and *antichiral* (or right) respectively. Each of them is spanned by half of the Grassmann coordinates. Due to this, a function which is defined on the chiral or the antichiral subspace have much shorter component expansion.

Consider, for instance, a *chiral superfield* $\Phi(x_{\mathrm{L}}, \theta)$. Its component expansion

$$\Phi(x_{\mathrm{L}}, \theta) = \phi(x_{\mathrm{L}}) + \sqrt{2}\,\theta^\alpha \psi_\alpha(x_{\mathrm{L}}) + \theta^2 F(x_{\mathrm{L}}) \tag{3.8.5}$$

includes one complex scalar field $\phi(x)$ (two bosonic states) and one complex Weyl spinor $\psi_\alpha(x)$ (two fermionic states) as well as the auxiliary F term which is non-propagating : as we will see shortly, this field will appear in Lagrangian without a kinetic term. Chiral superfields are used for constructing matter sectors of various theories. Note that the chiral superfield $\Phi(x_{\mathrm{L}}, \theta)$, being expressed as a function of x^μ, depends on θ^α as well as on $\bar\theta^{\dot\alpha}$ variables.

In fact, the chiral superfield Φ or the antichiral superfield $\bar\Phi$ can be obtained as solutions of the covariant constraints [37]

$$\bar D_{\dot\alpha}\Phi = 0 \qquad \text{and} \qquad D_\alpha \bar\Phi = 0 \tag{3.8.6}$$

which follow from

$$\bar D_{\dot\alpha}\, x^\mu_{\mathrm{L}} = 0 \qquad \text{and} \qquad D_\alpha\, x^\mu_{\mathrm{R}} = 0. \tag{3.8.7}$$

Moreover, the covariant superderivatives (3.7.14) in the chiral basis $\{x^\mu_{\mathrm{L}},\, \theta^\alpha,\, \bar\theta^{\dot\alpha}\}$ are realized as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha}\frac{\partial}{\partial x^\mu_{\mathrm{L}}}, \qquad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}}, \tag{3.8.8}$$

while in the antichiral basis $\{x^\mu_{\mathrm{R}},\, \theta^\alpha,\, \bar\theta^{\dot\alpha}\}$ they are realized as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha}, \qquad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - 2i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu_{\mathrm{R}}}. \tag{3.8.9}$$

Thus, for instance, the chirality condition $\bar D_{\dot\alpha}\Phi = 0$ translates itself into

$$\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\Phi = 0, \tag{3.8.10}$$

from which the solution (3.8.5) is obvious.

Let us write also the induced transformations of the component fields in the chiral superfield (3.8.5) under the infinitesimal supertransformations (3.8.3) or, equivalently, (3.7.11) :

$$\begin{aligned}
&\phi \to \phi + \sqrt{2}\,\epsilon^\alpha \psi_\alpha, \\
&\psi_\alpha \to \psi_\alpha + i\sqrt{2}\,\partial_\mu \phi\, \sigma^\mu_{\alpha\dot\alpha}\,\bar\epsilon^{\dot\alpha} + \sqrt{2} F \epsilon_\alpha, \\
&F \to F - i\sqrt{2}\,\partial_\mu\left(\psi^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}\right).
\end{aligned} \tag{3.8.11}$$

Note that the F term transforms through a total space-time derivative.

3.8.2 Real superfield

Let us inspect Eq. (3.7.5). It gives a reducible representation of the supersymmetry algebra. We can simply constrain it with the reality condition $S^\dagger = S$. This gives what is called vector superfield :

$$V(x,\theta,\bar\theta) = \phi + \theta\psi + \bar\theta\bar\psi + \theta^2 F + \bar\theta^2 \bar F + \theta^\alpha A_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} + \theta^2(\bar\theta\bar\lambda) + \bar\theta^2(\theta\lambda) + \theta^2\bar\theta^2 D. \tag{3.8.12}$$

The superfield V is real, $V = V^\dagger$, implying that the bosonic fields ϕ, D and $A^\mu = \frac{1}{2}\left(\bar\sigma^\mu\right)^{\dot\alpha\alpha} A_{\alpha\dot\alpha}$ are real. The fields ψ_α, λ_α, F are complex, with $\bar\psi_{\dot\alpha} = (\psi_\alpha)^\dagger$, $\bar\lambda_{\dot\alpha} = (\lambda_\alpha)^\dagger$, $\bar F = F^*$.

The real superfield V is used in construction of supersymmetric gauge theories. In fact, the (super)gauge freedom (which for the Abelian gauge superfield is $V \to V + (\Lambda + \bar\Lambda)$, where Λ is an arbitrary chiral superfield) allows one to eliminate the unwanted components ϕ, ψ_α, $\bar\psi_{\dot\alpha}$, F, and $\bar F$, reducing the physical content of V to

$$V \to \theta^\alpha A_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} + \left\{\theta^2(\bar\theta\bar\lambda) + \bar\theta^2(\theta\lambda)\right\} + \theta^2\bar\theta^2 D \tag{3.8.13}$$

so that λ_α is a fermionic *superpartner* of the gauge field $A_{\alpha\dot\alpha}$, while D is an auxiliary field which is not dynamical and can be excluded from a corresponding Lagrangian by algebraic equations.

The supersymmetry transformations (3.7.11) induce the transformations of the component fields in V. Here, for later purposes, we quote only the corresponding transformation for the auxiliary field D :

$$D \to D + \frac{i}{2}\partial_\mu\left(\epsilon^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha} - \lambda^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}\right). \tag{3.8.14}$$

Note that, like in the chiral superfield case, it transforms through a total space-time derivative. This property is of a paramount importance for construction of supersymmetric theories.

3.8.3 Properties of superfields

Let us enumerate the main properties of superfields which are used in construction of supersymmetric Lagrangians.

- Linear combinations of superfields as well as products of superfields are again superfields. In general, a function of superfields is a superfield.

– If Φ is a chiral superfield, then $\bar{\Phi} = \Phi^\dagger$ is an antichiral superfield and *vice versa*.
– Given a superfield, one can use the space-time derivatives $\partial/\partial x^\mu$ to generate a new one. At the same time, the Grassmann derivatives $\partial/\partial\theta^\alpha$ and $\partial/\partial\bar{\theta}^{\dot{\alpha}}$, being applied to a superfield, do not produce a superfield. One can use the covariant superderivatives D_α and $\bar{D}_{\dot{\alpha}}$ instead since they anticommute with the supercharges.
– The squares of the covariant superderivatives $\bar{D}^2$, D^2, being applied on a generic superfield, produce a chiral or antichiral superfield, respectively. This immediately follows from the expression for $\bar{D}_{\dot{\alpha}}$ in (3.8.8) and from the expression for D_α in (3.8.9). For instance, $\bar{D}^2\, D_\alpha S$ (with arbitrary S) is a chiral superfield while $D_\alpha\Phi$ (with Φ chiral) is not a chiral superfield. Indeed,

$$\bar{D}_{\dot{\alpha}}\,(D_\alpha\Phi) = \left\{\bar{D}_{\dot{\alpha}},\, D_\alpha\right\}\Phi = -2i\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu\Phi \neq 0. \tag{3.8.15}$$

At the same time, $D^2\Phi$ is an antichiral superfield.

3.9 Building supersymmetric Lagrangians

With the extension of space-time to the superspace (3.7.1), the space-time integral $\int d^4x$ must also include the integration over the new coordinates θ^α and $\bar{\theta}^{\dot{\alpha}}$. As it is shown below, one can covariantly integrate superfields in the following three ways :

$$\int d^4x\, d^4\theta \quad \text{or} \quad \int d^2\theta\, d^4x_{\mathrm{L}}, \quad \text{or} \quad \int d^2\bar{\theta}\, d^4x_{\mathrm{R}}. \tag{3.9.1}$$

Do do so, one needs to define the rules of Grassmann integration. After that, manifestly supersymmetric Lagrangians can be straightforwardly constructed. We will consider the simplest example of a superfield action involving a single chiral superfield.

3.9.1 Rules of the Grassmann integration

The rules of integration over the Grassmann variables, also known as Berezin integrals [38], are the following. One-dimensional integrals are defined as

$$\int d\theta_\alpha = 0, \qquad \int \theta_\alpha\, d\theta_\beta = \delta_{\alpha\beta} \tag{3.9.2}$$

(and similarly for the Grassmann variables with dotted indices), while multi-dimensional integrals involving two and more Grassmann variables are to be understood as product of one-dimensional integrals.

Note that the Grassmann variables θ and $\bar{\theta}$ have the dimension of $[\text{length}]^{1/2}$, while the differentials $d\theta$ and $d\bar{\theta}$ have the dimension of $[\text{length}]^{-1/2}$. If c is a number, then $d(c\,\theta) = c^{-1}d\theta$. This follows from the right equation in (3.9.2).

We normalize the integral over all four Grassmann dimensions of the superspace,

$$\int d^4\theta \equiv \int d^2\theta\, d^2\bar{\theta} \sim \int d\theta_1\, d\theta_2\, d\bar{\theta}_{\dot{1}}\, d\bar{\theta}_{\dot{2}}, \tag{3.9.3}$$

in such a way that

$$\int d^2\theta\, d^2\bar{\theta}\;\theta^2\,\bar{\theta}^2 = 1. \tag{3.9.4}$$

Respectively, the integrals over the chiral and the antichiral subspaces are normalized as

$$\int d^2\theta\ \theta^2 = 1, \qquad \int d^2\bar{\theta}\ \bar{\theta}^2 = 1. \tag{3.9.5}$$

3.9.2 Kinetic terms for matter fields

Consider the integral

$$\int d^4x\, d^4\theta\, V(\dots) \tag{3.9.6}$$

with V being the real superfield (3.8.12) which can be a function of other superfields. The rules of Grassmann integration (3.9.2), (3.9.4) imply that $D = \int d^4\,\theta V$, there D is the coefficient in front of $\theta^2\bar{\theta}^2$ in V. Since the supertransformations change the D term through a total space-time derivative, Eq. (3.8.14), then the expression

$$\int d^4\theta\, V(\dots) \tag{3.9.7}$$

is superinvariant up to a total derivative.

Let us exploit the above idea and construct kinetic term

$$S_{\text{kin}} = \int d^4x\, d^4\theta\, \bar{\Phi}\Phi \tag{3.9.8}$$

for a chiral superfield $\Phi(x_{\text{L}}, \theta)$. The component expansion of Φ is given in Eq. (3.8.5). The product $\Phi\bar{\Phi}$, where $\bar{\Phi} = \Phi^\dagger$, is a real superfield. The calculation of the highest term in θ^α and $\bar{\theta}^{\dot{\alpha}}$ gives

$$\begin{aligned} \bar{\Phi}\,\Phi = \dots + \theta^2\bar{\theta}^2 \Big\{ &\frac{1}{2}\partial_\mu\bar{\phi}\,\partial^\mu\phi - \frac{1}{4}\bar{\phi}\,\partial^2\phi - \frac{1}{4}\phi\,\partial^2\bar{\phi} \\ &- \frac{i}{2}\,\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu\bar{\psi}^{\dot{\alpha}} + \frac{i}{2}\,\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\psi^\alpha\,\bar{\psi}^{\dot{\alpha}} + \bar{F}F \Big\}, \end{aligned} \tag{3.9.9}$$

where all the component fields depend on the space-time point x^μ, and the ellipsis denotes all lower-order terms in θ^α and $\bar{\theta}^{\dot{\alpha}}$. Substituting the last expression into (3.9.8), integrating over the Grassmann variables and integrating by parts reads

$$S_{\text{kin}} = \int d^4x \left\{ \partial_\mu\bar{\phi}\,\partial^\mu\phi - i\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu\bar{\psi}^{\dot{\alpha}} + \bar{F}F \right\}. \tag{3.9.10}$$

Thus, this expression presents the kinetic term for the complex field $\phi(x)$ and the kinetic term for the Weyl spinor $\psi_\alpha(x)$. As one can see, the field $F(x)$ appears in the Lagrangian with no derivatives and does not represent any physical (propagating) degrees of freedom. It can be eliminated from the action by virtue of the equations of motion.

3.9.3 Potential terms for matter fields

Similarly to the previous section, the integral

$$\int d^2\theta d^4x_{\text{L}}\, K(\dots), \tag{3.9.11}$$

where K is a chiral superfield (see Eq. (3.8.5) for its component expansion), which can be a function of other superfields, is invariant under the action of the supersymmetry group. To prove this statement, note that

$$\int d^2\theta\, K(\dots) = F_K(x) - \frac{i}{\sqrt{2}}\partial_\mu \left(\psi_K^\alpha(x)\,\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\right), \tag{3.9.12}$$

with F_K being the coefficient in front of θ^2, while ψ_K^α being the coefficient in front of $\sqrt{2}\theta^\alpha$ in the component expansion of the chiral superfield K. In addition, the last term in (3.9.12) is a full space-time derivative and thus vanishes if one integrates it over space-time. Thereby, we see that

$$\int d^2\theta d^4x_{\mathrm{L}}\, K(\dots) = \int d^4x\, d^2\theta\, K(\dots) = \int d^4x\, F_K(x). \tag{3.9.13}$$

Finally, according to Eq. (3.8.11), the F_K term transforms as a full space-time derivative under the action of the supersymmetry group, and thus the integral above is indeed superinvariant. It is also clear now that the integral

$$\int d^2\bar\theta d^4x_{\mathrm{R}}\, \bar K(\dots), \tag{3.9.14}$$

where $\bar K$ is an antichiral superfield, is superinvariant.

Potential terms for component fields in the Lagrangian are those which enter with no space-time derivatives. They can be produced for the chiral superfield Φ from the previous section by the following superfield action :

$$S_{\mathrm{int}} = \int d^2\theta\, d^4x_{\mathrm{L}}\, \mathcal{W}(\Phi) + \int d^2\bar\theta\, d^4x_{\mathrm{R}}\, \bar{\mathcal{W}}(\bar\Phi). \tag{3.9.15}$$

Here $\mathcal{W}(\Phi)$ is a function of the chiral superfield termed *superpotential*. The second integral which involves $\bar{\mathcal{W}}(\bar\Phi) = (\mathcal{W}(\Phi))^\dagger$ ensures that the expression above is real. After taking the integrals over the Grassmann variables, one obtains the following component action :

$$S_{\mathrm{int}} = \int d^4x \left\{F\,\mathcal{W}'(\phi) - \frac{1}{2}\mathcal{W}''(\phi)\,\psi^2 + \text{h.c.}\right\}, \tag{3.9.16}$$

where $\psi^2 = \psi^\alpha\psi_\alpha$, the prime denotes the derivative with respect to ϕ and h.c. means the Hermitian conjugated expression.

3.9.4 The Wess–Zumino model

Let us combine the results of the previous two sections and write the full superfield action involving a single chiral superfield Φ and its Hermitian conjugated superfield $\bar\Phi$:

$$S_{\mathrm{WZ}} = \int d^4x\, d^4\theta\, \Phi\bar\Phi + \int d^2\theta\, d^4x_{\mathrm{L}}\mathcal{W}(\Phi) + \int d^2\bar\theta\, d^4x_{\mathrm{R}}\bar{\mathcal{W}}(\bar\Phi). \tag{3.9.17}$$

This model of supersymmetric field theory was invented by Wess and Zumino [37] and bears their name. Let us remark that the first term in the superfield action is the integral over the full superspace, while the second and the third terms run over the chiral and antichiral subspaces, respectively.

In components, the corresponding Lagrangian is

$$\mathcal{L}_{\rm WZ} = \partial_\mu \bar{\phi}\, \partial^\mu \phi - i\psi^\alpha \sigma^\mu_{\alpha\dot\alpha}\, \partial_\mu \bar{\psi}^{\dot\alpha} + \bar{F}F + \left(F\,\mathcal{W}'(\phi) - \frac{1}{2}\mathcal{W}''(\phi)\,\psi^2 \right) + \text{h.c.} \qquad (3.9.18)$$

It is supersymmetry-invariant up to a total space-time derivative. The auxiliary field F is non-dynamical and can be eliminated by virtue of its classical equation of motion,

$$\bar{F} = -\mathcal{W}'(\phi). \qquad (3.9.19)$$

The final expression for the component Lagrangian is

$$\mathcal{L}_{\rm WZ} = \partial_\mu \bar{\phi}\, \partial^\mu \phi - i\psi^\alpha \sigma^\mu_{\alpha\dot\alpha}\, \partial_\mu \bar{\psi}^{\dot\alpha} - |\mathcal{W}'(\phi)|^2 - \frac{1}{2}\mathcal{W}''(\phi)\,\psi^\alpha\psi_\alpha - \frac{1}{2}\bar{\mathcal{W}}''(\bar\phi)\,\bar\psi_{\dot\alpha}\bar\psi^{\dot\alpha}. \quad (3.9.20)$$

It contains the scalar potential $|\mathcal{W}'(\phi)|^2$ describing the self-interaction of the complex field ϕ. For a renormalizable field theory in four dimensions, the superpotential $\mathcal{W}(\Phi)$ must be a polynomial function of Φ of power not higher than three. Then, it can be always reduced to the form

$$\mathcal{W}(\Phi) = \frac{m}{2}\,\Phi^2 - \frac{\lambda}{3}\Phi^3 \qquad (3.9.21)$$

with two *complex* constants m and λ. In fact, one can always choose the phases of the constants m and λ at will.

As a simplest example, let us consider the case $\lambda = 0$. Then the last three terms in the Lagrangian (3.9.20),

$$-|m|^2\phi\bar\phi - \frac{m}{2}\psi^2 - \frac{\bar m}{2}\bar\psi^2, \qquad (3.9.22)$$

are the mass terms for the fields ϕ and ψ_α. As expected, the masses of scalar and spinor particles are equal and are given by one and the same parameter $|m|$.

3.10 R-symmetries

The Coleman–Mandula theorem states that all global bosonic symmetries must commute with the Poincaré group (i.e. be Lorentz scalars). However, it is not necessary for them to commute with all generators of the super-Poincaré group. Indeed, the form of the super-Poincaré algebra (3.2.8), (3.5.2), (3.5.3), (3.5.4) allows one to multiply the supercharges by a constant phase in such a way that the superalgebra itself stays unchanged :

$$Q_\alpha \to e^{-i\varphi}Q_\alpha, \qquad \bar{Q}_{\dot\alpha} \to e^{i\varphi}\bar{Q}_{\dot\alpha}. \qquad (3.10.1)$$

The corresponding Hermitian U(1) generator R has the following commutation relations with the supercharges,

$$[R,\, Q_\alpha] = -Q_\alpha, \qquad [R,\, \bar{Q}_{\dot\alpha}] = \bar{Q}_{\dot\alpha}, \qquad (3.10.2)$$

and it can be realized as a differential operator in the superspace (3.7.1),

$$R = \theta^\alpha \frac{\partial}{\partial \theta^\alpha} - \bar{\theta}^{\dot{\alpha}} \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}. \tag{3.10.3}$$

This additional U(1) symmetry is called the *R-symmetry*. It transforms the Grassmann parameters,

$$\theta^\alpha \to e^{i\varphi}\theta^\alpha, \qquad \bar{\theta}^{\dot{\alpha}} \to e^{-i\varphi}\,\bar{\theta}^{\dot{\alpha}}, \tag{3.10.4}$$

and the measure of the Grassmann integration,

$$d^2\theta \to e^{-2i\varphi} d^2\theta, \qquad d^2\bar{\theta} \to e^{2i\varphi} d^2\bar{\theta}, \qquad d^4\theta \to d^4\theta. \tag{3.10.5}$$

Thus, we assign the *R-charge* $+1$ to θ^α ($R\,\theta^\alpha = +\theta^\alpha$) and the R-charge -1 to $\bar{\theta}^{\dot{\alpha}}$ ($R\,\bar{\theta}^{\dot{\alpha}} = -\bar{\theta}^{\dot{\alpha}}$), while the R-charges of $d^2\theta$ and $d^2\bar{\theta}$ are -2 and $+2$, respectively. The R-charge of $d^4\theta$ is zero.

The R-symmetry can be a symmetry of a given system. To see how it works, let us consider the Wess-Zumino model (3.9.17) of a chiral superfield Φ with the superpotential

$$\mathcal{W} = -\frac{\lambda}{3}\Phi^3. \tag{3.10.6}$$

Then, the action (3.9.17) is invariant under the R-symmetry, if the R-charge of the superfield Φ is 2/3,

$$\Phi \to e^{2i\varphi/3}\Phi. \tag{3.10.7}$$

Indeed, the first term in the action, $\int d^4\theta\, \Phi\bar{\Phi}$, is invariant since the antichiral superfield $\bar{\Phi}$ transforms with the complex conjugated phase factor. The potential term is also invariant :

$$\int d^2\theta\, \mathcal{W}(\Phi) \longrightarrow \int \left(e^{-2i\varphi}\, d^2\theta\right) \left(\mathcal{W}(\Phi)\, e^{2i\varphi}\right) = \int d^2\theta\, \mathcal{W}(\Phi). \tag{3.10.8}$$

Since the R-symmetry does not commute with supersymmetry, the component fields (3.8.5) of the superfield Φ do not all carry the same R-charge. It follows from Eqs. (3.10.4) and (3.10.7) that, under the R-symmetry, they are transformed as

$$\begin{aligned} \phi(x) &\to e^{\frac{2}{3}i\varphi}\,\phi(x), \\ \psi(x) &\to e^{\left(\frac{2}{3}-1\right)i\varphi}\,\psi(x), \\ F(x) &\to e^{\left(\frac{2}{3}-2\right)i\varphi}\,F(x). \end{aligned} \tag{3.10.9}$$

As one can see, the R-charge of the lowest component $\phi(x)$ coincides with the R-charge of the superfield Φ itself. The above formulas can be used to explicitly check that the component Lagrangian (3.9.20) of the Wess-Zumino model is R-symmetry invariant provided $m = 0$.

The R-symmetries play very important role in harmonic superspace approach, which is discussed in the case of supersymmetric quantum mechanics in the next chapter.

3.11 Extended supersymmetry in Minkowski space in four dimensions

As we already know, the Coleman-Mandula theorem can be circumvented by the introduction to the Poincaré algebra four complex supercharges Q_α, $\bar{Q}_{\dot{\alpha}}$ of fermionic nature. The supersymmetry algebra (3.2.8), (3.5.2), (3.5.3) and (3.5.4) is usually referred to as minimal supersymmetry or as $\mathcal{N} = 1$ supersymmetry. In fact, it can be extended even more, with more generators of fermionic nature,

$$Q^I_\alpha \qquad \text{and} \qquad \bar{Q}^J_{\dot{\alpha}}, \tag{3.11.1}$$

where new indices $I, J = 1, 2, \ldots, \mathcal{N}$ numerate the "flavours" of supercharges. Evidently, the defining relations (3.5.3) and (3.5.4) can be generalized with [4]

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2P_\mu \, (\sigma^\mu)_{\alpha\dot{\alpha}} \, \delta^{IJ}, \tag{3.11.3}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \{Q^I_{\dot{\alpha}}, Q^J_{\dot{\beta}}\} = 0, \tag{3.11.4}$$

while the relation (3.5.2) stays untouched since it defines how spinors transform under the Poincaré symmetry.

In $\mathcal{N} = 1$ supersymmetric renormalizable four-dimensional field theory which do now include gravity, the supermultiplets involve particles with spins (0, 1/2) or (1/2, 1), since the supercharges change a particle spin by 1/2. Similarly, with the introduction of *extended supersymmetry*, one can transform a particle with spin zero to a particle with spin one by using two supercharges of different flavours. Thus, such a theory must include particles with spins (0, 1/2, 1) in a supermultiplet, i.e. it must be a gauge theory.

Gauge theories of this type are $\mathcal{N} = 2$ and $\mathcal{N} = 4$ super-Yang–Mills theories. They are obtained by dimensional reduction from the minimal super-Yang–Mills theories in six and ten dimensions, respectively. These theories are unsuitable for phenomenology, because all fermion fields they contain are nonchiral. Nevertheless, they have rich dynamics the study of which provides deep insights into a large number of problems in mathematical physics.

All the properties of a supersymmetric field theory such as vanishing of vacuum energy and equal number of bosons and fermions in a supermultiplet, which were discussed in Section 3.6, remain intact. Moreover, the Abelian U(1) R-symmetry from Eq. (3.10.2) is extended to be non-Abelian U($\mathcal{N}$) symmetry. Simply speaking, the non-Abelian R-symmetry group just mixes the flavours of the supercharges (i.e. it mixes the supercharge flavour indices).

In addition, the superspace (3.7.1) is trivially extended to

$$\{x^\mu, \, \theta^{I\,\alpha}, \, \bar{\theta}^{J\,\dot{\alpha}}\}. \tag{3.11.5}$$

[4] The relations (3.11.4) do not include possible central charges. One can, for instance, introduce them as

$$\{Q^I_\alpha, Q^J_\beta\} = \varepsilon_{\alpha\beta} Z^{IJ} \tag{3.11.2}$$

with $Z^{IJ} = -Z^{JI}$. See also the discussion at the end of Section 3.5.

Note, however, that the usefulness of this $\mathcal{N} = 2$ or $\mathcal{N} = 4$ superspace is limited due to the fact that there are no chiral subspaces in it (which span over half of Grassmann coordinates). There exist a different superspace called harmonic superspace (HSS) which has these invariant subspaces [5]. Such an approach gives more adequate superfield description. The harmonic superspace is obtained from the superspace (3.11.5) by extending it with bosonic coordinates which parametrize the R-symmetry group U($\mathcal{N}$) space or a certain factor space of it.

The harmonic superspace approach is discussed in supersymmetric quantum mechanics in the next chapter.

Chapitre 4

Supersymmetry and harmonic superspace in quantum mechanics

This chapter is devoted to basic introduction to supersymmetry in quantum mechanics. We consider the main properties of any supersymmetric system, explain what is superspace and how the superfield formalism can be used in order to construct genuine supersymmetric Lagrangians. A simplest example of a supersymmetric system is considered. It involves a particle with spin moving in a potential field in one-dimensional space.

The second part of the chapter is devoted to harmonic superspace approach in supersymmetric quantum mechanics. The essential definitions and notations are introduced. They will be extensively used in the next chapter. In particular, the supermultiplets $(\mathbf{4},\mathbf{4},\mathbf{0})$ and $(\mathbf{3},\mathbf{4},\mathbf{1})$ are described. The former is relevant in the context of four-dimensional quantum mechanics, while the latter is used in construction of three-dimensional systems.

4.1 Supersymmetry in quantum mechanics

A quantum-mechanical system with traditional commutation relations between coordinates and momenta and a traditional Hilbert space of states is described by its Hamiltonian function H. We introduce a set of complex operators Q_i together with their Hermitian conjugated operators,

$$\bar{Q}^i = (Q_i)^\dagger \,. \tag{4.1.1}$$

The system with the Hamiltonian H and *the supercharges* Q_i, $\bar{Q}^j$ is supersymmetric, by definition, if

$$\left\{Q_i, \bar{Q}^j\right\} \;=\; 2\delta_i^j H \tag{4.1.2}$$

$$\left\{Q_i, Q_j\right\} \;=\; \left\{\bar{Q}^i, \bar{Q}^j\right\} = 0, \tag{4.1.3}$$

where, as usual, curly brackets denote the anticommutator. In particular, note the important property $Q_i^2 = \left(\bar{Q}^i\right)^2 = 0$ for any i. Another important consequence is that the supercharges commute with the Hamiltonian :

$$\left[H,\, Q_i\right] = \left[H,\, \bar{Q}^i\right] = 0, \tag{4.1.4}$$

which can be proven by direct computation. In this way, the supercharges Q_i and $\bar{Q}^j$ are considered as conserved spinorial operators in the system.

Several comments are relevant here. The Latin indices i, j denote supercharge numbers and vary in the following region :

$$i,\, j = 1,\, 2,\, \ldots,\, \mathcal{N}/2 \tag{4.1.5}$$

with number $\mathcal{N}$ being even integer, see below.

We distinguish the position of indices for the supercharge Q_i and its Hermitian conjugated supercharge $\bar{Q}^i$. This is done because the SU($\frac{\mathcal{N}}{2}$) subgroup of the R-symmetry group (in the $\mathcal{N} \geq 4$ case) acts on these indices differently, see Section 4.3 for details. This subgroup will be of special importance for us later, when we limit ourselves to $\mathcal{N} = 4$ case and deploy harmonic superspace approach. Spinor products and their complex conjugates are conveniently written in such notations. For instance, for any two *anticommuting* fields $\bar{\psi}^i$ and ξ_j

$$\bar{\psi}\,\xi \equiv \bar{\psi}^i\,\xi_i, \qquad \text{while} \qquad \left(\bar{\psi}\,\xi\right)^* = \bar{\xi}\,\psi \equiv \bar{\xi}^i\,\psi_i, \tag{4.1.6}$$

where $\bar{\psi}^i = (\psi_i)^*$ and $\bar{\xi}^i = (\xi_i)^*$.

Finally, one can always pass to a *real* basis in the vector space of supercharges, for instance, by using the definitions

$$S_A = \begin{cases} Q_A + \bar{Q}^A, & \text{for} \quad A = 1,\, 2,\, \ldots,\, \mathcal{N}/2, \\ i\left(Q_A - \bar{Q}^A\right), & \text{for} \quad A = \mathcal{N}/2 + 1,\, \ldots,\, \mathcal{N} \end{cases} \tag{4.1.7}$$

which give commutation relations

$$\{S_A,\, S_B\} = 4\,\delta_{AB}\,H. \tag{4.1.8}$$

The new indices A, B vary from 1 to $\mathcal{N}$. Thereby, $\mathcal{N}$ counts the number of linearly independent real supercharges in the system at hand. For instance, $\mathcal{N}=2$ corresponds to an ordinary supersymmetric quantum mechanics, while a system with $\mathcal{N}=4$ is endowed with extended supersymmetry.

Let us remark that there is also a different convention in quantum mechanics. According to it, $\mathcal{N}$ counts the number of linearly independent complex supercharges which is, of course, twice smaller than the number of real supercharges. Note that, in a four-dimensional field theory context, the supercharges represent complex Weyl doublets, and $\mathcal{N}$ counts the number of those doublets.

4.2 Properties of supersymmetric quantum mechanics

We already saw in the previous chapter that a supersymmetric system is endowed with additional common properties, namely, vanishing of the vacuum energy, energy positiveness of any definite-energy state, equal number of bosons and fermions in a supermultiplet, their equal masses. Any supersymmetric quantum-mechanical system has similar features.

Let us describe them in detail. To this end, let us denote a normalized vacuum state as ψ_{vac},

$$\langle \psi_{\text{vac}} | \psi_{\text{vac}} \rangle = 1, \tag{4.2.1}$$

while any other similarly normalized state as ψ. For simplicity, all the states will be treated as normalizable. Also, we may denote a state either as ψ or as $|\psi\rangle$ according to our convenience.

4.2.1 Every eigenstate has non-negative energy

This statement can be proven by sandwiching Eq. (4.1.2) (with $i=j$ being fixed) with an eigenstate ψ :

$$\begin{aligned} 2E &= \left\langle \psi \left| Q_i \left(Q_i\right)^{\dagger} \right| \psi \right\rangle + \left\langle \psi \left| \left(Q_i\right)^{\dagger} Q_i \right| \psi \right\rangle \\ &= \left\langle \left(Q_i\right)^{\dagger} \psi \left| \left(Q_i\right)^{\dagger} \psi \right.\right\rangle^{*} + \left\langle Q_i \psi \left| Q_i \psi \right.\right\rangle^{*}. \end{aligned} \tag{4.2.2}$$

The second line in this equality is always non-negative. Thus,

$$E \geq 0. \tag{4.2.3}$$

4.2.2 Vanishing of the vacuum energy

If supersymmetry is unbroken, the vacuum ψ_{vac} has exactly zero energy. This statement straightforwardly follows from Eq. (4.2.2) if one changes $E \to E_{\text{vac}}$ and $\psi \to \psi_{\text{vac}}$. Indeed, the minimum $E_{\text{vac}}=0$ is achieved when

$$Q_i \left|\psi_{\text{vac}}\right\rangle = \bar{Q}^i \left|\psi_{\text{vac}}\right\rangle = 0, \qquad \text{for all } i. \tag{4.2.4}$$

In fact, the conditions (4.2.4) exactly correspond to the case of unbroken supersymmetry. The simplest example of supersymmetry breaking in quantum mechanics was constructed in [39] (see [40] for a good pedagogical review).

4.2.3 Supermultiplets

The eigenstates of a supersymmetric Hamiltonian can be divided into *supermultiplets*. To this end, take a state ψ with *positive* energy E. One can act on this state with supercharges Q_i and $(Q_j)^\dagger$ and obtain new states with exactly the same energy E. Indeed, using Eq. (4.1.4), we conclude that

$$H\,|Q_i\,\psi\rangle = Q_i\,H\,|\psi\rangle = E\,|Q_i\psi\rangle\,, \tag{4.2.5}$$

and similarly for $(Q_i)^\dagger$.

A supermultiplet is defined as a vector space of states of the same energy $E > 0$ obtained by all possible actions of supercharges on a chosen reference state ψ. The basis in the supermultiplet can be explicitly constructed. To do so, consider the following states :

$$Q_{i_1}\,Q_{i_2}\,Q_{i_3}\,\ldots|\psi\rangle\,. \tag{4.2.6}$$

Due to the property $Q_i^2 = 0$ there exists a state

$$|\psi_{\text{low}}\rangle = Q_{k_1}\,Q_{k_2}\,\ldots Q_{k_n}\,|\psi\rangle\,, \qquad n \le \mathcal{N}/2 \tag{4.2.7}$$

which is not zero, $\langle\psi_{\text{low}}|\psi_{\text{low}}\rangle \sim 1$, but all the supercharges Q_i annihilate it :

$$Q_i\,|\psi_{\text{low}}\rangle = 0 \qquad \text{for all } i. \tag{4.2.8}$$

The initial reference state ψ can be restored from the state ψ_{low} in the following way :

$$(Q_{k_1})^\dagger\,(Q_{k_2})^\dagger\ldots(Q_{k_n})^\dagger\,|\psi_{\text{low}}\rangle = (2E)^n\,|\psi\rangle\,, \tag{4.2.9}$$

where Eqs. (4.1.2) and (4.2.8) were used.

Let us now show that the states

$$(Q_{i_1})^\dagger\,(Q_{i_2})^\dagger\ldots\left(Q_{i_f}\right)^\dagger|\psi_{\text{low}}\rangle\,, \qquad i_1 < i_2 < \cdots < i_f\,, \qquad f \le \mathcal{N}/2 \tag{4.2.10}$$

form the basis in the supermultiplet.

To this end, consider a state of type

$$Q_{i_1}\,(Q_{j_1})^\dagger\,Q_{i_2}\,Q_{i_3}\,(Q_{j_2})^\dagger\ldots|\psi\rangle \tag{4.2.11}$$

and plug into it the expression for ψ from Eq. (4.2.9). One can use the relations (4.1.2), (4.1.3) and order the supercharges in such a way that any $(Q_i)^\dagger$ stays to the left from any Q_j. After that, the identity (4.2.8) allows one to eliminate any terms which contain Q_i. For instance,

$$\begin{aligned} Q_1\,(Q_2)^\dagger\,(Q_1)^\dagger\,|\psi_{\text{low}}\rangle &= (Q_2)^\dagger\,Q_1\,(Q_1)^\dagger\,|\psi_{\text{low}}\rangle \\ &= -\,(Q_2)^\dagger\,(Q_1)^\dagger\,Q_1\,|\psi_{\text{low}}\rangle + (Q_2)^\dagger\left\{Q_1,\,(Q_1)^\dagger\right\}|\psi_{\text{low}}\rangle \\ &= 2E\,(Q_2)^\dagger\,|\psi_{\text{low}}\rangle\,. \end{aligned} \tag{4.2.12}$$

Thus, any state in the supermultiplet adds up to a linear combination of the states (4.2.10).

The states (4.2.10) are linearly independent, because their scalar products are zero. For instance, one has for the two states $(Q_1)^\dagger (Q_2)^\dagger |\psi_{\rm low}\rangle$ and $(Q_1)^\dagger (Q_3)^\dagger |\psi_{\rm low}\rangle$:

$$\begin{aligned}\left\langle (Q_1)^\dagger (Q_2)^\dagger \psi_{\rm low} \middle| (Q_1)^\dagger (Q_3)^\dagger \psi_{\rm low}\right\rangle &= \left\langle \psi_{\rm low} \middle| Q_2\, Q_1\, (Q_1)^\dagger (Q_3)^\dagger \middle| \psi_{\rm low}\right\rangle^* \\ = -\left\langle \psi_{\rm low} \middle| Q_2\, (Q_1)^\dagger Q_1\, (Q_3)^\dagger \middle| \psi_{\rm low}\right\rangle^* &+ \left\langle \psi_{\rm low} \middle| Q_2 \left\{Q_1,\, (Q_1)^\dagger\right\} (Q_3)^\dagger \middle| \psi_{\rm low}\right\rangle^* \\ &= 2E \left\langle \psi_{\rm low} \middle| \left\{Q_2,\, (Q_3)^\dagger\right\} \middle| \psi_{\rm low}\right\rangle^* = 0 \qquad (4.2.13)\end{aligned}$$

Evidently, the states (4.2.10) themselves have positive normalization which follows from the calculations similar to above.

Let us remark that we assume that the energy E is positive. If not, Eq. (4.2.9) does not allow us to express ψ through $\psi_{\rm low}$, and thus the considerations above cannot be applied.

As it follows from Eq. (4.2.10), the dimension of the supermultiplet with $E > 0$ is $2^{\mathcal{N}/2}$. In this way, in an $\mathcal{N} = 2$ system each positive-energy state is doubly degenerate, while for a system with extended $\mathcal{N} = 4$ supersymmetry each state with nonzero energy is four-times degenerate.

The states of zero energy – *vacuums* – are of special interest. Their number cannot be found from general considerations, and Eqs. (4.2.4) have to be solved for that. If there are no vacuums in a theory, the supersymmetry is spontaneously broken (see e.g. Refs. [39, 40] for an example).

4.2.4 Equal number of bosonic and fermionic states in a supermultiplet

In a field theory, supermultiplets involve bosonic and fermionic states. The same concerns supermultiplets in quantum mechanics.

Let us consider the states (4.2.10). Each of them is characterized by the number of supercharges f acting on $\psi_{\rm low}$. We can introduce then the "fermionic number operator" $\hat{N}_F$ with eigenvalues f,

$$\hat{N}_F\, (Q_{i_1})^\dagger (Q_{i_2})^\dagger \ldots (Q_{i_m})^\dagger |\psi_{\rm low}\rangle = f\, (Q_{i_1})^\dagger (Q_{i_2})^\dagger \ldots (Q_{i_m})^\dagger |\psi_{\rm low}\rangle\,. \qquad (4.2.14)$$

By definition, the states for which $\hat{N}_F$ is even are called "bosonic", while the states for which $\hat{N}_F$ is odd – "fermionic" [1]. In particular, the state $\psi_{\rm low}$ is bosonic.

We have the following equalities for bosonic and fermionic states respectively :

$$(-1)^{\hat{N}_F} |\psi_{\rm B}\rangle = +\,|\psi_{\rm B}\rangle\,, \qquad (-1)^{\hat{N}_F} |\psi_{\rm F}\rangle = -\,|\psi_{\rm F}\rangle\,. \qquad (4.2.15)$$

[1] In our general consideration this is a matter of mere convention : we could equally well call the states for which $\hat{N}_F$ is even fermionic. For example, consider $\mathcal{N} = 2$ supersymmetric quantum-mechanical system described by the Hamiltonian (4.4.34). It describes a particle with spin 1/2. The corresponding supermultiplets involve two basis states of different spin directions, and each of them can be equally well called bosonic or fermionic.

Using the definitions above, one can check that

$$\left\{(-1)^{\hat{N}_F}\,,\,Q_i\right\} = \left\{(-1)^{\hat{N}_F}\,,\,\bar{Q}^i\right\} = 0. \tag{4.2.16}$$

In this way, one obtains (the index i is fixed)

$$\begin{aligned}&\operatorname{Tr}\left((-1)^{\hat{N}_F}\right) = \\ &= \frac{1}{2E}\operatorname{Tr}\left((-1)^{\hat{N}_F}\left\{Q_i,\,\bar{Q}^i\right\}\right) = \frac{1}{2E}\operatorname{Tr}\left(-Q_i\,(-1)^{\hat{N}_F}\,\bar{Q}^i + (-1)^{\hat{N}_F}\,\bar{Q}^i\,Q_i\right) = 0,\end{aligned} \tag{4.2.17}$$

where the cyclic property of the trace was used. The trace is taken among the states of the (finite-dimensional) supermultiplet. For instance, it is the sum over the averages of the states (4.2.10).

Summarizing all above, one concludes that any supermultiplet with positive energy has equal number of bosonic and fermionic degrees of freedom.

4.3 R-symmetries

Let us remark that the supercharges Q_i and $\bar{Q}^j$ can be linearly transformed in such a way that the form of Eqs. (4.1.2), (4.1.3) does not change. The corresponding group of automorphisms of the supersymmetry algebra is called *R-symmetry group*. To deduce the most general form of such linear transformations, one takes the *ansatz*

$$Q'_i = U_i^{\ j}\,Q_j, \qquad \bar{Q}'^i = \bar{Q}^j\,(U^\dagger)_j^{\ i} \tag{4.3.1}$$

involving matrix $U_i^{\ j}$ and its Hermitian-conjugated matrix $(U^\dagger)_j^{\ i}$ and substitutes it back into Eqs. (4.1.2), (4.1.3). It appears that the matrix U must be unitary, $U\,U^\dagger = 1$. Thus, the R-symmetry group is $\mathrm{U}(\frac{\mathcal{N}}{2})$. It is clear now why the supercharge Q_i carries the superscript index, while the supercharge $\bar{Q}^j$ carries the subscript index : they belong to complex-conjugated representations of the R-symmetry group.

In particular case of the $\mathcal{N} = 2$ supersymmetry, the R-symmetry group is no more than just U(1) group of multiplications of the supercharges by a phase factor :

$$Q'_i = \mathrm{e}^{i\varphi}Q_i, \qquad \bar{Q}'^j = \mathrm{e}^{-i\varphi}\bar{Q}^j \tag{4.3.2}$$

with φ being an arbitrary real number.

In the $\mathcal{N} \geq 4$ case, the R-symmetry group is $\mathrm{U}(\frac{\mathcal{N}}{2}) = \mathrm{U}(1)\times\mathrm{SU}(\frac{\mathcal{N}}{2})$, i.e. it contains the same U(1) subgroup of multiplications by phase factor and also the non-Abelian subgroup $\mathrm{SU}(\frac{\mathcal{N}}{2})$.

4.4 Superspace and superfields

This section is devoted to superfield approach in supersymmetric quantum mechanics. It is shown how to construct systems endowed with supersymmetry in a rather universal way. As we already know from the previous chapter, supersymmetry can be realized as a geometrical symmetry which acts on the coordinates on certain extended space. In quantum mechanics, this space includes the time coordinate t. The other space "dimensions" are of Grassmann nature, i.e. the corresponding coordinates *anticommute* among each other.

Superfield technique is very general. It allows one to construct genuine supersymmetric systems. It also serves as an efficient instrument for generalizing already known supersymmetric systems.

The extended space in quantum mechanics – *superspace* – is described by the following coordinates :

$$\left\{t,\, \theta_i,\, \bar{\theta}^j\right\}, \qquad i,\, j = 1,\, 2,\, \ldots,\, \mathcal{N}/2 \tag{4.4.1}$$

with the identification $\bar{\theta}^i = (\theta_i)^*$: the Grassmann coordinates θ_i and $\bar{\theta}^j$ are complex. They anticommute with each other :

$$\left\{\theta_i,\, \theta_j\right\} = \left\{\theta_i,\, \bar{\theta}^j\right\} = \left\{\bar{\theta}^i,\, \bar{\theta}^j\right\} = 0. \tag{4.4.2}$$

Let us remind that the upper index on the conjugated Grassmann variable reflects the way how it transforms under the action of the R-symmetry group.

Having thus defined the superspace, one introduces *superfields* depending on the superspace coordinates. The property $\theta^2 = 0$ for any Grassmann variable θ limits the number of terms in the expansion of a general superfield $\Phi(t,\, \theta_i,\, \bar{\theta}^j)$ as series in Grassmann variables. Take, for instance, $\mathcal{N} = 2$ case. Omitting the indices on θ_1 and $\bar{\theta}^1$, one obtains

$$\Phi(t,\, \theta,\, \bar{\theta}) = \phi(t) + \bar{\psi}(t)\theta + \xi(t)\bar{\theta} + D(t)\theta\bar{\theta}. \tag{4.4.3}$$

Two complex fields $\phi(t)$, $D(t)$ and two complex Grassmann fields $\bar{\psi}(t)$, $\xi(t)$ depend only on the time variable and may represent physical degrees of freedom in supersymmetric quantum mechanics. However, usually this field content is excessive, i.e. it is possible to reduce the number of fields by additional constraints on the superfield Φ. Such constraints must be covariant with respect to supersymmetry transformations. For instance, such a constraint can be reality condition for the superfield Φ, see Section 4.4.3 for details.

4.4.1 Supersymmetry transformations and differential operators on superspace

Infinitesimal supersymmetry transformations in quantum mechanics are realized as shifts on the superspace (4.4.1),

$$\begin{aligned} &t \to t + i\left(\epsilon_i\bar{\theta}^i + \bar{\epsilon}^i\theta_i\right), \\ &\theta_i \to \theta_i + \epsilon_i, \\ &\bar{\theta}^i \to \bar{\theta}^i + \bar{\epsilon}^i \end{aligned} \tag{4.4.4}$$

with Grassmann parameter ϵ_i, $\bar{\epsilon}^i = (\epsilon_i)^*$. Such transformations are induced by the operator $i\left(\bar{\epsilon}^i Q_i - \epsilon_i \bar{Q}^i\right)$, where the supercharges

$$Q_i = -i\frac{\partial}{\partial\bar{\theta}^i} + \theta_i\frac{\partial}{\partial t}, \qquad \bar{Q}^i = i\frac{\partial}{\partial\theta_i} - \bar{\theta}^i\frac{\partial}{\partial t} \tag{4.4.5}$$

satisfy the relations

$$\left\{Q_i,\, Q_j\right\} \;=\; \left\{\bar{Q}^i,\, \bar{Q}^j\right\} \;=\; 0, \tag{4.4.6}$$

$$\left\{Q_i,\, \bar{Q}^j\right\} \;=\; 2\delta_i^j\, i\partial_t \tag{4.4.7}$$

with

$$\partial_t = \frac{\partial}{\partial t}. \tag{4.4.8}$$

The supercharges (4.4.5) together with the operator $H = i\partial_t$ realize a particular representation of the supersymmetry algebra (4.1.2), (4.1.3) on the superspace (4.4.1).

Let us also introduce *covariant superderivatives* D^i and $\bar{D}_i$ defined as

$$D^i = \frac{\partial}{\partial \theta_i} - i\bar{\theta}^i \frac{\partial}{\partial t}, \qquad \bar{D}_i = \frac{\partial}{\partial \bar{\theta}^i} - i\theta_i \frac{\partial}{\partial t}. \tag{4.4.9}$$

These operators are of special interest, because they anticommute with the supercharges from above,

$$\left\{D^i,\, Q_j\right\} = \left\{D^i,\, \bar{Q}^j\right\} = \left\{\bar{D}_i,\, Q_j\right\} = \left\{\bar{D}_i,\, \bar{Q}^j\right\} = 0, \tag{4.4.10}$$

meaning that they are covariant with respect to supertransformations (4.4.4). Thereby, they can be used in covariant constraints on superfields to reduce their number of independent components. The anticommutation relations for the superderivatives are

$$\left\{D^i,\, D^j\right\} = \left\{\bar{D}_i,\, \bar{D}_j\right\} = 0, \qquad \left\{D^i,\, \bar{D}_j\right\} = -2\delta_j^i\, i\partial_t. \tag{4.4.11}$$

Note that, for later convenience, we intentionally use a different convention for the superderivatives (4.4.9) as compared with the supercharges (4.4.5).

4.4.2 The existence of analytical subspace in $\mathcal{N} = 2$ SQM

Like in supersymmetric field theory case, the $\mathcal{N} = 2$ supersymmetric quantum-mechanical superspace $\left\{t,\, \theta,\, \bar{\theta}\right\}$ is endowed with two invariant subspaces — *chiral*,

$$\left\{t_{\mathrm{L}},\, \theta\right\}, \qquad t_{\mathrm{L}} = t - i\theta\bar{\theta}, \tag{4.4.12}$$

and *antichiral*,

$$\left\{t_{\mathrm{R}},\, \bar{\theta}\right\}, \qquad t_{\mathrm{R}} = t + i\theta\bar{\theta}. \tag{4.4.13}$$

These two subspaces are invariant with respect to supersymmetry transformations (4.4.4) :

$$\begin{aligned} t_{\mathrm{L}} &\to t_{\mathrm{L}} + 2i\bar{\epsilon}\,\theta, \qquad && \theta \to \theta + \epsilon, \\ t_{\mathrm{R}} &\to t_{\mathrm{R}} + 2i\epsilon\,\bar{\theta}, \qquad && \bar{\theta} \to \bar{\theta} + \bar{\epsilon}. \end{aligned} \tag{4.4.14}$$

Practically this means that superfields which depend only on chiral (or antichiral) coordinates have twice as less Grassmann coordinates and thus their component expansion in Grassmann variables is shorter and contains smaller number of component fields. This feature of $\mathcal{N} = 2$ superspace is of primary importance in construction of supersymmetric quantum-mechanical systems. Consider, for instance, a superfield $q\,(t_{\mathrm{L}},\,\theta)$ which depends only on the coordinates of the chiral subspace. It satisfies a covariant constraint

$$\bar{D}\,q\,(t_{\mathrm{L}},\,\theta) = \left(\frac{\partial}{\partial\bar{\theta}} - i\theta\frac{\partial}{\partial t}\right) q\,(t_{\mathrm{L}},\,\theta) = 0 \tag{4.4.15}$$

which is due to $\bar{D}\,t_{\mathrm{L}} = 0$. Its component expansion

$$q\,(t_{\mathrm{L}},\,\theta) = z(t) + \psi(t)\theta - \dot{z}(t)\,i\theta\bar{\theta} \tag{4.4.16}$$

contains one complex variable $z(t)$ and its *superpartner* – one complex Grassmann variable $\psi(t)$. This is indeed the minimum number of degrees of freedom one may have for such a superfield.

The supertransformations (4.4.14) of the chiral superspace induce the supertransformations of the chiral superfield $q\,(t_{\mathrm{L}},\,\theta)$ and its components : $q\,(t_{\mathrm{L}},\,\theta) \to q\,(t_{\mathrm{L}},\,\theta) + \delta q\,(t_{\mathrm{L}},\,\theta)$. Direct calculation yields :

$$\begin{aligned} z(t) &\to z(t) + \psi(t)\epsilon, \\ \psi(t) &\to \psi(t) + 2i\,\dot{z}(t)\bar{\epsilon} \end{aligned} \tag{4.4.17}$$

Note that the field ψ transformation involves full time derivative.

4.4.3 Real superfield in $\mathcal{N} = 2$ case

Along with the (anti)chiral superfields, real superfields are widely used in the $\mathcal{N} = 2$ supersymmetric quantum mechanics. Let us list their properties. We will use a real superfield below for the construction of an example of simple supersymmetric system.

The real superfield $v(t,\,\theta,\,\bar{\theta})$ is defined by the reality condition,

$$v = v^{\dagger}. \tag{4.4.18}$$

Its component expansion reads :

$$v(t,\,\theta,\,\bar{\theta}) = x(t) + \bar{\psi}(t)\theta + \bar{\theta}\psi(t) + D(t)\,\theta\bar{\theta}, \tag{4.4.19}$$

where $\bar{\psi} = \psi^*$ and the fields $x(t)$ and $D(t)$ are real. Under the supertransformations (4.4.4) the component fields transform as

$$\begin{aligned} x &\to x + \bar{\psi}\epsilon + \bar{\epsilon}\psi, \\ \psi &\to \psi - i\dot{x}\,\epsilon, \\ \bar{\psi} &\to \bar{\psi} + i\dot{x}\,\bar{\epsilon}, \\ D &\to D - i\left(\dot{\bar{\psi}}\epsilon + \dot{\psi}\bar{\epsilon}\right) \end{aligned} \tag{4.4.20}$$

Note that the D-term transforms with full time derivative. It is this fact which allows one to build a supersymmetric system to which we now proceed.

4.4.4 One-dimensional $\mathcal{N} = 2$ supersymmetric quantum mechanics

Let us illustrate the material above with an example of the simplest supersymmetric quantum-mechanical system of one real bosonic variable $x(t)$ and one complex Grassmann variable $\psi(t)$ [39]. To this end, we take the real superfield (4.4.19) and write the following action for it :

$$S = \int dt\, d\bar{\theta} d\theta \left\{ \frac{1}{2} \bar{D} v\, D v + \Lambda(v) \right\} \tag{4.4.21}$$

which involves the covariant derivatives from Eq. (4.4.9) and an arbitrary real function $\Lambda(v)$. The rules of Grassmann integration were discussed in Section 3.9.1. The integration over the Grassmann variables leaves only the D-term of the real superfield which stays under the integral in Eq. (4.4.21). Consequently, the action (4.4.21) is automatically supersymmetric. Indeed, under the supertransformations, the D-term transforms as a full time derivative, see Eq. (4.4.20).

As one may guess, the first term in the action (4.4.21) describes the kinetic term of the system, while the second term is the interaction term. The integration over the Grassmann variables can be straightforwardly performed. Using the definition $\int d\bar{\theta} d\theta\, \theta\bar{\theta} = 1$, one obtains

$$S = \int dt \left\{ \frac{1}{2} \dot{x}^2 + \frac{i}{2} \left(\bar{\psi} \dot{\psi} - \dot{\bar{\psi}} \psi \right) + \frac{1}{2} D^2 - D\, W(x) - W'(x)\, \bar{\psi}\psi \right\}, \tag{4.4.22}$$

where we have introduced $W(x) = \Lambda'(x)$, and the prime denotes the derivative with respect to x. Only the last two terms come from the function $\Lambda(v)$ in the action. We now see that the variable $D(t)$ does not have kinetic term and, thus, it is not dynamical. One can integrate it out with its classical equations of motion,

$$D = W(x). \tag{4.4.23}$$

Putting Eq. (4.4.23) back into the component action (4.4.22), one finally obtains the final Lagrangian of the system :

$$L = \frac{1}{2} \dot{x}^2 - \frac{1}{2} W^2(x) + \frac{i}{2} \left(\bar{\psi} \dot{\psi} - \dot{\bar{\psi}} \psi \right) - W'(x)\, \bar{\psi}\psi. \tag{4.4.24}$$

This Lagrangian defines a traditional one-dimensional system with the usual kinetic term $\frac{1}{2}\dot{x}^2$ and the usual potential term $\frac{1}{2}W^2(x)$. The coordinate x, however, is coupled to a fermion ψ. Note that here one has one bosonic degree of freedom and two fermionic degrees of freedom. Let us remark that the statement about the equality of the number of bosons and fermions concerns the number of states in a multiplet, not the number of fields in the system.

To understand better the nature of the fermionic degree of freedom, let us perform the Legendre transform of the Lagrangian to the Hamiltonian and quantize the system. The third term in the Lagrangian (4.4.24) involves only one time derivative on ψ and $\bar{\psi}$ and will not enter the Hamiltonian. Nevertheless, it does provide the anticommutation relations of ψ and $\bar{\psi}$. Indeed, if one considers ψ as a coordinate, then $\bar{\psi}$ would be the corresponding momentum and *vice versa*. Thus, the canonical anticommutation relations are

$$\left\{ \psi, \bar{\psi} \right\} = 1, \tag{4.4.25}$$

and the Hamiltonian reads :

$$H_{\rm cl} = \frac{1}{2}p^2 + \frac{1}{2}W^2(x) + W'(x)\,\bar{\psi}\psi, \tag{4.4.26}$$

where $p = \dot{x}$. This expression corresponds to the "classical" Hamiltonian, because the term with $\bar{\psi}\psi$ has ordering ambiguity. The recipe to solve this issue is known [41] : one must take the classical supercharges (which can be obtained with Nöether theorem from the Lagrangian (4.4.24)) and order the problematic terms with $\bar{\psi}\psi$, if any, in certain way. After that, one must apply the commutation relations (4.1.2) to obtain the "quantum" Hamiltonian which is indeed enjoys supersymmetry algebra. In fact, the supercharges do not have order ambiguity problem. Their expressions are as simple as

$$Q = \frac{1}{\sqrt{2}}\,(p + iW)\,\psi, \qquad \bar{Q} = \frac{1}{\sqrt{2}}\,(p - iW)\,\bar{\psi}. \tag{4.4.27}$$

The anticommutator of supercharges shows the difference between the classical and the quantum Hamiltonian, namely, the quantum Hamiltonian is obtained from the classical Hamiltonian (4.4.26) by the replacement

$$\bar{\psi}\psi \to \frac{1}{2}\left(\bar{\psi}\psi - \psi\bar{\psi}\right). \tag{4.4.28}$$

One can realize the coordinate/momentum pair $\hat{\psi}$, $\hat{\bar{\psi}}$ as differential operators acting on the space of functions $f(\psi)$ of the argument ψ in the following way [2] :

$$\hat{\psi} = \psi, \qquad \hat{\bar{\psi}} = \frac{\partial}{\partial\psi}. \tag{4.4.29}$$

Alternatively, these operators can be realized as matrices. Indeed, the space of functions $f(\psi)$ is two-dimensional : $f(\psi) \equiv a + b\psi$. Let us introduce the basis functions

$$1 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \psi \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4.4.30}$$

so that

$$f(\psi) = a + b\psi \equiv \begin{pmatrix} a \\ b \end{pmatrix}. \tag{4.4.31}$$

Thus, the operators (4.4.29) have the following matrix form :

$$\hat{\psi} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \hat{\bar{\psi}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.4.32}$$

Thereby,

$$\hat{\bar{\psi}}\hat{\psi} - \hat{\psi}\hat{\bar{\psi}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3, \tag{4.4.33}$$

[2]We put "hats" on the operators for clearness.

where σ_3 denotes the third Pauli matrix. Combining Eqs. (4.4.26), (4.4.28) and (4.4.33), one finally obtains the quantum Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}W^2(x) + \frac{1}{2}W'(x)\sigma_3, \tag{4.4.34}$$

which describes a particle with spin moving in one-dimensional space. This Hamiltonian is supersymmetric with supercharges introduced above. As a simple exercise, one can take $W(x) = \omega x$ which gives the system composed with non-interacting one-dimensional oscillator and one spin degree of freedom.

The reader is referred to Ref. [40] for further details on this system, where the study of energy spectrum and quantum states is performed and where the illustration of spontaneous breaking of supersymmetry is presented.

4.5 Harmonic superspace approach

Let us emphasize that the existence of (anti)chiral subspace in the superspace (4.4.1) which leads to the reduction of the component expansion of any (anti)chiral superfield is inherent only to the $\mathcal{N} = 2$ case. The superspace (4.4.1) in the $\mathcal{N} \geq 4$ quantum mechanics does not have this feature. One, however, may still use covariant constraints (e.g. reality condition or equations involving superderivatives) to reduce number of components in superfields. Still, this is sometimes not enough to reduce the number to the expected minimum. Moreover, the constraints may be rather sophisticated. Some successful examples of manipulation with superfields and their covariant constraints are described in Ref. [42].

It appeared that the case of $\mathcal{N} = 4$ is special : it also admits a superspace which has two invariant subspaces and allows one to reduce the number of Grassmann variables in superfields by a factor of two. This is the so called harmonic superspace (HSS) approach [5] invented by Galperin, Ivanov, Ogievetsky and Sokatchev. The key idea here is that the standard superspace (4.4.1) should be supplemented with additional coordinates of bosonic nature.

Let us limit ourselves to the case of $\mathcal{N} = 4$ supersymmetric quantum mechanics.

The harmonic superspace can be seen as one of the superspaces on which the supersymmetry group acts. The superspace (4.4.1) is one of possible choices. All the conceivable spaces on which supersymmetry acts can be described as a factor

$$\text{superspace} = \frac{\text{supersymmetry group}}{\text{one of its certain subgroups}}. \tag{4.5.1}$$

We remind that $\mathcal{N} = 4$ supersymmetry algebra in quantum mechanics is invariant under the SU(2) R-symmetry group (cf. Section 4.3). Till this moment, the latter was factored out from the considerations. However, while searching for conceivable superspaces, the R-symmetry group space can be added to the superspace (4.4.1). It appears that the superfields on the extended superspace which deliver minimal component field content will be functions on a two-sphere $\mathrm{S}^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ – a factor of the R-symmetry group with respect to one of its U(1) subgroups.

4.5.1 Notations

The harmonic superspace (HSS) approach in quantum mechanics was developed in Ref. [6]. The convention in this manuscript follows the convention of Ref. [2] and differs from the convention of Ref. [6] by the change of time direction $t \to -t$. With this, one reproduces the correct sign in the kinetic term for the spinor field in Eq. (5.2.5).

From now and below, in $\mathcal{N} = 4$ supersymmetry, we use a different notation for spinor indices : the indices from the beginning of the Greek alphabet

$$\alpha,\, \beta = 1, 2 \tag{4.5.2}$$

are used instead of the indices i, j. For instance, the ordinary $\mathcal{N} = 4$ superspace is

$$\left\{t,\, \theta_\alpha,\, \bar\theta^\beta\right\}, \qquad \bar\theta^\beta = (\theta_\beta)^*. \tag{4.5.3}$$

For later references, let us repeat here the expression for the supercharges of Eq. (4.4.5) :

$$Q_\alpha = -i\frac{\partial}{\partial\bar\theta^\alpha} + \theta_\alpha\frac{\partial}{\partial t}, \qquad \bar Q^\alpha = i\frac{\partial}{\partial\theta_\alpha} - \bar\theta^\alpha\frac{\partial}{\partial t}, \tag{4.5.4}$$

and the superderivatives of Eq. (4.4.9) :

$$D^\alpha = \frac{\partial}{\partial\theta_\alpha} - i\bar\theta^\alpha\frac{\partial}{\partial t}, \qquad \bar D_\alpha = \frac{\partial}{\partial\bar\theta^\alpha} - i\theta_\alpha\frac{\partial}{\partial t}. \tag{4.5.5}$$

The supersymmetry algebra (4.1.2), (4.1.3) is

$$\begin{aligned} &\left\{Q_\alpha, \bar Q^\beta\right\} = 2\delta_\alpha^\beta H \\ &\left\{Q_\alpha, Q_\beta\right\} = \left\{\bar Q^\alpha, \bar Q^\beta\right\} = 0, \end{aligned} \tag{4.5.6}$$

4.5.2 Raising and lowering spinor indices

The SU(2) R-symmetry group admits the possibility of raising and lowering spinor indices with the invariant antisymmetric Levi-Civita tensors $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$. By definition,

$$\begin{aligned} \varepsilon_{\alpha\beta} &= -\varepsilon_{\beta\alpha}, \qquad \varepsilon_{12} = 1, \\ \varepsilon^{\alpha\beta} &= -\varepsilon^{\beta\alpha}, \qquad \varepsilon^{12} = -1, \end{aligned} \tag{4.5.7}$$

so that, for example, one has for a spinor v_α :

$$v^\alpha = \varepsilon^{\alpha\beta} v_\beta, \qquad v_\alpha = \varepsilon_{\alpha\beta} v^\beta. \tag{4.5.8}$$

Due to the invariance of the Levi-Civita tensors with respect to the action of SU(2) R-symmetry group, the equations involving them are also invariant.

Below, we will also introduce SU(2) Pauli-Gürsey group [5] and dotted indices for it. Analogously, one can introduce Levi-Civita tensors with dotted indices :

$$\begin{aligned} \varepsilon_{\dot\alpha\dot\beta} &= -\varepsilon_{\dot\beta\dot\alpha}, \qquad \varepsilon_{\dot 1\dot 2} = 1, \\ \varepsilon^{\dot\alpha\dot\beta} &= -\varepsilon^{\dot\beta\dot\alpha}, \qquad \varepsilon^{\dot 1\dot 2} = -1. \end{aligned} \tag{4.5.9}$$

4.5.3 Dealing with the sphere S^2

Before discussing the harmonic superspace as a whole it is instructive to study the coordinates on the SU(2)/U(1) space which is a two-sphere. One could choose, for example, polar or stereographic coordinates on S^2. However, it turns out much more convenient to deal with the homogeneous coordinates on the SU(2) group space and constrain functions on it to live on the S^2 space.

To elaborate more on this point, introduce the homogeneous complex coordinates $u^\pm_\alpha$, $\alpha = 1, 2$ called *harmonics* on a three-sphere SU(2). They satisfy the defining relations

$$u^{+\alpha} u^-_\alpha = 1, \qquad u^-_\alpha = (u^{+\alpha})^*. \tag{4.5.10}$$

(We are using raised indices in the above, see the previous subsection ; for instance, $u^{+\alpha} = \varepsilon^{\alpha\beta} u^+_\beta$.) Note also an important identity

$$u^+_\alpha u^-_\beta - u^-_\alpha u^+_\beta = \varepsilon_{\alpha\beta}. \tag{4.5.11}$$

We are interested in considering functions on the SU(2)/U(1) space. Consider functions on a three-sphere which have definitive U(1) charge. For illustration, take a function $f^+(u)$ of charge $+1$. In can be expanded into series in harmonics $u^\pm_\alpha$:

$$f^+(u) = f^\alpha u^+_\alpha + f^{\alpha\beta\gamma} u^+_\alpha u^+_\beta u^-_\gamma + \ldots, \tag{4.5.12}$$

where the constants f^α, $f^{\alpha\beta\gamma}$, ... can always be taken symmetric in their indices. Indeed, using the relation (4.5.11), one can transform any product of harmonics to symmetric combinations plus products of harmonics of smaller orders. For example,

$$\begin{aligned}
u^+_\alpha u^+_\beta u^-_\gamma &= \frac{1}{3}\left(u^+_\alpha u^+_\beta u^-_\gamma + u^-_\alpha u^+_\beta u^+_\gamma + u^+_\alpha u^-_\beta u^+_\gamma\right) + \\
&\quad + \frac{1}{3}\left(u^+_\alpha u^+_\beta u^-_\gamma - u^-_\alpha u^+_\beta u^+_\gamma\right) + \\
&\quad + \frac{1}{3}\left(u^+_\alpha u^+_\beta u^-_\gamma - u^+_\alpha u^-_\beta u^+_\gamma\right) = \\
&= \frac{1}{3}\left(u^+_\alpha u^+_\beta u^-_\gamma + u^-_\alpha u^+_\beta u^+_\gamma + u^+_\alpha u^-_\beta u^+_\gamma\right) + \frac{1}{3}\varepsilon_{\alpha\gamma} u^+_\beta + \frac{1}{3}\varepsilon_{\beta\gamma} u^+_\alpha.
\end{aligned} \tag{4.5.13}$$

With respect to the action of R-symmetry group, the function $f^+(u)$ undergo homogeneous U(1) phase transformations (according to its overall charge) and thus is well defined on a two-sphere SU(2)/U(1). The same is also true for any function of harmonics with fixed U(1) charge.

In fact, the harmonics $u^\pm_\alpha$ are the fundamental spin 1/2 spherical harmonics familiar from quantum mechanics. This is why they are called harmonic variables.

The harmonics can be used for projection of spinor indices onto harmonic space. For instance, $f^+ = u^+_\alpha f^\alpha$ and $f^- = u^-_\alpha f^\alpha$. The original spinor can be restored using Eq. (4.5.11) :

$$f^\alpha = u^{+\alpha} f^- - u^{-\alpha} f^+. \tag{4.5.14}$$

4.5.4 Differential operators on $\mathrm{SU}(2)_\mathrm{R}$ group

The differential operators

$$D^{++} = u^+_\alpha \frac{\partial}{\partial u^-_\alpha}, \qquad D^{--} = u^-_\alpha \frac{\partial}{\partial u^+_\alpha}\,, \qquad D^0 = u^+_\alpha \frac{\partial}{\partial u^+_\alpha} - u^-_\alpha \frac{\partial}{\partial u^-_\alpha} \tag{4.5.15}$$

are called *harmonic derivatives*. The operator D^0 plays a role of the U(1) charge operator. One has for a function $f^{+q}(u)$ of definite U(1) charge $+q$:

$$D^0 f^{+q}(u) = q\, f^{+q}(u). \tag{4.5.16}$$

The coordinates u^+_α have charge 1, while the coordinates u^-_α have charge -1.

4.5.5 $\mathcal{N}=4$ harmonic superspace

The $\mathcal{N}=4$ harmonic superspace formalism in quantum mechanics was developed in Ref. [6]. In this formalism, the superfields depend on time t and on harmonics $u^{\pm\alpha}$ which parametrize the R-symmetry group SU(2) of the $\mathcal{N}=4$ superalgebra, and on Grassmann variables θ_α, $\bar\theta^\beta$. The superspace is

$$\left\{t,\, \theta_\alpha,\, \bar\theta^\beta,\, u^\pm_\gamma\right\}, \qquad \bar\theta^\beta = (\theta_\beta)^*. \tag{4.5.17}$$

This is the so called *standard basis* in harmonic superspace.

Usually, instead of spinors θ_α and $\bar\theta^\beta$ it is preferable to use the following harmonic projections :

$$\theta^\pm = u^\pm_\alpha \theta^\alpha, \qquad \bar\theta^\pm = u^\pm_\alpha \bar\theta^\alpha. \tag{4.5.18}$$

One can also define harmonic projections of superderivatives, $D^\pm = u^\pm_\alpha D^\alpha$, $\bar D^\pm = u^\pm_\alpha \bar D^\alpha$. One can check that

$$D^+ = \frac{\partial}{\partial\theta^-} - i\bar\theta^+ \frac{\partial}{\partial t}, \qquad D^- = -\frac{\partial}{\partial\theta^+} - i\bar\theta^- \frac{\partial}{\partial t}, \tag{4.5.19}$$

$$\bar D^+ = -\frac{\partial}{\partial\bar\theta^-} - i\theta^+ \frac{\partial}{\partial t}, \qquad \bar D^- = \frac{\partial}{\partial\bar\theta^+} - i\theta^- \frac{\partial}{\partial t}, \tag{4.5.20}$$

in the standard basis (4.5.17).

4.5.6 Analytical basis in harmonic superspace

The most striking feature of harmonic superspace is the presence of an *analytic subspace*

$$\left\{t_\mathrm{A},\, \theta^+,\, \bar\theta^+,\, u^{\pm\alpha}\right\} \tag{4.5.21}$$

in it (an analog of $\mathcal{N}=2$ chiral superspace) involving the "analytic time"

$$t_A = t + i(\theta^+\bar\theta^- + \theta^-\bar\theta^+) \tag{4.5.22}$$

and containing twice as less fermionic coordinates.

Let us elaborate more on this point. It is convenient to go over to the *analytic basis* in harmonic superspace,

$$\left\{t_{\mathrm{A}},\, \theta^{\pm},\, \bar\theta^{\pm},\, u^{\pm}_{\alpha}\right\}. \tag{4.5.23}$$

In this basis, the covariant spinor derivatives D^+, $\bar D^+$ are as simple as

$$D^+ = \frac{\partial}{\partial\theta^-}, \qquad \bar D^+ = -\frac{\partial}{\partial\bar\theta^-}. \tag{4.5.24}$$

It is this fact which allows one to translate superfield constraints $D^+ f = \bar D^+ f$ for some superfield f to be independent of θ^- and $\bar\theta^-$: $f = f(t_A,\, \theta^+,\, \bar\theta^+,\, u^{\pm}_{\alpha})$.

One can check directly that the subspace (4.5.21) is invariant with respect to $\mathcal{N}=4$ supersymmetry transformations. Indeed, using Eq. (4.4.4), one obtains

$$\begin{aligned} &t_A \to t_A + 2i\left(\epsilon^-\bar\theta^+ - \bar\epsilon^-\theta^+\right),\\ &\theta^{\pm} \to \theta^{\pm} + \epsilon^{\pm},\\ &\bar\theta^{\pm} \to \bar\theta^{\pm} + \bar\epsilon^{\pm},\\ &u^{\pm}_{\alpha} \to u^{\pm}_{\alpha}, \end{aligned} \tag{4.5.25}$$

where

$$\bar\epsilon^{\alpha} = (\epsilon_{\alpha})^*, \qquad \epsilon^{\pm} = u^{\pm}_{\alpha}\epsilon^{\alpha}, \qquad \bar\epsilon^{\pm} = u^{\pm}_{\alpha}\bar\epsilon^{\alpha}. \tag{4.5.26}$$

Finally, let us also write the form of harmonic derivatives D^{++} and D^{--} from Eq. (4.5.15) in the analytic basis :

$$D^{++} = u^+_{\alpha}\frac{\partial}{\partial u^-_{\alpha}} + \theta^+\frac{\partial}{\partial\theta^-} + \bar\theta^+\frac{\partial}{\partial\bar\theta^-} + 2i\theta^+\bar\theta^+\frac{\partial}{\partial t_{\mathrm{A}}}, \tag{4.5.27}$$

$$D^{--} = u^-_{\alpha}\frac{\partial}{\partial u^+_{\alpha}} + \theta^-\frac{\partial}{\partial\theta^+} + \bar\theta^-\frac{\partial}{\partial\bar\theta^+} + 2i\theta^-\bar\theta^-\frac{\partial}{\partial t_{\mathrm{A}}}. \tag{4.5.28}$$

Note that in the subspace (4.5.21) the second and the third terms vanish.

4.5.7 Involution symmetry

The superspace (4.5.17) admits an *involution symmetry* which commutes with supersymmetry transformations [6, 5]. We denote the involution with a sign $\tilde{}$, e.g. its action is $f \to \widetilde{f}$ for an arbitrary superfield $f(t_A,\, \theta^{\pm},\, \bar\theta^{\pm},\, u^{\pm}_{\alpha})$.

By definition, the involution transformation acts just as the ordinary complex conjugation *except* its action on the harmonics $u^{\pm}_{\alpha}$ for which it is

$$\widetilde{u^{\pm}_{\alpha}} = u^{\pm\alpha}, \qquad \widetilde{u^{\pm\alpha}} = -u^{\pm}_{\alpha}. \tag{4.5.29}$$

This gives

$$\widetilde{t_{\mathrm{A}}} = t_{\mathrm{A}}, \qquad \widetilde{\theta^{\pm}} = \bar\theta^{\pm}, \qquad \widetilde{\bar\theta^{\pm}} = -\theta^{\pm}. \tag{4.5.30}$$

The action of the involution transformation on harmonics can be seen as a composition of complex conjugation and point inversion on the sphere S^2. In general, the involution symmetry is very similar to the operation of complex conjugation, but is does not change U(1) charges of superfields. It allows one to put additional constraints on superfields (e.g. reality condition) and is used in construction of supersymmetric Lagrangians.

4.5.8 Supermultiplets of different dimensions

Let us now discuss possible minimal supermultiplets [3] which are commonly involved in the superfield description. All such $\mathcal{N}=4$ supermultiplets are usually referred to with three numbers, $(\mathbf{b},\mathbf{f},\mathbf{a})$, where

- $\mathbf{b}$ is the number of physical bosonic degrees of freedom ;
- $\mathbf{f}$ is the number of physical fermionic degrees of freedom ;
- $\mathbf{a}$ is the number of auxiliary nondynamical bosonic degrees of freedom, which are integrated out of final Lagrangians.

The widely used supermultiplets are $(\mathbf{4},\mathbf{4},\mathbf{0})$ and $(\mathbf{3},\mathbf{4},\mathbf{1})$ [43, 6, 42, 15] which usually describe four- and three-dimensional dynamics respectively. We discuss them in detail below. Also, the common way of obtaining $\mathcal{N}=8$ supersymmetry or higher dimensional theories with $\mathcal{N}=4$ supersymmetry is to take several superfields of such types.

Other supermultiplets include $(\mathbf{2},\mathbf{4},\mathbf{2})$ and $(\mathbf{1},\mathbf{4},\mathbf{3})$ [44, 45] which are usually used to build systems of many particles in two and one dimensions. These multiplets are not discussed in this manuscript.

In general, the number of physical fermions in all such multiplets is four, while the sum of physical and auxiliary bosonic degrees of freedom is also four. It is even possible to introduce the $(\mathbf{0},\mathbf{4},\mathbf{4})$ supermultiplet [6].

4.5.9 Supermultiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$

The derivative operators D^+, $\bar{D}^+$, D^{++} (anti)commute with each other and with supercharges. Because of this, it is possible to consider a superfield q^+ with U(1) charge $+1$ satisfying

$$D^+q^+=0, \qquad \bar{D}^+q^+=0, \qquad D^{++}q^+=0. \tag{4.5.31}$$

In the analytic superspace coordinates, the first and the second equations mean that q^+ depend only on θ^+ and $\bar{\theta}^+$, but not on θ^- and $\bar{\theta}^-$, see Eq. (4.5.24). In this way, the first and the second equations form the so-called *superfield analyticity conditions*.

When expanding the superfield $q^+(t_A,\,\theta^+,\,\bar{\theta}^+,\,u^\pm_\alpha)$ over spinor coordinates and the harmonics, one obtains an infinite set of physical fields. However, imposing also the condition $D^{++}q^+=0$ drastically reduces the number of such fields, making it finite. In the analytic basis (4.5.23), the solution of the constraints (4.5.31) reads

$$q^+=x^\alpha(t_\mathrm{A})u^+_\alpha-2\theta^+\chi(t_\mathrm{A})-2\bar{\theta}^+\bar{\chi}'(t_\mathrm{A})-2i\theta^+\bar{\theta}^+\partial_\mathrm{A}x^\alpha(t_\mathrm{A})u^-_\alpha \tag{4.5.32}$$

with

$$\partial_\mathrm{A}\equiv\frac{\partial}{\partial t_A} \tag{4.5.33}$$

and the factors -2 introduced for convenience. Thus, the $(\mathbf{4},\mathbf{4},\mathbf{0})$ superfield q^+ involves two complex bosonic coordinates x^α and two complex fermions – χ, $\bar{\chi}'$.

[3] In Section 4.2, we discussed *supermultiplets of quantum states* (characterized by their wave functions), whereas here by a *supermultiplet* we mean a superfield with certain “minimal” superfield content. In particular, the statement about the equality of the number of bosons and fermions (Section 4.2.4) is not applicable to the number of physical bosonic and fermionic fields in a system. This property was already observed in the example of the simplest quantum mechanics, see Section 4.4.4.

Let us also remark that the above property is specific only to quantum mechanics.

The constraints $D^+q^+ = \bar{D}^+q^+ = 0$ are akin to the discussed previously chirality constraints in $\mathcal{N} = 1$ four-dimensional supersymmetric field theories. Such constraints appear naturally in the HSS formalism and are common also in four-dimensional field theories. A possibility to impose the extra constraint $D^{++}q^+ = 0$ is specific for quantum mechanics only, where it has a pure kinematic nature. In $\mathcal{N} = 2$ supersymmetric field theories, the relation $D^{++}q^+ = 0$ is not a kinematic constraint, it is the equation of motion for the *free hypermultiplet* derived from the action $S = \int d^4x\, du\, d^4\theta^+\, \widetilde{q^+} D^{++}q^+$ [5].

The constraints (4.5.31) admit an involution symmetry $q^+ \to \widetilde{q^+}$ which commutes with supersymmetry transformations [6, 5] :

$$\widetilde{q^+} = [x_\alpha(t_{\rm A})]^*\, u^+_\alpha - 2\theta^+\bar{\chi}'^*(t_{\rm A}) + 2\bar{\theta}^+\chi^*(t_{\rm A}) - 2i\theta^+\bar{\theta}^+\partial_{\rm A}\,[x_\alpha(t_{\rm A})]^*\, u^-_\alpha. \tag{4.5.34}$$

It is straightforward to see that the field $\widetilde{q^+}$ satisfies the same constraints (4.5.31) as the field q^+.

As we will use the $(\mathbf{4},\mathbf{4},\mathbf{0})$ supermultiplet to construct supersymmetric quantum-mechanical system with four space dimensions, it will be more convenient for us to use the q^+ supermultiplet in different form. Namely, let us introduce the supermultiplet

$$q^{+\dot{\alpha}} = \left\{q^+,\, \widetilde{q^+}\right\}, \qquad \dot{\alpha} = 1, 2. \tag{4.5.35}$$

The involution symmetry can used to impose the pseudoreality condition on the field $q^{+\dot{\alpha}}$,

$$q^{+\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\widetilde{\left(q^{+\dot{\beta}}\right)}, \tag{4.5.36}$$

which is in fact equivalent to Eq. (4.5.35). In components,

$$q^{+\dot{\alpha}} = x^{\alpha\dot{\alpha}}(t_{\rm A})u^+_\alpha - 2\theta^+\chi^{\dot{\alpha}}(t_{\rm A}) - 2\bar{\theta}^+\bar{\chi}^{\dot{\alpha}}(t_{\rm A}) - 2i\theta^+\bar{\theta}^+\partial_{\rm A}x^{\alpha\dot{\alpha}}(t_{\rm A})u^-_\alpha. \tag{4.5.37}$$

The constraint (4.5.36) implies

$$x^{\alpha\dot{\alpha}} = -\left(x_{\alpha\dot{\alpha}}\right)^*, \qquad \bar{\chi}^{\dot{\alpha}} = \left(\chi_{\dot{\alpha}}\right)^*. \tag{4.5.38}$$

The form of the supermultiplet $q^{+\dot{\alpha}}$ suggests that one can associate the SU(2) group related to the dotted index $\dot{\alpha}$. This Pauli-Gürsey group [5] is also realized on the q^+ supermultiplet, but not manifestly.

Consequently, the quantum-mechanical system which will be discussed in the next chapter, inherits the $\mathrm{SO}(4) = \mathrm{SU}(2)_{\rm R} \times \mathrm{SU}(2)_{\rm PG}$ group composed from the R-symmetry group and the Pauli-Gürsey group. In general, this rotational SO(4) group is completely broken by the presence of a four-dimensional gauge field, see next chapter.

4.5.10 Supermultiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$

Instead of the coordinate superfield $q^{+\dot{\alpha}}$ one can deal with the analytic superfield L^{++} of charge +2 which encompass the supermultiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ and is subjected to the constraints

$$\begin{gathered} D^+L^{++} = \bar{D}^+L^{++} = 0, \\ D^{++}L^{++} = 0, \qquad \widetilde{(L^{++})} = -L^{++}, \end{gathered} \tag{4.5.39}$$

They restrict the analytic superfield L^{++} to have the appropriate off-shell component field content, namely $(\mathbf{3}, \mathbf{4}, \mathbf{1})$:

$$L^{++} = \ell^{\alpha\beta} u^+_\alpha u^+_\beta + 2i\theta^+ \chi^\alpha u^+_\alpha + 2i\bar\theta^+ \bar\chi^\alpha u^+_\alpha + \theta^+\bar\theta^+ [F - 2i\dot\ell^{\alpha\beta} u^+_\alpha u^-_\beta] \tag{4.5.40}$$

with

$$(\ell_{\alpha\beta})^* = -\ell^{\alpha\beta}, \qquad (\chi^\alpha)^* = \bar\chi_\alpha. \tag{4.5.41}$$

The multiplet L^{++} involves the 3-dimensional target space coordinates $\ell^{\alpha\beta} = \ell^{\beta\alpha}$, their fermionic partners χ^α, $\bar\chi^\alpha$ and a real auxiliary field F.

4.5.11 Harmonic integrals

The invariant actions involve the harmonic integral $\int du$. To find such integral of any function $f(u^\pm_\alpha)$, one should expand f in the harmonic Taylor series and, for each term, do the integrals using the rules

$$\int du\, 1 = 1, \qquad \int du\, u^+_{\{\alpha_1} \ldots u^+_{\alpha_k} u^-_{\alpha_{k+1}} \ldots u^-_{\alpha_{k+\ell}\}} = 0\ , \tag{4.5.42}$$

where the integrand in the right equation is symmetrized over all indices. The values of the integrals of all other harmonic monoms (for example, $\int du\, u^+_\alpha u^-_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}$) follow from (4.5.42) and the definitions (4.5.10), (4.5.11).

4.5.12 Notations in four-dimensional mechanics

We keep the notation of the previous chapter for the four-dimensional Euclidean space vector indices,

$$\mu, \nu = 0, 1, 2, 3. \tag{4.5.43}$$

The Euclidean four-dimensional sigma-matrices, however, are different ; we use the following SO(4) notation (compare with Eq. (3.1.6)) :

$$(\sigma_\mu)_{\alpha\dot\alpha} = \{i, \vec\sigma\}_{\alpha\dot\alpha}\,, \qquad \left(\sigma^\dagger_\mu\right)^{\dot\alpha\alpha} = \{-i, \vec\sigma\}^{\dot\alpha\alpha}\,, \tag{4.5.44}$$

where $\vec\sigma$ are ordinary Pauli matrices. (These are more or less the conventions of [33] rotated to Euclidean space.) The matrix $\sigma^\dagger_\mu$ is obtained from the matrix σ_μ by the operation of raising of indices :

$$\left(\sigma^\dagger_\mu\right)^{\dot\alpha\alpha} = -\varepsilon^{\dot\alpha\dot\gamma}\varepsilon^{\alpha\gamma}(\sigma_\mu)_{\gamma\dot\gamma}\,. \tag{4.5.45}$$

The matrices σ_μ, $\sigma^\dagger_\mu$ satisfy the identities

$$\begin{gathered} \sigma_\mu\sigma^\dagger_\nu + \sigma_\nu\sigma^\dagger_\mu = \sigma^\dagger_\mu\sigma_\nu + \sigma^\dagger_\nu\sigma_\mu = 2\delta_{\mu\nu}, \\ \sigma^\dagger_\mu\sigma_\nu - \sigma^\dagger_\nu\sigma_\mu = 2i\,\eta^a_{\mu\nu}\sigma_a, \\ \sigma_\mu\sigma^\dagger_\nu - \sigma_\nu\sigma^\dagger_\mu = 2i\,\bar\eta^a_{\mu\nu}\sigma_a, \end{gathered} \tag{4.5.46}$$

where $\eta^a_{\mu\nu}$, $\bar\eta^a_{\mu\nu}$ are the 't Hooft symbols,

$$\eta^a_{ij} = \bar\eta^a_{ij} = \varepsilon_{aij}, \quad \eta^a_{i0} = -\eta^a_{0i} = \bar\eta^a_{0i} = -\bar\eta^a_{i0} = \delta_{ai} \tag{4.5.47}$$

(σ_a – Pauli matrices, indices a, i, j run from 1 to 3). They are self-dual and anti-self-dual respectively,

$$\eta^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}\eta^a_{\rho\lambda}, \qquad \bar{\eta}^a_{\mu\nu} = -\frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}\bar{\eta}^a_{\rho\lambda}, \tag{4.5.48}$$

with the convention

$$\varepsilon_{0123} = -1. \tag{4.5.49}$$

Another useful identity is

$$\sigma_2\sigma^T_\mu\sigma_2 = \ -\sigma^\dagger_\mu. \tag{4.5.50}$$

The $(\mathbf{4},\mathbf{4},\mathbf{0})$ supermultiplet (4.5.37) involves the bosonic field $x^{\alpha\dot\alpha}(t)$. In fact, such bosonic field is equivalent to a real four-vector in the Euclidean space. Let us describe the transformation between the spinor notation $v^{\alpha\dot\alpha}$ and the corresponding vector notation v^μ for an arbitrary field v :

$$\begin{aligned} v_{\alpha\dot\alpha} &= v_\mu(\sigma_\mu)_{\alpha\dot\alpha}, \\ v_\mu &= \tfrac{1}{2}v_{\alpha\dot\alpha}(\sigma^\dagger_\mu)^{\dot\alpha\alpha} \ = \ -\tfrac{1}{2}v^{\alpha\dot\alpha}(\sigma_\mu)_{\alpha\dot\alpha}, \\ v^{\alpha\dot\alpha} &= \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}v_{\beta\dot\beta} \ = \ -v_\mu(\sigma^\dagger_\mu)^{\dot\alpha\alpha}. \end{aligned} \tag{4.5.51}$$

Particularly, it is straightforward to check that for the field $x^{\alpha\dot\alpha}$ with the constraint (4.5.38) the corresponding vector field x^μ is real.

4.5.13 Relations for the η symbols

Here we give a list of relations for the 't Hooft symbols $\eta^a_{\mu\nu}$ and $\bar\eta^a_{\mu\nu}$, defined by Eqs. (4.5.46), (4.5.47) :

$$\begin{aligned} &\eta^a_{\mu\nu} = -\eta^a_{\nu\mu}, \\ &\eta^a_{\mu\nu}\eta^a_{\mu\lambda} = 3\delta_{\nu\lambda}, \\ &\eta^a_{\mu\nu}\eta^b_{\mu\nu} = 4\delta^{ab}, \\ &\eta^a_{\mu\nu}\eta^a_{\gamma\lambda} = \delta_{\mu\gamma}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\gamma} + \varepsilon_{\mu\nu\gamma\lambda}, \\ &\varepsilon_{\mu\nu\lambda\sigma}\eta^a_{\gamma\sigma} = \delta_{\gamma\mu}\eta^a_{\nu\lambda} - \delta_{\gamma\nu}\eta^a_{\mu\lambda} + \delta_{\gamma\lambda}\eta^a_{\mu\nu}, \\ &\eta^a_{\mu\nu}\eta^b_{\mu\lambda} = \delta^{ab}\delta_{\nu\lambda} + \varepsilon^{abc}\eta^c_{\nu\lambda}, \\ &\varepsilon^{abc}\eta^b_{\mu\nu}\eta^c_{\gamma\lambda} = \delta_{\mu\gamma}\eta^a_{\nu\lambda} - \delta_{\mu\lambda}\eta^a_{\nu\gamma} - \delta_{\nu\gamma}\eta^a_{\mu\lambda} + \delta_{\nu\lambda}\eta^a_{\mu\gamma}, \\ &\eta^a_{\mu\nu}\bar\eta^b_{\mu\nu} = 0, \\ &\eta^a_{\gamma\mu}\bar\eta^b_{\gamma\lambda} = \eta^a_{\gamma\lambda}\bar\eta^b_{\gamma\mu}. \end{aligned} \tag{4.5.52}$$

To pass from the relations for $\eta^a_{\mu\nu}$, to those for $\bar\eta^a_{\mu\nu}$ it is necessary to make the substitution

$$\eta^a_{\mu\nu} \to \bar\eta^a_{\mu\nu}, \quad \varepsilon_{\mu\nu\gamma\delta} \to -\varepsilon_{\mu\nu\gamma\delta}. \tag{4.5.53}$$

Chapitre 5 New supersymmetric models of quantum mechanics

This is the central chapter of the manuscript describing certain new SQM models discussed and studied in the papers [2, 3, 4]. The explicit form of the corresponding superfield and component actions, as well as of the quantum Hamiltonians and supercharges is given. The brief summary of the results is the following.

It is shown that the Hamiltonian $H = \not{\mathcal{D}}^2$, where $\not{\mathcal{D}}$ is flat four-dimensional Dirac operator in an external *self-dual* gauge background, Abelian or non-Abelian, is supersymmetric with $\mathcal{N} = 4$ supersymmetry. A generalization of this Hamiltonian to the motion on a curved conformally flat four-dimensional manifold exists. For an *Abelian* self-dual background, the corresponding Lagrangian can be derived from certain harmonic superspace expressions.

If the Hamiltonian involves a *non-Abelian* self-dual gauge field, one can construct the Lagrangian formulation by introducing auxiliary bosonic variables with Wess-Zumino type action. For a special class of such Lagrangians when the gauge group is SU(2) and the gauge field is expressed in the ‘t Hooft ansatz form, it is possible to give a superfield description using the harmonic superspace formalism. As a new explicit example, the $\mathcal{N} = 4$ mechanics with *Yang monopole* in $\mathbb{R}^5$ (which coincides with an instanton on S^4) is considered.

Independently, a similar system with $\mathcal{N} = 4$ supersymmetry in *three dimensions* also admits the superfield description. Although the three-dimensional system involves different superfields as compared with the four-dimensional case, its component Lagrangian and Hamiltonian appear to be the three-dimensional reduction of the mentioned four-dimensional system. The off-shell $\mathcal{N} = 4$ supersymmetry requires the gauge field to be a static form of the ’t Hooft ansatz for the four-dimensional self-dual SU(2) gauge fields, that is a particular solution of Bogomolny equations for *BPS monopoles*.

5.1 Fermions in four-dimensional self-dual background

5.1.1 Matrix description

Consider the Dirac operator in flat four-dimensional Euclidean space

$$\not{\mathcal{D}} = \sum_{\mu=0,1,2,3} \mathcal{D}_\mu \gamma_\mu \, , \tag{5.1.1}$$

where $\mathcal{D}_\mu = \partial_\mu - i\mathcal{A}_\mu$ with $\mathcal{A}_\mu$ being a gauge field and γ_μ are Euclidean anti-Hermitian gamma–matrices,

$$\gamma_\mu = \begin{pmatrix} 0 & -\sigma_\mu^\dagger \\ \sigma_\mu & 0 \end{pmatrix}, \qquad \{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}. \tag{5.1.2}$$

The matrices σ_μ and $\sigma_\mu^\dagger$ were introduced in Eq. (4.5.44). The Hamiltonians we are going to construct enjoy $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ covariance such that the undotted spinor index refers to the first SU(2) factor, while the dotted one to the second.

Consider the operator

$$H = \frac{1}{2}\not{\mathcal{D}}^2 = -\frac{1}{2}\mathcal{D}^2 - \frac{i}{4}\mathcal{F}_{\mu\nu}\gamma_\mu\gamma_\nu \, , \tag{5.1.3}$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i\,[\mathcal{A}_\mu, \mathcal{A}_\nu]$ is the gauge field strength. It is well known that nonzero eigenvalues of the Euclidean Dirac operator come in pairs $(-\lambda, \lambda)$ and hence the spectrum of the Hamiltonian H is double-degenerate for all excited states. This means that, for any external field $\mathcal{A}_\mu$, this Hamiltonian is supersymmetric [9] admitting two different anticommuting real supercharges : $\not{\mathcal{D}}$ and $i\not{\mathcal{D}}\gamma_5$ $(\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3)$. Suppose now that the background field is self-dual,

$$\mathcal{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\delta}\mathcal{F}_{\rho\delta} \quad \longleftrightarrow \quad \mathcal{F}_{\mu\nu} = \eta^a_{\mu\nu} B_a \, , \tag{5.1.4}$$

where $\eta^a_{\mu\nu}$ are the 't Hooft symbols defined in Eq. (4.5.46). One can be easily convinced that in this case the Hamiltonian admits *four* different Hermitian square roots S_A that satisfy the extended supersymmetry algebra (4.1.8) ; we repeat these relations here :

$$\{S_A, S_B\} = 4\delta_{AB} H. \tag{5.1.5}$$

One of the choices is

$$\begin{aligned} S_1 &= \not{\mathcal{D}} = \gamma_0\mathcal{D}_0 + \gamma_1\mathcal{D}_1 + \gamma_2\mathcal{D}_2 + \gamma_3\mathcal{D}_3, \\ S_2 &= \gamma_0\mathcal{D}_3 + \gamma_1\mathcal{D}_2 - \gamma_2\mathcal{D}_1 - \gamma_3\mathcal{D}_0, \\ S_3 &= \gamma_0\mathcal{D}_2 - \gamma_1\mathcal{D}_3 - \gamma_2\mathcal{D}_0 + \gamma_3\mathcal{D}_1, \\ S_4 &= \gamma_0\mathcal{D}_1 - \gamma_1\mathcal{D}_0 + \gamma_2\mathcal{D}_3 - \gamma_3\mathcal{D}_2. \end{aligned} \tag{5.1.6}$$

Introducing the complex supercharges

$$\begin{aligned} Q_1 &= (S_1 - iS_2)/2, \qquad & Q_2 &= (S_3 - iS_4)/2, \\ \bar{Q}^1 &= (S_1 + iS_2)/2, \qquad & \bar{Q}^2 &= (S_3 + iS_4)/2, \end{aligned} \tag{5.1.7}$$

we obtain the standard $\mathcal{N} = 4$ supersymmetry algebra (4.5.6). Correspondingly, the excited spectrum of H is four-fold degenerate, while the spectrum of $\not{\mathcal{D}}$ consists of the quartets involving two degenerate positive and two degenerate negative eigenvalues. Note that, in contrast to $\not{\mathcal{D}}$, the operator $\not{\mathcal{D}}\gamma_5$ is not expressed into a linear combination of S_A. In other words, the $\mathcal{N} = 2$ supersymmetry algebra with the operators $\not{\mathcal{D}}(1 \pm \gamma_5)$ is not a subalgebra of the $\mathcal{N} = 4$ algebra (4.5.6).

The algebra (5.1.5) with supercharges (5.1.6) holds for any self-dual field, irrespectively of whether it is Abelian or non-Abelian. Thus, the additional 2-fold degeneracy of the spectrum of the Dirac operator mentioned above should be there for a generic self-dual field. One particular example of a non-Abelian self-dual field is the instanton solution, where this degeneracy was observed back in [10] (see Eqs. (4.15) there). The generalization of the Dirac operator and (anti)self-duality conditions to higher-dimensional manifolds was considered in Ref. [14].

5.1.2 Covariant description

To make contact with the Lagrangian (and, especially, superfield) description, it is convenient to introduce holomorphic fermion variables $\psi_{\dot\alpha}$ and $\bar\psi^{\dot\alpha}$, which satisfy the standard anticommutation relations

$$\{\psi_{\dot\alpha}, \psi_{\dot\beta}\} \;=\; \{\bar\psi^{\dot\alpha}, \bar\psi^{\dot\beta}\} \;=\; 0, \qquad \{\bar\psi^{\dot\alpha}, \psi_{\dot\beta}\} = \delta^{\dot\alpha}_{\dot\beta} \tag{5.1.8}$$

and are realized as matrices in the following way :

$$\begin{aligned} \psi_{\dot 1} &= \frac{-\gamma_0 + i\gamma_3}{2}, \qquad & \bar\psi^{\dot 1} &= \frac{\gamma_0 + i\gamma_3}{2}, \\ \psi_{\dot 2} &= \frac{\gamma_2 + i\gamma_1}{2}, \qquad & \bar\psi^{\dot 2} &= \frac{-\gamma_2 + i\gamma_1}{2}. \end{aligned} \tag{5.1.9}$$

Then two complex supercharges (5.1.7) are expressed in a very simple way, namely

$$\begin{aligned} Q_\alpha &= \left(\sigma_\mu \bar\psi\right)_\alpha (\hat p_\mu - \mathcal{A}_\mu)\,, \\ \bar Q^\alpha &= \left(\psi \sigma^\dagger_\mu\right)^\alpha (\hat p_\mu - \mathcal{A}_\mu)\,, \end{aligned} \tag{5.1.10}$$

with $\hat p_\mu = -i\partial_\mu$. The Hamiltonian (5.1.3) is expressed in these terms as

$$H = \frac{1}{2}\,(\hat p_\mu - \mathcal{A}_\mu)^2 + \frac{i}{4}\,\mathcal{F}_{\mu\nu}\,\psi\sigma^\dagger_\mu\sigma_\nu\bar\psi\;. \tag{5.1.11}$$

It is clear now why the spinor indices in Eqs. (4.5.6) are undotted, while in Eq. (5.1.9) they are dotted : the supercharges are rotated by the first SU(2) and the variables $\psi_{\dot\alpha}$ by the second [1]. A careful distinction between two different SU(2) factors allows one to understand better the reason why the supercharges (5.1.10) satisfy the simple algebra (4.5.6) in a self-dual background. The self-dual field density $\mathcal{F}$ carries in the spinor

[1] Note that complex conjugation leaves the spinors in the same representation, the symmetry group here is SO(4) rather than SO(1, 3).

notation only dotted indices. Therefore any expression involving $\mathcal{F}, \psi, \bar{\psi}$ is a scalar with respect to undotted SU(2). The only such scalar that can appear in the right hand side of the anticommutators of the supercharges $\{Q_\alpha, \bar{Q}^\beta\}$ is the structure which is proportional to δ_α^β, i.e. the Hamiltonian. No other operator is allowed.

In the *Abelian* case, the supercharges (5.1.10) and the Hamiltonian (5.1.11) are scalar operators not carrying matrix indices anymore. This allows one to derive the Lagrangian,

$$L = \frac{1}{2}\, \dot{x}_\mu \dot{x}_\mu + \mathcal{A}_\mu(x)\dot{x}_\mu + i\bar{\psi}^{\dot{\alpha}}\dot{\psi}_{\dot{\alpha}} - \frac{i}{4}\mathcal{F}_{\mu\nu}\, \psi\sigma_\mu^\dagger\sigma_\nu\bar{\psi} \ . \qquad (5.1.12)$$

In the non-Abelian case, the expressions (5.1.10) and (5.1.11) still keep their color matrix structure, and one cannot derive the Lagrangian in a so straightforward way. One of the ways to handle the matrix structure is to introduce a set of color fermion variables (say, in the fundamental representation of the group) and impose the extra constraint considering only the sector with unit fermion charge [9]. An alternative non-Abelian construction of the Lagrangian is presented in Section 5.3. In this section, we limit ourselves only to Lagrangians for Abelian fields.

5.1.3 The generalization to conformally flat metric

As will be demonstrated explicitly in Section 5.2, the component Lagrangian (5.1.12) follows from the superfield action written earlier by Ivanov and Lechtenfeld in the framework of harmonic superspace approach [6]. We will see that one can naturally derive in this way a σ-model type generalization of the Lagrangian (5.1.12) describing the motion over the manifold with nontrivial conformally flat metric $ds^2 = \{f(x)\}^{-2}\, dx_\mu dx_\mu$. It is written as follows :

$$\begin{aligned} L \ = \ & \frac{1}{2}f^{-2}\, \dot{x}_\mu\dot{x}_\mu + i\bar{\psi}^{\dot{\alpha}}\dot{\psi}_{\dot{\alpha}} + \frac{1}{4}\left\{3\left(\partial_\mu f\right)^2 - f\partial^2 f\right\}\psi^4 + \frac{i}{2}f^{-1}\partial_\mu f\, \dot{x}_\nu\, \psi\sigma^\dagger_{[\mu}\sigma_{\nu]}\bar{\psi} \\ & + \mathcal{A}_\mu(x)\dot{x}^\mu - \frac{i}{4}f^2\mathcal{F}_{\mu\nu}\, \psi\sigma^\dagger_\mu\sigma_\nu\bar{\psi}, \end{aligned} \qquad (5.1.13)$$

The corresponding quantum Nöether supercharges and the Hamiltonian are (see also Section 5.2.3 for details)

$$\begin{aligned} Q_\alpha &= f\left(\sigma_\mu\bar{\psi}\right)_\alpha(\hat{p}_\mu - \mathcal{A}_\mu) - \psi_{\dot{\gamma}}\bar{\psi}^{\dot{\gamma}}\left(\sigma_\mu\bar{\psi}\right)_\alpha i\partial_\mu f, \\ \bar{Q}^\alpha &= \left(\psi\sigma^\dagger_\mu\right)^\alpha(\hat{p}_\mu - \mathcal{A}_\mu)\, f + i\partial_\mu f\left(\psi\sigma^\dagger_\mu\right)^\alpha \cdot \psi_{\dot{\gamma}}\bar{\psi}^{\dot{\gamma}}, \end{aligned} \qquad (5.1.14)$$

$$\begin{aligned} H = & \frac{1}{2}f\left(\hat{p}_\mu - \mathcal{A}_\mu\right)^2 f + \frac{i}{4}f^2\, \mathcal{F}_{\mu\nu}\, \psi\sigma^\dagger_\mu\sigma_\nu\bar{\psi} \\ & - \frac{1}{2}fi\partial_\mu f\left(\hat{p}_\nu - \mathcal{A}_\nu\right)\psi\sigma^\dagger_{[\mu}\sigma_{\nu]}\bar{\psi} + f\partial^2 f\left\{\psi_{\dot{\gamma}}\bar{\psi}^{\dot{\gamma}} - \frac{1}{2}\left(\psi_{\dot{\gamma}}\bar{\psi}^{\dot{\gamma}}\right)^2\right\}. \end{aligned} \qquad (5.1.15)$$

On the other hand, one can explicitly calculate the anticommutators of the supercharges (5.1.14) for any self-dual [2] field $\mathcal{A}_\mu(x)$, Abelian or non-Abelian, and verify that the algebra

[2]Anti-self-duality conditions are obtained when one interchanges σ_μ and $\sigma_\mu^\dagger$ in all the formulas. This is equivalent to the interchange of two spinor representations of SO(4).

(4.5.6) holds. While doing this, the use of the following Fierz identity

$$\left(\bar{\psi}\sigma_\mu^\dagger\right)^\beta \left(\sigma_\nu \psi\right)_\alpha - \left(\sigma_\mu \bar{\psi}\right)_\alpha \left(\psi \sigma_\nu^\dagger\right)^\beta = \delta_\alpha^\beta \, \bar{\psi}\sigma_\mu^\dagger \sigma_\nu \psi \ , \tag{5.1.16}$$

which can be proven using (4.5.50), is convenient.

Note that, with a nontrivial factor $f(x)$, the Dirac operator $\not{\mathcal{D}}$ in a conformally flat background can be expressed as a linear combination of Q_α and $\bar{Q}^\alpha$ *only* if one also adds a certain *torsion* proportional to the derivatives of $f(x)$ [46]. The Hamiltonian (5.1.15) would also coincide with $\not{\mathcal{D}}^2/2$ in this case. In fact, generalization of the system with the Hamiltonian (5.1.11) to the conformally flat case, which preserves $\mathcal{N} = 4$ supersymmetry, as in Eq. (5.1.15) always involves the torsion field. The Hamiltonian (5.1.11) also admits another type of $\mathcal{N} = 4$ extension to a curved space endowed with Hyper-Kähler metric and without a torsion [14]. In its most generality, the Hamiltonian involves weak HKT geometry – the generalization of Hyper-Kähler metric obtained by introduction of conformal factor and torsion [46].

The model (5.1.13), (5.1.14), (5.1.15) is a close relative to the model constructed in Ref. [12] (see Eqs. (30) and (31) there), which describes the motion on a *three*-dimensional conformally flat manifold in external magnetic field and a scalar potential. In fact, the latter model can be obtained from the former, if assuming that the metric and the vector potential $\mathcal{A}_\mu \equiv (\Phi, \vec{\mathcal{A}})$ depend only on three spatial coordinates x_i. If assuming further that the metric is flat, one is led to the Hamiltonian [11]

$$H \ = \ \frac{1}{2}\left(\hat{\vec{p}} - \vec{\mathcal{A}}\right)^2 + \frac{1}{2}U^2 + \vec{\nabla}U \, \psi\vec{\sigma}\bar{\psi}, \tag{5.1.17}$$

which is supersymmetric under the condition $\mathcal{F}_{ij} = \varepsilon_{ijk}\partial_k U$ (the 3-dimensional reduction of the four-dimensional self-duality condition). It was noticed in Ref. [12] that the effective Hamiltonian of a chiral supersymmetric electrodynamics in finite spatial volume belongs to this class with $U \propto 1/|\vec{A}|$. The vector potential $\vec{\mathcal{A}}(\vec{A})$ describes in this case a Dirac magnetic monopole such that the Berry phase appears. The three dynamical variables $\vec{A}$ (do not confuse with curly $\vec{\mathcal{A}}$!) have in this case the meaning of the zero Fourier harmonic of the vector potential in the original field theory. In the leading order, the metric is flat. When higher loop corrections are included, a (conformally flat!) metric on the moduli space $\{\vec{A}\}$ appears.

Performing the Hamiltonian reduction of Eq. (5.1.15) with non-Abelian $\mathcal{A}_\mu$, a non-Abelian generalization of Eq. (5.1.17) can easily be derived. It keeps the gauge structure of Eq. (5.1.17) with matrix-valued $\vec{\mathcal{A}}$ and U satisfying the condition $\mathcal{F}_{ij} = \varepsilon_{ijk}\mathcal{D}_k U$. Note that such Hamiltonian does *not* coincide with the non-Abelian 3-dimensional Hamiltonian derived in Ref. [21].

5.1.4 Example of constant gauge field

As an illustration, consider the system described by the Hamiltonian (5.1.11) in a constant self-dual Abelian background. The constant self-dual field strength $\mathcal{F}_{\mu\nu} = \eta^a_{\mu\nu} B_a$ is parametrized by three independent components. Let us direct B^a along the third axis, $B_a = (0, 0, B)$, and choose the gauge

$$\mathcal{A}_0 = Bx_3, \qquad \mathcal{A}_2 = Bx_1, \qquad \mathcal{A}_1 = \mathcal{A}_3 = 0. \tag{5.1.18}$$

The Hamiltonian (5.1.11) acquires the form

$$H = \left\{ \frac{1}{2} \left(\hat{p}_0 - Bx_3\right)^2 + \frac{1}{2}\hat{p}_3^2 + B\left(\chi_1\bar{\chi}^1 - \frac{1}{2}\right)\right\} + \left\{ \frac{1}{2} \left(\hat{p}_2 - Bx_1\right)^2 + \frac{1}{2}\hat{p}_1^2 + B\left(\chi_2\bar{\chi}^2 - \frac{1}{2}\right)\right\}. \quad (5.1.19)$$

For convenience, we have introduced notations $\chi_1 = \bar{\psi}^{\dot{1}}$, $\bar{\chi}^1 = \psi_{\dot{1}}$, $\chi_2 = \psi_{\dot{2}}$, $\bar{\chi}^2 = \bar{\psi}^{\dot{2}}$. The Hamiltonian is thus reduced to the sum $H_1 + H_2$ of two independent (acting in different Hilbert spaces) supersymmetric Hamiltonians, each describing the 2-dimensional motion of an electron in homogeneous orthogonal to the plane magnetic field $\vec{B}$. The bosonic sector of each such Hamiltonian corresponds to the spin projection $\vec{s}\vec{B}/|\vec{B}| = -1/2$, and the fermionic sector to the spin projection $\vec{s}\vec{B}/|\vec{B}| = 1/2$. This is the first and the simplest supersymmetric quantum problem ever considered [47]. The energy levels for each Hamiltonian are $\varepsilon_i = B\left(n_i + \frac{1}{2} + s_i\right)$, $n_i \geq 0$ – integers, $s_i = \pm\frac{1}{2}$. Each level of H_i is doubly degenerate. Besides, there is an infinite degeneracy associated with the positions of the center of the orbit along the axes 1 and 3 that are proportional to the integrals of motion p_2 and p_0. The full spectrum

$$E = B\left(n_1 + n_2 + 1 + s_1 + s_2\right) \quad (5.1.20)$$

is thus 4-fold degenerate at each level (except for the singlet state with $E = 0$) for given p_0, p_2.

It might be instructive to explicitly associate this degeneracy with the action of supercharges (5.1.10). Let us assume for definiteness $B > 0$. One can represent Q_α as

$$Q_1 = \sqrt{2B}\left(b\chi_1 + a^\dagger\bar{\chi}^2\right), \qquad Q_2 = \sqrt{2B}\left(a\chi_1 - b^\dagger\bar{\chi}^2\right), \quad (5.1.21)$$

where $a^\dagger$, $b^\dagger$ and a, b are the creation and annihilation operators,

$$a = \frac{1}{\sqrt{2B}}\left(\hat{p}_1 - iBx_1 \ + ip_2\right), \qquad b = \frac{1}{\sqrt{2B}}\left(\hat{p}_3 - iBx_3 \ + ip_0\right), \quad (5.1.22)$$

$$\left[a, a^\dagger\right] = 1, \qquad \left[b, b^\dagger\right] = 1. \quad (5.1.23)$$

In these notations, the Hamiltonian (5.1.19) takes a very simple form

$$H = B\left\{a^\dagger a + b^\dagger b + \chi_1\bar{\chi}^1 + \chi_2\bar{\chi}^2\right\}. \quad (5.1.24)$$

Obviously, the energy levels of the Hamiltonian (5.1.19) are defined by two integrals of motion $p_{2,0}$, two oscillator excitation numbers $n_{1,2}$ and two spins $s_{1,2}$, as in Eq. (5.1.20). For each p_2, p_0, there is a unique ground zero energy state $|0\rangle$ annihilated by all supercharges. A quartet of excited states can be represented as

$$|n_1, n_2\rangle, \quad Q_1^\dagger|n_1, n_2\rangle, \quad Q_2^\dagger|n_1, n_2\rangle, \quad Q_1^\dagger Q_2^\dagger|n_1, n_2\rangle, \quad (5.1.25)$$

where the state

$$|n_1, n_2\rangle \equiv \chi_1 \cdot \left(a^\dagger\right)^{n_1}\left(b^\dagger\right)^{n_2}|0\rangle$$

of energy $E = B(n_1 + n_2 + 1)$ is annihilated by both Q_1 and Q_2.

For each p_2, p_0, there are N such quartets at the energy level $E = BN$.

5.2 Harmonic superspace description in the Abelian case

In this section, we derive the Hamiltonian (5.1.15) from the harmonic superspace approach. The introduction to the harmonic superspace and its salient features and definitions in application to quantum mechanical problems were already discussed in the previous chapter. The relevant superfield action was written in [6]. Let us show here that the corresponding component Lagrangian coincides with (5.1.13). The corresponding supercharges (5.1.14) and the Hamiltonian (5.1.15) involve an Abelian self-dual gauge field $\mathcal{A}_\mu(x)$. The non-Abelian field case is discussed later in this chapter.

5.2.1 Superfield content

Let us introduce a doublet of superfields $q^{+\dot\alpha}$ with charge +1 ($D^0 q^{+\dot\alpha} = q^{+\dot\alpha}$) satisfying the constraints (4.5.31), (4.5.36). The index $\dot\alpha$ is the fundamental representation index of an additional external (Pauli-Gürsey) group SU(2). The solution for these constraints in the analytical basis was written in Eqs. (4.5.37), (4.5.38). It can be presented in the central basis (4.5.17) as $q^{+\dot\alpha} = u^+_\alpha q^{\alpha\dot\alpha}$, where $q^{\alpha\dot\alpha}$ does not depend on $u^\pm_\alpha$ (the latter follows from the constraint $D^{++}q^{+\dot\alpha} = 0$ and the definition $D^{++} = u^+_\alpha \frac{\partial}{\partial u^-_\alpha}$). It is convenient to go over to the four-dimensional vector notation (4.5.51), introducing

$$q_\mu = -\frac{1}{2} \left(\sigma_\mu\right)_{\alpha\dot\alpha} q^{\alpha\dot\alpha}, \qquad q^{+\dot\alpha} = -q_\mu \left(\sigma^\dagger_\mu\right)^{\dot\alpha\alpha} u^+_\alpha. \tag{5.2.1}$$

Now, q_μ is a vector with respect to the group SO(4) = $\mathrm{SU_R}(2) \times \mathrm{SU_{PG}}(2)$, with the first factor representing the $\mathcal{N} = 4$ R-symmetry group and the second one being the Pauli-Gürsey global SU(2) group which rotates the dotted "flavor" indices.

The pseudoreality condition (4.5.36) implies that the superfield q_μ is real. The latter is expressed in components as follows :

$$q_\mu = x_\mu + \theta\sigma_\mu\chi + \bar\theta\sigma_\mu\bar\chi - \frac{i}{2}\dot x_\nu\, \bar\theta\sigma_{[\mu}\sigma^\dagger_{\nu]}\theta + \frac{i}{2}\bar\theta\sigma_\mu\dot\chi\,\theta^2 - \frac{i}{2}\theta\sigma_\mu\dot{\bar\chi}\,\bar\theta^2 - \frac{1}{4}\ddot x_\mu\,\theta^4 \ , \tag{5.2.2}$$

where $\theta^2 \equiv \theta^\alpha\theta_\alpha$, $\bar\theta^2 \equiv \bar\theta^\alpha\bar\theta_\alpha$, $\theta^4 \equiv \theta^2\bar\theta^2$. Moreover, the first equality in Eq. (4.5.38) implies that

$$x_\mu = -\frac{1}{2} x^{\alpha\dot\alpha}(\sigma_\mu)_{\alpha\dot\alpha} \tag{5.2.3}$$

is also real, and we are left with four dynamic bosonic variables.

5.2.2 Superfield action

The classical $\mathcal{N} = 2$ supersymmetric action for the superfield q_μ can now be written. It consists of two parts, $S = S_{\rm kin} + S_{\rm int}$. The kinetic part,

$$S_{\rm kin} = \int dt\, d^4\theta du\, R'_{\rm kin}(q^{+\dot\alpha}, q^{-\dot\beta}, u^\pm_\gamma) = \int dt\, d^4\theta\, R_{\rm kin}(q_\mu), \tag{5.2.4}$$

depends on an arbitrary function $R_{\rm kin}(q_\mu)$. Note that one can forget the harmonic superspace coordinates here and work in an ordinary superspace. In other words, the kinetic term $S_{\rm kin}$ does not require the additional coordinates $u^\pm_\alpha$ of the harmonic superspace. Plugging (5.2.2) into (5.2.4) and adding/subtracting proper total derivatives, one obtains

$$S_{\rm kin} = \int dt \left\{ \frac{1}{2} g(x)\, \dot{x}_\mu \dot{x}_\mu + \frac{i}{2} g(x) \left(\bar{\chi}^{\dot\alpha} \dot{\chi}_{\dot\alpha} - \dot{\bar{\chi}}^{\dot\alpha} \chi_{\dot\alpha} \right) \right.$$
$$\left. + \frac{1}{8} \partial^2 g(x)\, \chi^4 - \frac{i}{4} \partial_\mu g(x)\, \dot{x}_\nu\, \chi \sigma^\dagger_{[\mu} \sigma_{\nu]} \bar{\chi} \right\}, \quad (5.2.5)$$

where $g(x) = \frac{1}{2} \partial^2_x R_{\rm kin}(x)$ and $\chi^4 = \chi^{\dot\alpha} \chi_{\dot\alpha}\, \bar{\chi}^{\dot\beta} \bar{\chi}_{\dot\beta}$.

To couple x_μ to an external gauge field, one should add the interaction term $S_{\rm int}$ that represents an integral over *analytic* superspace in the harmonic superspace,

$$S_{\rm int} = \int dt\, du\, d\bar{\theta}^+ d\theta^+ R^{++}_{\rm int} \left(q^{+\dot\alpha}(t_{\rm A}, \theta^+, \bar{\theta}^+), u^\pm_\gamma \right). \quad (5.2.6)$$

We choose $R^{++}_{\rm int}$ (it carries the charge +2, $D^0 R^{++}_{\rm int} = 2R^{++}_{\rm int}$) satisfying the condition $\widetilde{R^{++}_{\rm int}} = -R^{++}_{\rm int}$ (the involution operation $\widetilde{X}$ was defined in Section 4.5.7) such that the action (5.2.6) is real. In contrast to the kinetic term, the interaction term involves the dependence on harmonics $u^\pm_\gamma$ and thus cannot be written in terms of superfields of ordinary superspace (4.5.3).

To do the integral over θ^+ and $\bar{\theta}^+$, we substitute Eq. (4.5.37) into (5.2.6) and expand the latter in Taylor series over θ^+, $\bar{\theta}^+$, keeping only terms $\sim \theta^+ \bar{\theta}^+$:

$$R^{++}_{\rm int}(q^{+\dot\alpha}, u^\pm_\gamma) = \partial_{+\dot\alpha} R^{++}_{\rm int} \cdot \left(-2i \theta^+ \bar{\theta}^+ u^-_\alpha \dot{x}^{\alpha\dot\alpha} \right)$$
$$+ 2 \partial_{+\dot\alpha} \partial_{+\dot\beta} R^{++}_{\rm int} \cdot \theta^+ \bar{\theta}^+ \left(\chi^{\dot\alpha} \bar{\chi}^{\dot\beta} + \chi^{\dot\beta} \bar{\chi}^{\dot\alpha} \right) + \ldots \quad (5.2.7)$$

(ellipsis denote terms not proportional to $\theta^+ \bar{\theta}^+$) with

$$\partial_{+\dot\alpha} R^{++}_{\rm int}(x, u) \equiv \frac{\partial R^{++}_{\rm int}(x^{+\dot\gamma}, u^\pm_\gamma)}{\partial x^{+\dot\alpha}} \quad (5.2.8)$$

and similarly for $\partial_{+\dot\alpha} \partial_{+\dot\beta} R^{++}_{\rm int}$. Let us also pass to vector notation x_μ for the coordinates $x^{\alpha\dot\alpha}$, see Eq. (5.2.3). Consequently,

$$x^{+\dot\alpha} \equiv x^{\alpha\dot\alpha} u^+_\alpha = -x_\mu \left(\sigma^\dagger_\mu \right)^{\dot\alpha\alpha} u^+_\alpha. \quad (5.2.9)$$

Then

$$S_{\rm int} = \int dt\, du \left\{ 2i \left(\sigma^\dagger_\mu \right)^{\dot\alpha\alpha} \partial_{+\dot\alpha} R^{++}_{\rm int}\, u^-_\alpha \cdot \dot{x}_\mu - 4 \chi^{\dot\alpha} \bar{\chi}^{\dot\beta}\, \partial_{+\dot\alpha} \partial_{+\dot\beta} R^{++}_{\rm int} \right\}. \quad (5.2.10)$$

Now, define the gauge field,

$$\mathcal{A}_\mu(x) \equiv \int du \left\{ 2i \left(\sigma^\dagger_\mu \right)^{\dot\alpha\alpha} \partial_{+\dot\alpha} R^{++}_{\rm int}\, u^-_\alpha \right\}. \quad (5.2.11)$$

As the action (5.2.10) is real, the field $\mathcal{A}_\mu(x)$ is also real. It automatically has zero divergence,

$$\partial_\mu \mathcal{A}_\mu = 0. \quad (5.2.12)$$

The field strength is expressed as

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu = -2\eta^a_{\mu\nu} \int du\, \partial_{+\dot\alpha}\partial_{+\dot\beta} R^{++}_{\rm int}\, \varepsilon^{\dot\alpha\dot\gamma} \left(\sigma_a\right)^{\dot\beta}_{\ \dot\gamma} \tag{5.2.13}$$

(the identities (4.5.46) were used). It is obviously self-dual because the 't Hooft symbols are self-dual, see Eq. (4.5.48). With the definitions (5.2.11) and (5.2.13) in hand, one can represent the interaction term (5.2.10) simply as

$$S_{\rm int} = \int dt \left\{ \mathcal{A}_\mu(x)\dot x_\mu - \frac{i}{4}\mathcal{F}_{\mu\nu}\, \chi\sigma^\dagger_\mu\sigma_\nu\bar\chi \right\}. \tag{5.2.14}$$

Finally, one can get rid of the factor $g(x)$ in the fermion kinetic term (5.2.5) by introducing canonically conjugated

$$\psi_{\dot\alpha} = f^{-1}(x)\chi_{\dot\alpha}, \qquad \bar\psi^{\dot\alpha} = f^{-1}(x)\bar\chi^{\dot\alpha} \tag{5.2.15}$$

with

$$f(x) = g^{-1/2}(x) \equiv \left[\frac{1}{2}\,\partial^2_\mu R_{\rm kin}(x)\right]^{-1/2}. \tag{5.2.16}$$

Adding the kinetic term in (5.2.5) to the interaction term (5.2.14), one can explicitly check that the Lagrangian $L = L_{\rm kin} + L_{\rm int}$ coincides, up to a total derivative, with (5.1.13).

As was noticed, the field A_μ naturally obtained in the HSS framework satisfies the constraint $\partial_\mu \mathcal{A}_\mu = 0$ [6]. This does not really impose a restriction, however, because gauge transformations of A_μ that shift it by the gradient of an arbitrary function amount to adding a total derivative in the Lagrangian (5.2.14).

5.2.3 Supertransformations, quantization and Weyl ordering

By construction, the action with the Lagrangian (5.1.13) is invariant under the following supersymmetry transformations :

$$\begin{aligned} x_\mu &\to x_\mu + f\epsilon\sigma_\mu\psi + f\bar\epsilon\sigma_\mu\bar\psi, \\ f\psi_{\dot\alpha} &\to f\psi_{\dot\alpha} + i\dot x_\mu\left(\bar\epsilon\sigma_\mu\right)_{\dot\alpha}, \\ f\bar\psi^{\dot\alpha} &\to f\bar\psi^{\dot\alpha} - i\dot x_\mu\left(\sigma^\dagger_\mu\epsilon\right)^{\dot\alpha}. \end{aligned} \tag{5.2.17}$$

The Nöether classical supercharges expressed in terms of $\psi_{\dot\alpha}$ and $\bar\psi^{\dot\alpha}$, x_μ and their canonical momenta,

$$p_\mu \ = \ f^{-2}\dot x_\mu + \mathcal{A}_\mu - \frac{i}{2}f^{-1}\partial_\nu f\, \psi\sigma^\dagger_{[\mu}\sigma_{\nu]}\bar\psi\ , \tag{5.2.18}$$

are

$$\begin{aligned} Q_\alpha &= f\left(\sigma_\mu\bar\psi\right)_\alpha (p_\mu - \mathcal{A}_\mu) - i\partial_\mu f\,\psi_{\dot\gamma}\bar\psi^{\dot\gamma}\left(\sigma_\mu\bar\psi\right)_\alpha, \\ \bar Q^\alpha &= [\text{complex conjugate}]. \end{aligned} \tag{5.2.19}$$

The quantization procedure of the corresponding classical Hamiltonian has order ambiguity problem for the bosonic operators $\hat p_\mu = -i\partial_\mu$ and x^μ as well as for the fermionic operators $\hat\psi_{\dot\alpha}$ and $\hat{\bar\psi}^{\dot\alpha}$ with anticommutation relations (5.1.8). (We temporary restore "hats" on fermionic operators in order to distinguish them from the corresponding classical anticommuting variables $\psi_{\dot\alpha}$ and $\bar\psi^{\dot\alpha}$.) One must thus define an ordering procedure in such a way that the supersymmetry algebra (4.5.6) would hold. The solution to this problem is known [41] and it prescribes to order the supercharges in a certain way, while the quantum Hamiltonian should be obtained from the anticommutator $\left\{Q_\alpha,\, \bar Q^\alpha\right\}$.

It is prescribed by Ref. [41] to order the operators in the classical supercharges (5.2.19) with the so called *Weyl ordering* procedure : any product of operators must be substituted with its totally symmetrized expression, taking into account the commuting/anticommuting nature of the operators. For instance, the expression $x^1x^2p_3$, upon quantization, becomes

$$x^1x^2p_3 \quad\longrightarrow\quad \frac{1}{6}\left(x^1x^2\hat p_3 + x^2x^1\hat p_3 + x^1\hat p_3x^2 + x^2\hat p_3x^1 + \hat p_3x^1x^2 + \hat p_3x^2x^1\right), \tag{5.2.20}$$

while the expression $\psi_{\dot\gamma}\bar\psi^{\dot\gamma}\,\bar\psi^{\dot\beta}$ becomes

$$\begin{aligned}\psi_{\dot\gamma}\bar\psi^{\dot\gamma}\,\bar\psi^{\dot\beta} \quad\longrightarrow\quad & \frac{1}{6}\Big(\hat\psi_{\dot\gamma}\hat{\bar\psi}^{\dot\gamma}\hat{\bar\psi}^{\dot\beta} + (-1)\hat\psi_{\dot\gamma}\hat{\bar\psi}^{\dot\beta}\hat{\bar\psi}^{\dot\gamma} + (-1)\hat{\bar\psi}^{\dot\gamma}\hat\psi_{\dot\gamma}\hat{\bar\psi}^{\dot\beta} \\ & + \hat{\bar\psi}^{\dot\beta}\hat\psi_{\dot\gamma}\hat{\bar\psi}^{\dot\gamma} + \hat{\bar\psi}^{\dot\gamma}\hat{\bar\psi}^{\dot\beta}\hat\psi_{\dot\gamma} + (-1)\hat{\bar\psi}^{\dot\beta}\hat{\bar\psi}^{\dot\gamma}\hat\psi_{\dot\gamma}\Big).\end{aligned} \tag{5.2.21}$$

Let us elaborate more on the ordering of the expression $f(x)p_\mu$. For this, consider Weyl ordering of the expression x^np, where x is any of the coordinates x^μ, while p is the corresponding conjugated momentum which, upon quantization, becomes $\hat p = -i\partial/\partial x$. One has :

$$\begin{aligned} x^np \quad\longrightarrow\quad & \frac{1}{n+1}\left(x^n\hat p + x^{n-1}\hat p x + x^{n-2}\hat p x^2 + \cdots + \hat p x^n\right) \\ & = x^n\hat p + \frac{1}{n+1}x^{n-1}\left(1+2+\cdots+n\right)\cdot[\hat p,\, x] \\ & = x^n\hat p + \frac{1}{2}nx^{n-1}\,[\hat p,\, \hat x] \end{aligned} \tag{5.2.22}$$

Thus, in fact, Weyl ordering of the product $f(x)p_\mu$ gives a simple formula :

$$f(x)p_\mu \quad\longrightarrow\quad \frac{1}{2}\Big[f(x)\hat p_\mu + \hat p_\mu\, f(x)\Big] = f(x)\hat p_\mu - \frac{1}{2}i\,\partial_\mu f(x). \tag{5.2.23}$$

In the same way, the expression in Eq. (5.2.21) can be simplified which gives

$$\psi_{\dot\gamma}\bar\psi^{\dot\gamma}\,\bar\psi^{\dot\beta} \quad\longrightarrow\quad \hat\psi_{\dot\gamma}\hat{\bar\psi}^{\dot\gamma}\,\hat{\bar\psi}^{\dot\beta} - \frac{1}{2}\hat{\bar\psi}^{\dot\beta}. \tag{5.2.24}$$

Finally, combining Eqs. (5.2.23), (5.2.24), one obtains :

$$fp_\mu\bar\psi^{\dot\beta} - i\partial_\mu f\,\psi_{\dot\gamma}\bar\psi^{\dot\gamma}\,\bar\psi^{\dot\beta} \quad\longrightarrow\quad f\hat p_\mu\hat{\bar\psi}^{\dot\beta} - i\partial_\mu f\,\hat\psi_{\dot\gamma}\hat{\bar\psi}^{\dot\gamma}\,\hat{\bar\psi}^{\dot\beta}. \tag{5.2.25}$$

This gives the quantum supercharges (5.1.14). One can check that the anticommutator $\{Q_\alpha, \bar Q^\alpha\}$ gives the quantum Hamiltonian (5.1.15).

5.3 The component Lagrangian in the non-Abelian case

For a matrix-valued non-Abelian self-dual field $\mathcal{A}_\mu$, the (scalar) Lagrangian cannot be straightforwardly derived from the Hamiltonian (5.1.15) by a Legendre transformation as it was done in the Abelian case, Eq. (5.1.13). Nevertheless, this can be done in the case of SU(N) gauge group by introducing extra "semi-dynamical" fields φ_i in the fundamental representation of SU(N) and the auxiliary U(1) gauge field $B(t)$. The second line in (5.1.13) is then generalized to

$$L_{\rm int}^{{\rm SU}(N)} = i\bar\varphi^i\left(\dot\varphi_i + iB\varphi_i\right) + kB + \mathcal{A}_\mu^a T^a\,\dot x_\mu - \frac{i}{4}f^2\mathcal{F}^a_{\mu\nu}T^a\,\psi\sigma^\dagger_\mu\sigma_\nu\bar\psi \qquad (5.3.1)$$

with integer k and

$$T^a = \bar\varphi^i\left(t^a\right)_i{}^j\varphi_j, \qquad (5.3.2)$$

t^a being standard SU(N) algebra generators. The interaction Lagrangian (5.3.1) possesses $\mathcal{N}=4$ supersymmetry. The corresponding supersymmetry transformations are written below in Eqs. (5.4.16). It is not difficult to check that it is invariant with respect to the non-Abelian gauge transformations of the target space :

$$\begin{aligned} &\mathcal{A}^a_\mu\,t^a \to U^\dagger\mathcal{A}^a_\mu\,t^aU + iU^\dagger\partial_\mu U \\ &\varphi_i \to \left(U^\dagger\varphi\right)_i, \quad \bar\varphi^i \to \left(\bar\varphi U\right)^i, \end{aligned} \qquad (5.3.3)$$

where $U(x)\in{\rm SU}(N)$. In addition, the expression (5.3.1) is also invariant with respect to the following gauge transformations of auxiliary fields $B(t)$ and φ_i :

$$B(t) \to \; B(t) + \frac{d\alpha(t)}{dt}, \qquad \varphi_i(t) \to e^{-i\alpha(t)}\varphi_i(t). \qquad (5.3.4)$$

It is not immediately clear how to extend the Abelian superfield description to a general non-Abelian case, i.e. to the gauge group SU(N) . We succeeded in constructing such a description for the particular case of SU(2) self-dual or anti-self-dual gauge fields expressed in the form

$$\mathcal{A}^a_\mu \;=\; -\bar\eta^a_{\mu\nu}\partial_\nu\ln h(x) \quad \text{or} \quad \mathcal{A}^a_\mu \;=\; -\eta^a_{\mu\nu}\partial_\nu\ln h(x) \qquad (5.3.5)$$

respectively, with harmonic function $h(x)$,

$$\partial^2_\mu h(x) = \text{a finite sum of delta functions.} \qquad (5.3.6)$$

This is the so called 't Hooft ansatz for a multi-instanton SU(2) solution [16] with the 't Hooft symbols $\eta^a_{\mu\nu}$ defined previously in Eq. (4.5.46). If one takes the function $h(x)$ to be vanishing at $|x|\to\infty$, then this function can be presented as the following sum over instantons :

$$h(x) = 1 + \sum_I \frac{c_I}{(x^\mu - b^\mu_I)^2}. \qquad (5.3.7)$$

It involves particular instanton positions b^μ_I as well as the numbers c_I associated with each instanton.

5.3.1 Quantization of auxiliary fields in the SU(2) gauge group case

Let us understand how the interaction Lagrangian (5.3.1) gives rise to the matrix Hamiltonian (5.1.15). This is achieved upon quantization of the auxiliary variables φ_α and $\bar{\varphi}^\beta$. We consider only the particular case of SU(2) gauge group when the indices i, j for the auxiliary variables take only two values, 1 and 2, and are denoted as α, β. See Section 5.4.5 for a general discussion of SU(N) gauge group.

Observe that the variables φ_α enter the Lagrangian with only one time derivative. Thus, they are not full-fledged dynamic variables (like x_μ) and not auxiliary fields (like $B(t)$ field or $\omega_{1,2}$ fields in Eq. (5.4.5), see below). They have a kind of intermediate nature. In the context of $\mathcal{N} = 4$ SQM models, such variables (together with their analytic superfield carriers $v^+, \widetilde{v^+}$, see Eqs. (5.4.5), (5.4.6) below) were introduced in [19, 20]. See also [21] for a recent application. To understand better the nature of the auxiliary fields, perform the quantization.

The canonical commutation relations following from the action (5.3.1) through the standard Dirac prescription are

$$[\varphi_\alpha, \bar{\varphi}^\beta] = \delta_\alpha^\beta \,, \qquad\qquad [\varphi_\alpha, \varphi_\beta] = [\bar{\varphi}^\alpha, \bar{\varphi}^\beta] = 0. \tag{5.3.8}$$

The fact that k must be integer leads to the finite representations of the operator algebra φ_α, $\bar{\varphi}^\alpha$. Indeed, consider the constraint

$$\bar{\varphi}^\alpha \varphi_\alpha = k \,, \tag{5.3.9}$$

which follows from (5.3.1) by varying with respect to B. All real positive values of k are classically allowed. As we will shortly see, in the quantum theory, k must be integer, but not necessary positive.

Consider the case of positive values of the integer k. In quantum theory, one can choose $\varphi_\alpha \equiv \partial/\partial\bar{\varphi}^\alpha$ and impose (5.3.9) on the wave functions :

$$\bar{\varphi}^\alpha \varphi_\alpha \Psi = \bar{\varphi}^\alpha \frac{\partial}{\partial \bar{\varphi}^\alpha} \Psi = k\Psi \ . \tag{5.3.10}$$

In other words, the wave functions represent homogeneous polynomials of $\bar{\varphi}^\alpha$ of (an integer) degree k. In the case $k < 0$ the algebra (5.3.8) is the same, but one must choose $\bar{\varphi}^\alpha = -\partial/\partial\varphi_\alpha$ and consider polynomials of φ_α of degree $|k|$. The number of such (linearly independent) polynomials is $|k| + 1$. Moreover, it is also easy to see that the operators (5.3.2) (which enter the interaction Lagrangian (5.3.1)) satisfy the following algebra :

$$[T^a, T^b] \ = \ i\varepsilon^{abc} T^c. \tag{5.3.11}$$

In addition, assuming $k > 0$ and taking into account (5.3.10), one derives

$$T^a T^a \ = \ \frac{1}{4}\left[(\bar{\varphi}^\alpha \varphi_\alpha)^2 + 2(\bar{\varphi}^\alpha \varphi_\alpha)\right] \ = \ \frac{k}{2}\left(\frac{k}{2} + 1\right). \tag{5.3.12}$$

In other words, T^a can be treated as the generators of SU(2) in the representation of spin $k/2$.

This way of quantizing semi-dynamical variables $\varphi_\alpha, \bar\varphi^\alpha$ was employed in Ref. [20]. Alternatively, one could interpret $\varphi_\alpha, \bar\varphi^\alpha$ with the constraint (5.3.9) as a kind of the target harmonic variables representing a sphere S^2, solve (5.3.9) in terms of the stereographic projection coordinates and quantize the system (see, for example, Ref. [48]).

A nice feature is that this gauge SU(2) group is in fact the R-symmetry group of $\mathcal{N} = 4$ supersymmetry algebra.

The crucial role of the constraint (5.3.10) is to restrict the space of quantum states of the considered model to the *finite* set of irreducible SU(2) multiplets of fixed spins (e.g., of the spin $k/2$ in the bosonic sector). This is an essential difference of this approach from that employed, e.g., in [18] (and later in [21, 17]) where no analog of the constraints (5.3.9) and (5.3.10) was imposed, thus allowing for the space of states to involve an *infinite* number of SU(2) multiplets of all spins. The quantization scheme which we follow here was earlier used in the SQM context in [19, 20] and can be traced back to the work [49].

5.3.2 Why the number k must be integer

Let us restrict ourselves by the first two terms in the interaction Lagrangian (5.3.1). The action

$$S \;=\; \int dt \left[i\bar\varphi^i(\dot\varphi_i + iB\varphi_i) \;+\; kB\right] \tag{5.3.13}$$

much resembles the three-dimensional Chern-Simons action,

$$S_{\mathrm{CS}} = \kappa \int \left(A \wedge dA - \frac{2i}{3} A \wedge A \wedge A\right). \tag{5.3.14}$$

In both systems, the canonical Hamiltonian is zero, the canonical momenta are algebraically expressed through coordinates, and the quantization consists in imposing certain second class constraints (for a nice review of the classical and quantum aspects of the Chern-Simons theory, see [50]). Another well-known feature of CS theory is the quantization of the coupling, $k_{\mathrm{CS}} = 4\pi\kappa =$ integer. This follows from the requirement for the Euclidean path integral to be invariant with respect to large (topologically nontrivial) gauge transformations. As was mentioned above, in our case the coefficient k is also quantized. This can be derived following a similar reasoning.

Consider the Euclidean version of the action (5.3.13), where one changes the time t to the Euclidean time τ by

$$t = -i\tau \tag{5.3.15}$$

and regularize it in the infrared by putting it on a finite Euclidean interval $\tau \in (0, \beta)$ and imposing the periodic boundary conditions. This is of course equivalent to do calculations at finite temperature $T = 1/\beta$.

Notice first that the action (5.3.13) is invariant with respect to gauge transformations (5.3.4) which, in the Euclidean version of the theory, become

$$B(\tau) \to \; B(\tau) + i\frac{d\alpha(\tau)}{d\tau}, \qquad \varphi_i(\tau) \to e^{-i\alpha(\tau)}\varphi_i(\tau). \tag{5.3.16}$$

Let us discover topologically nontrivial gauge transformation in the Euclidean version of this theory with the periodic boundary conditions

$$B(\beta) = B(0), \qquad \varphi_i(\beta) = \varphi_i(0). \tag{5.3.17}$$

The only admissible gauge transformations (5.3.16) are those which do not break these periodicity conditions. We see that the transformation with $\alpha(\tau) = 2\pi\tau/\beta$ is topologically nontrivial : it cannot be reduced to a chain of infinitesimal transformations. This transformation shifts the Euclidean version of the last term in the action (5.3.13) by an *imaginary* constant, $\Delta S_{\rm FI} = -2\pi i k$. The requirement that the Euclidean path integrals (involving the factor $e^{-S_{\rm FI}}$) are not changed leads [49] to the quantization condition

$$k \ = \ \text{integer} \tag{5.3.18}$$

Thus, a benign quantum theory can only be defined if this requirement is fulfilled.

5.4 Harmonic superspace Lagrangian with non-Abelian gauge fields

5.4.1 Superfield content

To construct the action involving non-Abelian gauge fields, introduce, as earlier, a doublet of superfields $q^{+\dot\alpha}$ with charge $+1$ satisfying the constraints (4.5.31), (4.5.36). *On top of that*, we introduce an analytic gauge superfield V^{++} of charge $+2$ satisfying the constraints

$$D^+V^{++} = \bar D^+V^{++} = 0\ , \qquad V^{++} = \widetilde{V^{++}} \tag{5.4.1}$$

and the "matter" superfield v^+ of charge $+1$. The constraints it satisfies,

$$D^+v^+ = 0, \qquad \bar D^+v^+ = 0, \qquad (D^{++} + iV^{++})v^+ = 0\,, \tag{5.4.2}$$

differ from (4.5.31) by the presence of the covariant harmonic derivative $\mathcal{D}^{++} = D^{++} + iV^{++}$ [5]. The constraint $\mathcal{D}^{++}v^+ = 0$ is covariant with respect to gauge transformations

$$V^{++} \ \to\ V^{++} + D^{++}\Lambda, \qquad v^+ \ \to\ e^{-i\Lambda}v^+, \qquad D^+\Lambda = \bar D^+\Lambda = 0\ . \tag{5.4.3}$$

We can use this gauge freedom to eliminate almost all components from V^{++} and to present it as

$$V^{++} \ = \ 2i\,\theta^+\bar\theta^+B, \tag{5.4.4}$$

where the gauge field $B(t)$ is real. This is a one-dimensional counterpart of the familiar Wess-Zumino gauge in four-dimensional theories. Observe also that Eq. (5.3.4) is a remnant of gauge transformations (5.4.3), which survives in the Wess-Zumino gauge (5.4.4).

Then the superfield v^+ is expressed in the analytical basis as

$$v^+ = \phi^\alpha u^+_\alpha - 2\theta^+\omega_1 - 2\bar\theta^+\bar\omega_2 - 2i\theta^+\bar\theta^+(\dot\phi^\alpha + iB\phi^\alpha)u^-_\alpha\ , \tag{5.4.5}$$

from which it follows that

$$\widetilde{v^+} = \bar\phi^\alpha u^+_\alpha - 2\theta^+\omega_2 + 2\bar\theta^+\bar\omega_1 - 2i\theta^+\bar\theta^+(\dot{\bar\phi}^\alpha - iB\bar\phi^\alpha)u^-_\alpha \tag{5.4.6}$$

with $\bar\phi^\alpha = (\phi_\alpha)^*$. Thus, the fields ϕ_α and $\bar\phi^\alpha$ carry nonzero opposite U(1) charges associated with the auxiliary gauge field B.

5.4.2 Superfield action

The $\mathcal{N}=4$ supersymmetry-invariant action consists of three parts, $S = S_{\rm kin} + S_{\rm int} + S_{\rm FI}$. The kinetic part is more convenient to express in the central basis $\{t,\, \theta_\alpha,\, \bar\theta^\beta\}$. It has the same form as in Eq. (5.2.4) and its component expansion coincides with the first line in Eq. (5.1.13), where the same change of variables (5.2.15) and (5.2.16) is performed as in the Abelian case.

The interaction part is taken as

$$S_{\rm int} = -\frac{1}{2}\int dt\, du\, d\bar\theta^+ d\theta^+\, K\left(q^{+\dot\alpha}, u^\pm_\beta\right) v^+ \widetilde{v^+}, \tag{5.4.7}$$

where the condition $\widetilde{K} = K$ is imposed to ensure the action to be real. Finally, we add the Fayet-Iliopoulos term

$$S_{\rm FI} \;=\; -\frac{ik}{2}\int dt\, du\, d\bar\theta^+ d\theta^+\, V^{++} \;=\; k\int dt\, B, \tag{5.4.8}$$

which is invariant under gauge transformations (5.4.3).

Let us concentrate on the interaction part. It is convenient to introduce new variables

$$\varphi_\alpha \;=\; \phi_\alpha \sqrt{h(x)}\;, \tag{5.4.9}$$

where

$$h(x) = \int du\, K(x^{+\dot\alpha}, u^\pm_\beta) \qquad \left(x^{+\dot\alpha} = x^{\alpha\dot\alpha}u^+_\alpha\right), \tag{5.4.10}$$

is a harmonic function [3]. Indeed,

$$\partial^2_\mu h(x) = 4\,\varepsilon^{\dot\alpha\dot\beta}\int du\, \partial_{+\dot\alpha}\partial_{-\dot\beta} K(x^{+\dot\gamma}, u^\pm_\beta) = 0\,. \tag{5.4.11}$$

Substituting (4.5.37), (5.4.5) and (5.4.6) into (5.4.7) and eliminating the auxiliary fermionic degrees of freedom $\omega_{1,2}$, $\bar\omega_{1,2}$ by their algebraic equations of motion, we derive after some algebra

$$L_{\rm int} \;=\; i\bar\varphi^\alpha(\dot\varphi_\alpha + iB\varphi_\alpha) - \frac{1}{2}\bar\varphi^\beta\varphi_\gamma\left(\mathcal{A}_{\alpha\dot\alpha}\right)_\beta{}^\gamma\, \dot x^{\alpha\dot\alpha} - \frac{i}{4}\left(\mathcal{F}_{\dot\alpha\dot\beta}\right)_\beta{}^\gamma \chi^{\dot\alpha}\bar\chi^{\dot\beta}\bar\varphi^\beta\varphi_\gamma\,. \tag{5.4.12}$$

Here

$$(\mathcal{A}_{\alpha\dot\alpha})_\beta{}^\gamma = -\frac{2i}{\int du\, K}\int du\, \partial_{+\dot\alpha}K\left(u^{+\gamma}\varepsilon_{\alpha\beta} - \frac{1}{2}u^+_\alpha\delta^\gamma_\beta\right) = \frac{i}{h}\left(\varepsilon_{\alpha\beta}\,\partial^\gamma{}_{\dot\alpha}h - \frac{1}{2}\delta^\gamma_\beta\,\partial_{\alpha\dot\alpha}h\right) \tag{5.4.13}$$

($\partial_{\alpha\dot\alpha} \equiv (\sigma_\mu)_{\alpha\dot\alpha}\,\partial_\mu = -2\partial/\partial x^{\alpha\dot\alpha}$) is a Hermitian traceless matrix – the gauge field, and

$$\left(\mathcal{F}_{\dot\alpha\dot\beta}\right)_\beta{}^\gamma = (\mathcal{F}_{\mu\nu})_\beta{}^\gamma\left(\sigma^\dagger_\mu\sigma_\nu\right)_{\dot\alpha\dot\beta} = \partial_{\delta\dot\alpha}(\mathcal{A}^\delta{}_{\dot\beta})_\beta{}^\gamma - i(\mathcal{A}_{\delta\dot\alpha})_\beta{}^\lambda(\mathcal{A}^\delta{}_{\dot\beta})_\lambda{}^\gamma\; + (\dot\alpha \leftrightarrow \dot\beta) \tag{5.4.14}$$

is its self-dual part. It is easy to check explicitly, that the anti-self-dual part of the gauge field $\mathcal{A}_\mu$ vanishes,

$$(\mathcal{F}_{\alpha\beta})_\gamma{}^\delta = (\mathcal{F}_{\mu\nu})_\gamma{}^\delta\left(\sigma_\mu\sigma^\dagger_\nu\right)_{\alpha\beta} = -\partial_{\alpha\dot\alpha}(\mathcal{A}_\beta{}^{\dot\alpha})_\gamma{}^\delta + i(\mathcal{A}_{\alpha\dot\alpha})_\gamma{}^\lambda(\mathcal{A}_\beta{}^{\dot\alpha})_\lambda{}^\delta + (\alpha\leftrightarrow\beta) = 0. \tag{5.4.15}$$

[3] We assume here that $h(x) > 0$. The case $h(x) < 0$ is treated similarly, if one redefines $h(x) \to -h(x)$.

Thus, the field strength $\mathcal{F}^a_{\mu\nu}$ is self-dual and belongs to the representation $(0,1)$ of SO(4) = SU(2) × SU(2). Passing to $\mathcal{A}^a_\mu$ such that $(\mathcal{A}_\mu)_\beta{}^\gamma = \mathcal{A}^a_\mu\,(\sigma_a)_\beta{}^\gamma/2$, we find that the representation (5.4.13) precisely amounts to the self-dual 't Hooft ansatz (5.3.5), left equation. The anti-self-dual expression from the right equation arises if one interchanges dotted and undotted indices, i.e. effectively interchanges σ_μ and $\sigma^\dagger_\mu$. This also implies passing to the harmonics $u^\pm_{\dot\alpha}$ and in fact to another $\mathcal{N}=4$ supersymmetry, with the second SU(2) (acting on dotted indices) as the R-symmetry group.

Finally, substituting $\bar\varphi^\beta\varphi_\gamma = T^a\,(\sigma_a)_\gamma{}^\beta$ and $\chi_{\dot\alpha} = f\psi_{\dot\alpha}$ into (5.4.12), where T^a is defined in (5.3.2) with $t^a = \frac{1}{2}\sigma_a$, one convinces himself that the interaction term together with the FI term (5.4.8) yields just (5.3.1) for the SU(2) gauge group case, and the quantum Hamiltonian derived from the Lagrangian $L_{\rm kin} + L_{\rm int} + L_{\rm FI}$ has the form (5.1.15) with $\mathcal{A}_\mu \equiv \mathcal{A}^a_\mu T^a$ and $\mathcal{F}_{\mu\nu} \equiv \mathcal{F}^a_{\mu\nu}T^a$.

5.4.3 Supersymmetry transformations

The full Lagrangian in the non-Abelian case representing the sum of Eq. (5.3.1) and the first line of Eq. (5.1.13) is invariant, up to a total derivative, with respect to $\mathcal{N}=4$ supersymmetry transformations (in the infinitesimal form) of Eqs. (5.2.17) supplemented with transformations for the auxiliary fields :

$$\begin{aligned} \varphi_i &\to \varphi_i + if\,(t^a\varphi)_i\,\mathcal{A}^a_\mu\left(\epsilon\sigma_\mu\psi + \bar\epsilon\sigma_\mu\bar\psi\right), \\ \bar\varphi^i &\to \bar\varphi^i - if\,(\bar\varphi t^a)^i\,\mathcal{A}^a_\mu\left(\epsilon\sigma_\mu\psi + \bar\epsilon\sigma_\mu\bar\psi\right). \end{aligned} \tag{5.4.16}$$

Note that the above formulas are written for the case of the gauge group SU(N) when the semi-dynamical fields φ_i, $\bar\varphi^j$ belong to a fundamental representation of SU(N).

5.4.4 $\mathcal{N}=4$ supersymmetry with Yang monopole

We have constructed the superfield action for the $\mathcal{N}=4$ supersymmetric quantum mechanics corresponding to the Hamiltonian (5.1.15) with a non-Abelian SU(2) gauge field $\mathcal{A}_\mu$ which lives on a conformally flat 4-manifold and is representable in the 't Hooft ansatz form (5.3.5).

As an example of such a field, let us quote the instanton solution on S^4. Generically, it depends on the radius R of the sphere and the instanton size ρ. The configurations of maximal size, $\rho = R$, present a particular interest. In the stereographic coordinates on S^4,

$$ds^2 = \frac{4R^4 dx_\mu^2}{(x^2+R^2)^2}, \tag{5.4.17}$$

they are expressed by the same formulas as the flat instantons in the singular gauge,

$$\mathcal{A}^a_\mu = \frac{2R^2\bar\eta^a_{\mu\nu}x_\nu}{x^2(x^2+R^2)} \quad \text{or} \quad (\mathcal{A}_{\alpha\dot\alpha})_\beta{}^\gamma = -\frac{2i\,R^2}{x^2(x^2+R^2)}\left(\varepsilon_{\alpha\beta}x^\gamma_{\dot\alpha} - \frac{1}{2}\,\delta^\gamma_\beta\,x_{\alpha\dot\alpha}\right), \tag{5.4.18}$$

and

$$(\mathcal{F}_{\dot\alpha\dot\beta})_\beta{}^\gamma = \frac{8i\,R^2}{x^2(x^2+R^2)^2}\left(x^\gamma_{\dot\beta}x_{\beta\dot\alpha} + x^\gamma_{\dot\alpha}x_{\beta\dot\beta}\right). \tag{5.4.19}$$

The corresponding functions in Eq. (5.4.10) are taken in the form

$$K(x^{+\dot\alpha},u^\pm_\beta)=1+\frac{1}{\left(c^-_{\dot\alpha}x^{+\dot\alpha}\right)^2}\,,\qquad h(x)\equiv\int du\,K(x^{+\dot\alpha},u^\pm_\beta)=1+\frac{R^2}{x^2_\mu}\,,\tag{5.4.20}$$

where $c^-_{\dot\alpha}=c^\alpha{}_{\dot\alpha}u^-_\alpha$, $c^{\alpha\dot\alpha}$ – constant vector and $R^2=1/c^2_\mu$. The integral on the right hand side of Eq. (5.4.20) can be calculated as the power series in $c^-_{\dot\alpha}c^{+\dot\alpha}=-c^2_\mu$ or directly after noting that the form of this integral is SO(4) invariant and putting $c_\mu=(c,0,0,0)$, $x_\mu=(x_1,x_2,0,0)$ [4].

The field $\mathcal{A}^a_\mu$ can be brought to the nonsingular gauge

$$\mathcal{A}^a_\mu=\frac{2\eta^a_{\mu\nu}x_\nu}{x^2+R^2}\,,\qquad \mathcal{F}^a_{\mu\nu}\;=\;-\frac{4R^2\eta^a_{\mu\nu}}{(x^2+R^2)^2}\,,\tag{5.4.25}$$

by the gauge transformations (5.3.3) with $U(x)=-i\sigma_\mu x_\mu/\sqrt{x^2}$ (this particular $U(x)$ form is prompted by the form of the field strength (5.4.19)). The action density $\sim\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ is the same in this case at all points of S^4. It is worth noting that the singular gauge transformation converts the undotted gauge group indices into the dotted ones : the self-dual gauge potential and the field strength in the spinorial notation become

$$(\mathcal{A}_{\alpha\dot\alpha})^{\dot\gamma}{}_{\dot\beta}=\frac{2i}{x^2+R^2}\left(\varepsilon_{\dot\alpha\dot\beta}x^{\dot\gamma}_\alpha-\frac12\,\delta^{\dot\gamma}_{\dot\beta}\,x_{\alpha\dot\alpha}\right),\quad (\mathcal{F}_{\dot\alpha\dot\beta})^{\dot\gamma}{}_{\dot\delta}=-\frac{8i\,R^2}{(x^2+R^2)^2}\left(\varepsilon_{\dot\alpha\dot\delta}\delta^{\dot\gamma}_{\dot\beta}+\varepsilon_{\dot\beta\dot\delta}\delta^{\dot\gamma}_{\dot\alpha}\right)\tag{5.4.26}$$

and, also, $\varphi_\alpha\rightarrow\varphi^{\dot\alpha}=-i\varphi_\alpha\,x^{\alpha\dot\alpha}/\sqrt{x^2}$, $\bar\varphi^\alpha\rightarrow\bar\varphi_{\dot\alpha}=-i\bar\varphi^\alpha\,x_{\alpha\dot\alpha}/\sqrt{x^2}\,$.

Note that, the field (5.4.18), (5.4.25) describes the Yang monopole living in $\mathbb{R}^5$ [51]. The potential (5.4.25) has a nice group-theoretical meaning as one of the two SU(2) connections on the coset manifold SO(5)/[SU(2) × SU(2)] ∼ S^4 (see e.g. [52]). It coincides with the flat self-dual instanton only in the conformally flat parametrization of S^4 as in (5.4.17). When coupled to the world-line through our semi-dynamical variables $\varphi_\alpha,\bar\varphi^\alpha$, the 5-dimensional Yang monopole is reduced to this SU(2) connection defined on S^4.

[4] Let us describe the latter possibility in more detail. If c_μ and x_μ are chosen as mentioned above, this gives

$$c^-_{\dot\alpha}x^{+\dot\alpha}=-c\,x^0+ic\,x^1\left(u^+_1u^-_1-u^+_2u^-_2\right).\tag{5.4.21}$$

To calculate the integral in Eq. (5.4.20), one realizes the harmonics $u^\pm_{\dot\alpha}$ in the familiar stereographic parametrization [5] :

$$\begin{pmatrix}u^+_1 & u^-_1\\ u^+_2 & u^-_2\end{pmatrix}=\frac{1}{\sqrt{1+t\bar t}}\begin{pmatrix}e^{i\psi} & -\bar t\,e^{-i\psi}\\ t\,e^{i\psi} & e^{-i\psi}\end{pmatrix},\tag{5.4.22}$$

where $t\in\mathbb{C}$, $0\le\psi<2\pi$. It is also necessary to define the measure of integration,

$$\int du\quad\longrightarrow\quad\frac{i}{4\pi^2}\int_0^{2\pi}d\psi\int\frac{dt\,d\bar t}{(1+t\bar t)^2}.\tag{5.4.23}$$

After this, the computation of the integral

$$\frac{i}{2\pi}\int\frac{dt\,d\bar t}{(1+t\bar t)^2}\left[1+\frac{(1+t\bar t)^2}{\left(c\,x^0(1+t\bar t)+ic\,x^1(t+\bar t)\right)^2}\right]\tag{5.4.24}$$

gives (5.4.20).

Let us elaborate on this point in more detail, choosing, without loss of generality, $R = 1$ in the above formulas. Consider the following $d = 1$ bosonic Lagrangian with the $\mathbb{R}^5$ target space and an additional coupling to Yang monopole

$$L_{\mathbb{R}^5} = \frac{1}{2}\left(\dot{y}_5\dot{y}_5 + \dot{y}_\mu\dot{y}_\mu\right) + \mathcal{B}^a_\mu(y)T^a\,\dot{y}_\mu\,. \tag{5.4.27}$$

Here, $\mathcal{B}^a_\mu$ is the standard form of the Yang monopole in the $\mathbb{R}^5$ coordinates,

$$\mathcal{B}^a_\mu = \frac{\eta^a_{\mu\nu}y_\nu}{r(r+y_5)}\,, \quad r = \sqrt{y_5^2 + y_\mu^2}\,, \tag{5.4.28}$$

T^a are defined as in (5.3.2) with $t^a = \frac{1}{2}\sigma^a$, and the action for the semi-dynamical variables $\varphi_\alpha, \bar{\varphi}^\alpha$ is omitted. Now one passes to the polar decomposition of $\mathbb{R}^5$ into a radius r and the angular part S^4 , $(y_5, y_\mu) \;\to\; (r, \tilde{y}_5, \tilde{y}_\mu)$, $\tilde{y}_5 = \sqrt{1-\tilde{y}_\mu^2}$, and rewrites (5.4.27) as

$$L_{\mathbb{R}^5} = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\left(\dot{\tilde{y}}_5\dot{\tilde{y}}_5 + \dot{\tilde{y}}_\mu\dot{\tilde{y}}_\mu\right) + \frac{\eta^a_{\mu\nu}\tilde{y}_\nu\dot{\tilde{y}}_\mu\,T^a}{1+\sqrt{1-\tilde{y}_\mu^2}}\,. \tag{5.4.29}$$

The coordinates $\tilde{y}_\mu$ give a particular parametrization of S^4. Passing to the stereographic coordinates is accomplished by the redefinition

$$\tilde{y}_\mu = 2\,\frac{x_\mu}{1+x^2}\,,$$

which casts (5.4.29) into the form

$$L_{\mathbb{R}^5} = \frac{1}{2}\left\{\dot{r}^2 + 4r^2\frac{\dot{x}_\mu\dot{x}_\mu}{(1+x^2)^2}\right\} + \frac{2\eta^a_{\mu\nu}x_\nu\dot{x}_\mu\,T^a}{1+x^2}\,. \tag{5.4.30}$$

One sees that the S^4 metric (5.4.17) (with $R = 1$) and the instanton vector potential (5.4.25) appear.

Thus, current approach, as a by-product, provides a solution to the long-standing problem of constructing $\mathcal{N} = 4$ supersymmetric quantum mechanics with Yang monopole (see e.g. [8] and references therein). Obviously, the component Lagrangian (5.1.13) (with the relevant function $f(x)$) is just the S^4 part of the Lagrangian (5.4.30) with the "frozen" radial variable $r = 1$. Presumably, one can restore the full 5-dimensional kinetic part in (5.4.30) by adding a coupling to the appropriately constrained scalar $\mathcal{N} = 4$ zero-charge superfield $X(t,\theta,\bar{\theta})$ which describes an off-shell multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ with one physical bosonic field [44], such that $X|_{\theta=\bar{\theta}=0} = r$.

5.4.5 Some remarks in the non-Abelian case

The problem of finding a superfield formulation for a generic $\mathrm{SU}(N)$ self-dual field is more complicated and is still an open question. However, by introducing extra variables φ_i, it is always possible to write a *component* Lagrangian (5.3.1) (together with the first line in (5.1.13)) corresponding to the matrix Hamiltonian (5.1.15).

This observation has actually nothing to do with supersymmetry. It boils down to the following. Consider the eigenvalue problem for a usual Hermitian matrix H_{jk}. It can be treated as a Schrödinger problem $\hat{H}\Psi(\varphi_j) = \lambda\Psi(\varphi_j)$ with the constraint $\hat{G}\Psi = 0$, where

$$\hat{H} \;=\; \varphi_j H_{jk}\frac{\partial}{\partial\varphi_k}\,, \qquad\qquad \hat{G} = \varphi_j\frac{\partial}{\partial\varphi_j} - 1. \tag{5.4.31}$$

The corresponding Lagrangian is

$$L \;=\; i\bar{\varphi}_j\dot{\varphi}_j - B(\bar{\varphi}_j\varphi_j - 1) - \bar{\varphi}_j H_{jk}\varphi_k, \tag{5.4.32}$$

where $\bar{\varphi}_j = (\varphi_j)^*$. This easily generalizes to the case where H is an operator depending on a set of canonically conjugated variables $\{p_\mu, x_\mu\}$. The only difference is that $-H_{jk}$ is now replaced by the matrix L_{jk} obtained from H_{jk} by the appropriate Legendre transformation [5].

The initial goal was to find a Lagrangian representation for the Hamiltonian (5.1.15) with matrix-valued $\mathcal{A}_\mu$, $\mathcal{F}_{\mu\nu}$. The construction just described, with φ_i in the fundamental representation of SU(N), leads to the $N\times N$ matrix Hamiltonian. The Lagrangian (5.4.32) coincides in this case with the Lagrangian (5.3.1) with the choice $k = 1$, to which the first line from Eq. (5.1.13) is also added.

Obviously, one can describe the Hamiltonians in higher representations of SU(N) in a similar way, by choosing the number of components φ_i equal to the dimension of the representation. We have seen, however, that in the SU(2) case one can be more economic, introducing only a couple of dynamic variables φ_α and multiplying the term proportional to B in the Lagrangian by an arbitrary integer k. This leads to the Hamiltonian in the representation of spin $|k|/2$. Certain SU(N) representations (namely, the symmetric products of $|k|$ fundamental or $|k|$ antifundamental representations) can also be attained in this way.

One can also construct in this way a $\mathcal{N} = 2$ supersymmetric Lagrangian for the Hamiltonian (5.1.15) with generic (not necessarily self-dual) $\mathcal{A}_\mu$. A similar construction (but with extra fermionic rather than bosonic variables) was in fact discussed in Ref. [9]. A beauty of the harmonic superspace approach explored here is, however, that such extra variables and the constraint (5.3.10) are not introduced by hand, but arise naturally from the manifestly off-shell supersymmetric superfield actions.

5.5 Three-dimensional SQM in non-Abelian monopole background

The Hamiltonian (5.1.15) presents the generalization of the Hamiltonian (5.1.11) to the conformally flat metric case in *four dimensions*. We succeeded in the construction of this generalization using the superfield formalism. In this section, we employ similar

[5] This elementary observation should be well known, for example, in matrix models. Surprisingly, it is not found it in such a "chemically pure" form in the literature, but similar constructions were discussed, e.g., in Refs. [49, 53].

construction and generalize the *three-dimensional* system described by with the Hamiltonian (5.1.17) to the conformally flat case. Although the resulting Hamiltonian, the supercharges and the component Lagrangian appear to be just the three-dimensional reduction of the four-dimensional counterpart, the superfield formalism in the three-dimensional case involves a different superfield for the space coordinates x^i and is thus implemented differently.

5.5.1 Superfield content and superfield action

In this section, instead of the coordinate superfield $q^{+\dot\alpha}$ one deals with the analytic superfield L^{++} which encompass the multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ and is subjected to the constraints (4.5.39). The superfield V^{++} and the auxiliary superfields v^+ and $\widetilde{v^+}$ are defined by the same Eqs. (5.4.1), (5.4.2). The superfield V^{++} in the Wess-Zumino gauge (5.4.4) is expressed through one independent component $B(t)$. We remind that $B(t)$ is a real one-dimensional "gauge field" which transforms as $B \to B + \dot\lambda$, with $\lambda(t)$ being the parameter of the residual gauge U(1) symmetry.

The explicit expressions for the superfields $q^{+\dot\alpha}$, v^+, $\widetilde{v^+}$ and V^{++} are written in Eqs. (4.5.37), (5.4.5), (5.4.6) and (5.4.4) respectively. The component expansion of the analytic superfield L^{++} can be found in Eqs. (4.5.40), (4.5.41). The multiplet L^{++} involves the three-dimensional target space coordinates $\ell^{\alpha\beta} = \ell^{\beta\alpha}$, their fermionic partners χ^α, $\bar\chi^\alpha$ and a real auxiliary field F. Let us remark that the three-dimensional case involves only one SU(2) (R-symmetry) group and thus no dotted indices present in the description.

Note also that the constraint $\bar\chi_\alpha = (\chi^\alpha)^*$ involves different position of spinor indices compared to Eq. (4.5.38) in the four-dimensional case (see Section 5.5.7 below). The transition from the spinor notation $\ell^{\alpha\beta}$ to the vector notation ℓ^i,

$$\ell_\alpha^\beta = \ell_i\,(\sigma_i)_\alpha^{\ \beta}\,, \qquad \ell_i = \frac{1}{2}\ell_\beta^\alpha\,(\sigma_i)_\alpha^{\ \beta}\,, \qquad i = 1,2,3 \tag{5.5.1}$$

(σ_i are Pauli matrices and, as usual, spinor indices are raised and lowered with antisymmetric Levi-Civita tensors $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$), for the three-dimensional coordinates is considered in Section 5.5.3. The condition (4.5.41) ensures that the coordinates ℓ_i are real.

The full Lagrangian $\mathcal{L}$ entering the $\mathcal{N}=4$ invariant off-shell action $S=\int dt\mathcal{L}$ consists of the three pieces [6]

$$\mathcal{L} = \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm int} + \mathcal{L}_{\rm FI} = \int du\, d^4\theta\, R_{\rm kin}(L^{++}, L^{+-}, L^{--}, u)$$
$$- \frac{1}{2}\int du\, d\bar\theta^+ d\theta^+\, K(L^{++}, u) v^+ \widetilde{v^+} - \frac{ik}{2}\int du\, d\bar\theta^+ d\theta^+\, V^{++}, \tag{5.5.2}$$

where $L^{+-} = \frac{1}{2}D^{--}L^{++}$ and $L^{--} = D^{--}L^{+-}$. The superfield functions $R_{\rm kin}$ and K bear an arbitrary dependence on their arguments [7]. The meaning of three terms in (5.5.2) is explained below.

[6] The first superfield formulation of general $(\mathbf{3},\mathbf{4},\mathbf{1})$ SQM without background gauge field couplings was given in [15].

[7] The superfield Lagrangian (5.5.2) is written in the non-Abelian case. In the Abelian case, the super-

5.5.2 From harmonic superspace to components

The first, sigma-model-type term in Eq. (5.5.2), after integrating over Grassmann and harmonic variables, yields the generalized kinetic terms for $\ell^{\alpha\beta}, \chi^\alpha, \bar\chi_\alpha$:

$$\mathcal{L}_{\rm kin} = \frac{1}{8} f^{-2}\left(-2\dot\ell_{\alpha\beta}\dot\ell^{\alpha\beta} + F^2\right) + \frac{i}{2} f^{-2}\left(\bar\chi_\alpha\dot\chi^\alpha - \dot{\bar\chi}_\alpha\chi^\alpha\right) + \frac{1}{4}\left(\partial_{\alpha\beta}\partial^{\alpha\beta} f^{-2}\right)\chi^4$$
$$+ \frac{i}{f^3}\dot\ell^{\alpha\beta}\Big\{\partial_{\alpha\gamma} f\chi_\beta\bar\chi^\gamma + \partial_{\beta\gamma} f\chi^\gamma\bar\chi_\alpha\Big\} - \frac{1}{f^3} F\chi^\alpha\bar\chi^\beta\partial_{\alpha\beta} f, \quad (5.5.4)$$

where $\chi^4 = \chi^\alpha\chi_\alpha\,\bar\chi^\beta\bar\chi_\beta$, $\partial_{\alpha\beta} \equiv \frac{\partial}{\partial\ell^{\alpha\beta}}$ and $f(\ell)$ is a conformal factor. The calculations are most easily performed in the central basis, where $L^{++} = u^+_\alpha u^+_\beta L^{\alpha\beta}\left(t, \theta_\gamma, \bar\theta^\delta\right)$. Then

$$f^{-2}(\ell) = -\partial_{\alpha\beta}\partial^{\alpha\beta}\int R_{\rm kin}\left(\ell^{\alpha\beta}u^+_\alpha u^+_\beta, \ell^{\alpha\beta}u^+_\alpha u^-_\beta, \ell^{\alpha\beta}u^-_\alpha u^-_\beta\right) du.$$

The fermionic kinetic term can be brought to the canonical form by the change of variables

$$\chi^\alpha = f\psi^\alpha, \qquad \bar\chi_\alpha = f\bar\psi_\alpha. \quad (5.5.5)$$

It is worth pointing out that the R-symmetry SU(2) group amounts to the rotational SO(3) group in the $\mathbb{R}^3$ target space parametrized by ℓ^i from Eq. (5.5.1). The conformal factor $f(\ell)$ can bear an arbitrary dependence on $\ell^{\alpha\beta}$, so this SO(3) can be totally broken in the Lagrangian (5.5.4).

The second piece in Eq. (5.5.2) describes the coupling to an external non-Abelian gauge field. Performing the integration over θ^+, $\bar\theta^+$ and $u^\pm_\alpha$, eliminating the auxiliary fermionic fields $\omega_{1,2}$ and, finally, rescaling the bosonic doublet variables as $\varphi_\alpha = \phi_\alpha\sqrt{h(\ell)}$, where

$$h(\ell) = \int du\, K\left(\ell^{\alpha\beta}u^+_\alpha u^+_\beta, u^\pm_\gamma\right), \quad (5.5.6)$$

after some algebra one obtains

$$\mathcal{L}_{\rm int} = i\bar\varphi^\alpha\left(\dot\varphi_\alpha + iB\varphi_\alpha\right) + \bar\varphi^\gamma\varphi^\delta\frac{1}{2}\left(\mathcal{A}_{\alpha\beta}\right)_{\gamma\delta}\dot\ell^{\alpha\beta} - \frac{1}{2}F\,\bar\varphi^\gamma\varphi^\delta\,U_{\gamma\delta} + \chi^\alpha\bar\chi^\beta\bar\varphi^\gamma\varphi^\delta\nabla_{\alpha\beta}U_{\gamma\delta}. \quad (5.5.7)$$

Here the non-Abelian background gauge field and the scalar (matrix) potential are fully specified by the function h :

$$\left(\mathcal{A}_{\alpha\beta}\right)_{\gamma\delta} = \frac{i}{2h}\Big\{\varepsilon_{\gamma\beta}\partial_{\alpha\delta}h + \varepsilon_{\gamma\alpha}\partial_{\beta\delta}h + \varepsilon_{\delta\beta}\partial_{\alpha\gamma}h + \varepsilon_{\delta\alpha}\partial_{\beta\gamma}h\Big\}, \qquad U_{\gamma\delta} = \frac{1}{h}\partial_{\gamma\delta}h\,. \quad (5.5.8)$$

---

field Lagrangian is simpler as it does not involve the auxiliary superfields V^{++}, v^+, $\widetilde{v^+}$:

$$\mathcal{L} = \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm int} = \int du\, d^4\theta\, R_{\rm kin}(L^{++}, L^{+-}, L^{--}, u) + \int du\, d\bar\theta^+ d\theta^+\, K^{++}(L^{++}, u). \quad (5.5.3)$$

Here the interaction term is defined by a function $K^{++}(L^{++}, u)$ of charge $+2$. One can check that although the corresponding component Lagrangian as well as the expression for the Abelian gauge field differ from that in the non-Abelian case, the quantum Hamiltonian and the supercharges in the Abelian case have exactly the same form as in the non-Abelian case.

By its definition, the function h obeys the 3-dimensional Laplace equation,

$$\partial^{\alpha\beta}\partial_{\alpha\beta}\, h = 0\,. \tag{5.5.9}$$

Using the explicit expressions (5.5.8), it is straightforward to check the relation

$$\left(\mathcal{F}_{\alpha\beta}\right)_{\gamma\delta} = 2i\nabla_{\alpha\beta}U_{\gamma\delta}, \tag{5.5.10}$$

where

$$\left(\mathcal{F}_{\alpha\beta}\right)_{\gamma\delta} = -2\partial_{\alpha}^{\lambda}\left(\mathcal{A}_{\lambda\beta}\right)_{\gamma\delta} + i\left(\mathcal{A}_{\alpha}{}^{\lambda}\right)_{\gamma\sigma}\left(\mathcal{A}_{\lambda\beta}\right)_{\delta}^{\sigma} + (\alpha\leftrightarrow\beta)\,, \tag{5.5.11}$$

$$\nabla_{\alpha\beta}U_{\gamma\delta} = -2\partial_{\alpha\beta}U_{\gamma\delta} + i\left(\mathcal{A}_{\alpha\beta}\right)_{\gamma\lambda}U_{\delta}{}^{\lambda} + i\left(\mathcal{A}_{\alpha\beta}\right)_{\delta\lambda}U_{\gamma}{}^{\lambda}, \tag{5.5.12}$$

and $\left(\mathcal{F}_{\alpha\beta}\right)_{\gamma\delta}$ is related to the standard gauge field strength in the vector notation, see below. As we shall see soon, the condition (5.5.10) is none other than the static form of the general self-duality condition for the SU(2) Yang-Mills field on $\mathbb{R}^4$ (see Eq. (5.5.19)), i.e. the Bogomolny equations for BPS monopoles [22], while (5.5.8) provides a particular solution to these equations, being a static form of the 't Hooft ansatz [16].

Note that the relation (5.5.10) is covariant and the Lagrangian (5.5.7) is form-invariant under the "target space" SU(2) gauge transformations written in Eq. (5.3.3). This is not a genuine symmetry; rather, it is a reparametrization of the Lagrangian which allows one to cast the background potentials (5.5.8) in some different equivalent forms. It is worth noting that the gauge group indices coincide with those of the R-symmetry group, like in the four-dimensional case. Nevertheless, the "gauge" reparametrizations (5.3.3) do not affect the doublet indices of the target space coordinates $\ell^{\alpha\beta}$ and their superpartners present in the superfield L^{++}. They act only on the semi-dynamical spin variables $\varphi_{\alpha}, \bar{\varphi}^{\alpha}$ and gauge and scalar potentials (5.5.8).

Finally, the last piece in Eq. (5.5.2) yields the Fayet-Iliopoulos term,

$$\mathcal{L}_{\rm FI} = kB\,. \tag{5.5.13}$$

In the quantum case, the coefficient k is quantized, $k\in\mathbb{Z}$, on the same grounds as in the 4-dimensional case, see Section 5.3.2.

5.5.3 Vector notations in three dimensions

It is instructive to rewrite the above relations and expressions, including the full Lagrangian (5.5.2), in the vector notations. To this end, let us pass from $\ell^{\alpha\beta}$ to ℓ_i as in Eq. (5.5.1) and associate the vector $\mathcal{A}_i$ to the gauge field $\mathcal{A}_{\alpha\beta}$ (with additional matrix indices which are omitted here for simplicity) by the rule

$$\mathcal{A}_{\alpha}^{\beta} = \mathcal{A}_i\left(\sigma_i\right)_{\alpha}{}^{\beta}, \qquad \mathcal{A}_i = \frac{1}{2}\mathcal{A}_{\beta}^{\alpha}\left(\sigma_i\right)_{\alpha}{}^{\beta}, \qquad i=1,2,3. \tag{5.5.14}$$

One can check that the coordinates ℓ_i are real while the matrix $\left(\mathcal{A}_i\right)_{\gamma}{}^{\delta}$ is Hermitian. Note also the relation between the partial derivatives $\partial_{\alpha\beta} = \partial/\partial\ell^{\alpha\beta}$ and $\partial_i = \partial/\partial\ell_i$:

$$\partial_{\alpha\beta} = -\frac{1}{2}\left(\sigma_i\right)_{\alpha\beta}\partial_i, \qquad \partial_i = -\left(\sigma_i\right)_{\alpha}{}^{\beta}\partial_{\beta}^{\alpha}\,. \tag{5.5.15}$$

Let us also make a similar conversion of the gauge group indices,

$$M_\gamma{}^\delta = \frac{1}{2} M^a \left(\sigma_a\right)_\gamma{}^\delta, \qquad M^a = M_\delta{}^\gamma \left(\sigma_a\right)_\gamma{}^\delta, \qquad a = 1,2,3, \tag{5.5.16}$$

for any Hermitian traceless 2×2 matrix M.

In the new notations, the total Lagrangian (5.5.2) takes the following form :

$$\begin{aligned} \mathcal{L} = \frac{1}{2} f^{-2} \dot{\ell}_i^2 + \mathcal{A}_i^a T^a \dot{\ell}_i + i \bar{\varphi}^\alpha \left(\dot{\varphi}_\alpha + i B \varphi_\alpha\right) + k B + i \bar{\psi}_\alpha \dot{\psi}^\alpha + f^2 \nabla_i U^a T^a \, \psi \sigma_i \bar{\psi} \\ + \frac{1}{4} \left\{ f \partial_i^2 f - 3 \left(\partial_i f\right)^2 \right\} \psi^4 + 2 f^{-1} \varepsilon_{ijk} \partial_i f \, \dot{\ell}_j \, \psi \sigma_k \bar{\psi} \\ + \frac{1}{8} f^{-2} F^2 + \frac{1}{2} F \left(U^a T^a - f^{-1} \partial_i f \, \psi \sigma_i \bar{\psi} \right). \end{aligned} \tag{5.5.17}$$

where T^a defined in Eq. (5.3.2). Here

$$\nabla_i U^a = \partial_i U^a + \varepsilon^{abc} \mathcal{A}_i^b U^c \tag{5.5.18}$$

and the Bogomolny equations (5.5.10) relating $\mathcal{A}_i^a$ and U^a are equivalently rewritten in the more familiar form,

$$\mathcal{F}_{ij}^a = \varepsilon_{ijk} \nabla_k U^a, \tag{5.5.19}$$

where $\mathcal{F}_{ij}^a = \partial_i \mathcal{A}_j^a - \partial_j \mathcal{A}_i^a + \varepsilon^{abc} \mathcal{A}_i^b \mathcal{A}_j^c$. Finally, the gauge field and the matrix potential defined in (5.5.8) are rewritten as

$$\mathcal{A}_i^a = -\varepsilon_{ija} \partial_j \ln h, \qquad U^a = -\partial_a \ln h, \qquad \Delta h = 0. \tag{5.5.20}$$

After eliminating the auxiliary field F by its equation of motion,

$$F = 2 f^2 \left(f^{-1} \partial_i f \, \psi \sigma_i \bar{\psi} - U^a T^a \right), \tag{5.5.21}$$

the Lagrangian (5.5.17) takes the form

$$\begin{aligned} \mathcal{L} = \frac{1}{2} f^{-2} \dot{\ell}_i^2 + \mathcal{A}_i^a T^a \dot{\ell}_i + i \bar{\varphi}^\alpha \left(\dot{\varphi}_\alpha + i B \varphi_\alpha\right) + k B + i \bar{\psi}_\alpha \dot{\psi}^\alpha + f^2 \psi \sigma_i \bar{\psi} \left(\nabla_i + f^{-1} \partial_i f \right) U^a T^a \\ + \frac{1}{4} \left\{ f \partial_i^2 f - 4 \left(\partial_i f\right)^2 \right\} \psi^4 + 2 f^{-1} \varepsilon_{ijk} \partial_i f \, \dot{\ell}_j \, \psi \sigma_k \bar{\psi} - \frac{1}{2} f^2 (U^a T^a)^2. \end{aligned} \tag{5.5.22}$$

We see that this Lagrangian involves three physical bosonic fields ℓ_i and four physical fermionic fields ψ_α. It is fully specified by two independent functions : the metric conformal factor $f(\ell)$ which can bear an arbitrary dependence on ℓ_i and the function $h(\ell)$ which satisfies the 3-dimensional Laplace equation and determines the background non-Abelian gauge and scalar potentials. The representation (5.5.6) for h in terms of the analytic function $K(\ell^{++}, u)$ yields in fact a general solution of the 3-dimensional Laplace equation [54]. If one takes the function $h(\ell)$ to be vanishing at $|\vec{\ell}| \to \infty$, then this function can be presented as the following sum over monopoles :

$$h(\ell) = 1 + \sum_M \frac{c_M}{\left| \vec{\ell} - \vec{b}_M \right|}. \tag{5.5.23}$$

It involves particular monopole positions $\vec{b}_M$ as well as the numbers c_M associated with each monopole.

The Lagrangian (5.5.22) also contains the "semi-dynamical" spin variables $\varphi_\alpha, \bar{\varphi}^\alpha$, the role of which is the same as in the four-dimensional case : after quantization they ensure that T^a defined in (5.3.2) become matrix SU(2) generators corresponding to the spin $|k|/2$ representation.

5.5.4 Supertransformations of component fields

The component action corresponding to the Lagrangian (5.5.17) is partly on shell since we have already eliminated the fermionic fields of the auxiliary v^+ multiplet by their algebraic equations of motion. The fields of the coordinate multiplet L^{++} are still off shell. The $\mathcal{N}=4$ transformations leaving invariant the action $S=\int dt\,\mathcal{L}$ look most transparent being expressed in terms of the component fields $\ell_i, F, \chi^\alpha, \bar{\chi}^\alpha, \phi^\beta, \bar{\phi}^\beta$:

$$\begin{aligned}
&\ell_i \to \ell_i + i\epsilon\sigma_i\chi + i\bar{\epsilon}\sigma_i\bar{\chi},\\
&F \to F - 2\epsilon^\alpha\dot{\chi}_\alpha - 2\bar{\epsilon}^\alpha\dot{\bar{\chi}}_\alpha,\\
&\chi^\alpha \to \chi^\alpha - \tfrac{1}{2}iF\bar{\epsilon}^\alpha - (\bar{\epsilon}\sigma_i)^\alpha\,\dot{\ell}_i,\\
&\bar{\chi}^\alpha \to \bar{\chi}^\alpha + \tfrac{1}{2}iF\epsilon^\alpha + (\epsilon\sigma_i)^\alpha\,\dot{\ell}_i,\\
&\phi^\alpha \to \phi^\alpha - i\Big(\epsilon^\alpha\chi\sigma_i\phi + \bar{\epsilon}^\alpha\bar{\chi}\sigma_i\phi\Big)\partial_i\ln h,\\
&\bar{\phi}^\alpha \to \bar{\phi}^\alpha - i\left(\epsilon^\alpha\chi\sigma_i\bar{\phi} + \bar{\epsilon}^\alpha\bar{\chi}\sigma_i\bar{\phi}\right)\partial_i\ln h.
\end{aligned} \tag{5.5.24}$$

These transformations can be deduced from the analytic subspace realization of $\mathcal{N}=4$ supersymmetry (4.4.4), with taking into account the compensating U(1) gauge transformations of the superfields $v^+, \widetilde{v}^+$ and V^{++} needed to preserve the WZ gauge (5.4.4). Note that $\delta B = 0$ under $\mathcal{N}=4$ supersymmetry. This transformation law matches with the $\mathcal{N}=4$ superalgebra in WZ gauge, taking into account that the translation of B looks as a particular U(1) gauge transformation of the latter.

The Lagrangian (5.5.22) is invariant, modulo a total time derivative, under the transformations (5.5.24) in which F is expressed from (5.5.21).

5.5.5 Hamiltonian and supercharges

The Lagrangian (5.5.22) is the point of departure for setting up the Hamiltonian formulation of the model under consideration and quantizing the latter. After substitution of SU(2) spin-$k/2$ generators instead of T^a, the quantum Hamiltonian of this system takes the form

$$\begin{aligned}
H = {}& \frac{1}{2}f\left(\hat{p}_i - \mathcal{A}_i\right)^2 f + \frac{1}{2}f^2U^2 - f^2\nabla_i U\psi\sigma_i\bar{\psi}\\
&+ \Big\{\varepsilon_{ijk}f\partial_i f\left(\hat{p}_j - \mathcal{A}_j\right) - f\partial_k f U\Big\}\psi\sigma_k\bar{\psi} + f\partial^2 f\left\{\psi^\gamma\bar{\psi}_\gamma - \frac{1}{2}\left(\psi^\gamma\bar{\psi}_\gamma\right)^2\right\},
\end{aligned} \tag{5.5.25}$$

which is just a static 3-dimensional reduction of the 4-dimensional Hamiltonian given by Eq. (5.1.15). In this expression, the gauge field $\mathcal{A}_i = \mathcal{A}_i^a T^a$ and the scalar potential

$U = U^a T^a$ are SU(2) matrices subjected to the constraint (5.5.19). It is also easy to find the supercharges $Q_\alpha, \bar{Q}^\beta$:

$$
\begin{aligned}
Q_\alpha &= f\left(\sigma_i \bar{\psi}\right)_\alpha (\hat{p}_i - \mathcal{A}_i) - \psi^\gamma \bar{\psi}_\gamma \left(\sigma_i \bar{\psi}\right)_\alpha i\partial_i f - ifU\bar{\psi}_\alpha, \\
\bar{Q}^\alpha &= (\psi \sigma_i)^\alpha (\hat{p}_i - \mathcal{A}_i) f + i\partial_i f (\psi\sigma_i)^\alpha \psi^\gamma \bar{\psi}_\gamma + ifU\psi^\alpha,
\end{aligned}
\tag{5.5.26}
$$

The ordering ambiguity arising in the case of the general conformal factor $f(\ell)$ can be fixed, as usual, by Weyl ordering procedure [41], see Section 5.2.3 for details.

Let us emphasize that the only condition required from the background matrix fields $\mathcal{A}_i$ and U for the generators Q_α and $\bar{Q}^\beta$ to form $\mathcal{N} = 4$ superalgebra (4.5.6) is that these fields satisfy the Bogomolny equations (5.5.19). Thus the expressions (5.5.25) and (5.5.26) define the $\mathcal{N} = 4$ SQM model in the field of *arbitrary* BPS monopole, not necessarily restricted to the ansatz (5.5.20). Also, one can extend the gauge group SU(2) to SU(N) in (5.5.25) and (5.5.26).

Let us remark that the three-dimensional Hamiltonian (5.5.25) and the supercharges (5.5.26) were considered for the first time in Ref. [12] (in the Abelian case).

5.5.6 $\mathcal{N} = 4$ supersymmetry with Wu-Yang monopole

Finally, as a simple example of the monopole background consistent with the off- and on-shell $\mathcal{N} = 4$ supersymmetry, let us consider a particular 3-dimensional spherically symmetric case. It corresponds to the most general SO(3) invariant solution of the Laplace equation for the function h,

$$
h_{\mathrm{so}(3)}(\ell) = c_0 + c_1 \frac{1}{\sqrt{\ell^2}} \,. \tag{5.5.27}
$$

The corresponding potentials calculated according to Eqs. (5.5.20) read

$$
\mathcal{A}_i^a = \varepsilon_{ija} \frac{\ell_j}{\ell^2} \frac{c_1}{c_1 + c_0\sqrt{\ell^2}}, \qquad U^a = \frac{\ell_a}{\ell^2} \frac{c_1}{c_1 + c_0\sqrt{\ell^2}}. \tag{5.5.28}
$$

This configuration becomes the Wu-Yang monopole [24] for the choice $c_0 = 0$. It is easy to find the analytic function $K(\ell^{++}, u)$ which generates the solution (5.5.27) (see [6]) :

$$
\begin{aligned}
&h_{\mathrm{so}(3)}(\ell) = \int du\, K_{\mathrm{so}(3)}(\ell^{++}, u)\,, \quad K_{so(3)}(\ell^{++}, u) = c_0 + c_1 \left(1 + a^{--}\hat{\ell}^{++}\right)^{-\frac{3}{2}}\,, \\
&\ell^{++} \equiv \hat{\ell}^{++} + a^{++}\,, \quad a^{\pm\pm} = a^{\alpha\beta} u^\pm_\alpha u^\pm_\beta\,, \quad a^\alpha_\beta a^\beta_\alpha = 2\,.
\end{aligned}
\tag{5.5.29}
$$

One could equally choose as $h(\ell)$, e.g., the well-known multi-center solution to the Laplace equation, with the broken SO(3). Note that the $\mathcal{N} = 4$ mechanics with coupling to Wu-Yang monopole was recently constructed in [21], proceeding from a different approach, with the built-in SO(3) invariance and the treatment of spin variables in the spirit of Ref. [18]. Our general consideration shows, in particular, that the demand of SO(3) symmetry is not necessary for the existence of $\mathcal{N} = 4$ SQM models with non-Abelian monopole backgrounds.

5.5.7 Relation to four-dimensional $\mathcal{N}=4$ SQM system

It is instructive to show that (5.5.20) can indeed be viewed as a 3-dimensional reduction of the 't Hooft ansatz for the solution of general self-duality equation in $\mathbb{R}^4$ for the gauge group SU(2), with the identification $U^a = \mathcal{A}_0^a$, while the condition (5.5.19) is the 3-dimensional reduction of this equation.

To establish this relationship, we use the following rules of correspondence between the SO(4) = SU(2) × SU(2) spinor formalism and its SU(2) reduction :

$$\begin{aligned} &(\sigma_\mu)_{\alpha\dot\beta} \to \left\{ i\delta_\alpha^\beta, (\sigma_i)_\alpha{}^\beta \right\}, \\ &\varepsilon^{\dot\alpha\dot\beta} \to -\varepsilon_{\alpha\beta}, \qquad \varepsilon_{\dot\alpha\dot\beta} \to -\varepsilon^{\alpha\beta}, \\ &x_{\alpha\dot\beta} \to \ell_\alpha^\beta, \qquad x^{\alpha\dot\beta} \to -\ell_\beta^\alpha, \\ &\psi_{\dot\alpha} \to \psi^\alpha. \end{aligned} \tag{5.5.30}$$

This reflects the fact that the R-symmetry SU(2) in the $(\mathbf{3},\mathbf{4},\mathbf{1})$ models can be treated as a diagonal subgroup in the symmetry group SO(4) = SU(2) × SU(2) of the $(\mathbf{4},\mathbf{4},\mathbf{0})$ models, with the SU(2) factors acting, respectively, on the undotted and dotted indices.

The self-dual $\mathbb{R}^4$ SU(2) gauge field in the 't Hooft ansatz is written in the spinor notation in Eq. (5.4.13). Then, using the rules (5.5.30), one performs the reduction $\mathbb{R}^4 \to \mathbb{R}^3$ as

$$\begin{aligned} &(\mathcal{A}_{\alpha\dot\beta})_\gamma{}^\delta \;\to\; iU_\gamma{}^\delta\delta_\alpha^\beta + (\mathcal{A}_\alpha^\beta)_\gamma{}^\delta, \qquad (\mathcal{A}_\alpha^\alpha)_\gamma{}^\delta = 0, \\ &h(x) \;\to\; h(\ell), \qquad \partial_\beta^\alpha\partial_\alpha^\beta h = 0. \end{aligned} \tag{5.5.31}$$

Upon this reduction, the four-dimensional ansatz (5.4.13) yields precisely (5.5.8), while the general self-duality condition (5.4.15) goes over into the Bogomolny equations (5.5.10). Of course, the same reduction can be performed in the vector notation, with $\mathcal{F}_{\mu\nu} \to \{\mathcal{F}_{ij}, \mathcal{F}_{0k} = \nabla_k U\}$, and Eqs. (5.5.19), (5.5.20) as an output.

Thus, the general gauge field background prescribed by the off-shell $\mathcal{N}=4$ supersymmetry in this $(\mathbf{3},\mathbf{4},\mathbf{1})$ system is a static form of the 't Hooft ansatz for the self-dual SU(2) gauge field in $\mathbb{R}^4$. This suggests that the above bosonic target space reduction has its superfield counterpart relating the four-dimensional system described in Section 5.4 to the one considered here.

Indeed, the superfield $(\mathbf{3},\mathbf{4},\mathbf{1})$ action (5.5.2) can be obtained from the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet action composed from Eqs. (5.2.4), (5.4.7), (5.4.8) via the "automorphic duality" [55] by considering a restricted class of the $(\mathbf{4},\mathbf{4},\mathbf{0})$ actions with U(1) isometry and performing a superfield gauging of this isometry by an extra gauge superfield $V^{++\prime}$ along the general line of Ref. [56]. Actually, the bosonic target space reduction we have just described corresponds to the shift isometry of the analytic superfield $q^{+\dot\alpha}$ accommodating the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet, namely, to $q^{+\dot\alpha} \to q^{+\dot\alpha} + \omega u^{+\dot\alpha}$. It is the invariant projection $q^{+\dot\alpha}u^+_{\dot\alpha}$ which is going to become the $(\mathbf{3},\mathbf{4},\mathbf{1})$ superfield L^{++} upon gauging this isometry and choosing the appropriate manifestly $\mathcal{N}=4$ supersymmetric gauge.

An important impact of this superfield reduction on the structure of the component action is the appearance of the new induced potential bilinear in the gauge group generators $\sim U^2 = U^aU^bT^aT^b$. It comes out as a result of eliminating the auxiliary field F in the off-shell $(\mathbf{3},\mathbf{4},\mathbf{1})$ multiplet, and so is necessarily prescribed by $\mathcal{N}=4$ supersymmetry.

It is interesting that analogous potential terms were introduced in [57] at the bosonic level for ensuring the existence of some hidden symmetries in the models of 3-dimensional particle in a non-Abelian monopole background.

The same reduction $\mathbb{R}^4 \to \mathbb{R}^3$ can be performed at the level of Hamiltonian and supercharges. In particular, the reduction of the Hamiltonian of the four-dimensional system of Eq. (5.1.15) yields the 3-dimensional Hamiltonian (5.5.25).

Conclusion

We studied some rather general off-shell $\mathcal{N} = 4$ supersymmetric coupling of the $d = 1$ coordinate supermultiplets $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ and $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ to an external self-dual (or anti-self-dual) Abelian gauge field and discussed the $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ case in details. Our main framework was the harmonic superspace approach.

The use of an analytic "semi-dynamical" multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ with the Wess-Zumino type action allowed us to make coupling of the coordinate multiplets $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ and $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ to an external SU(2) gauge field. This auxiliary multiplet incorporates SU(2) doublet of bosonic spin variables which are crucial for arranging couplings to non-Abelian gauge fields. In the four-dimensional case, the off-shell $\mathcal{N} = 4$ supersymmetry restricts the non-Abelian gauge field to be self-dual (or anti-self-dual) and in a form of the 't Hooft ansatz for SU(2) gauge field. In the three-dimensional case, the non-Abelian gauge field is a three-dimensional reduction of this 't Hooft ansatz, i.e. a particular solution of the Bogomolny monopole equations. Additionally, in three dimensions, at the component level, the coupling to a gauge field is necessarily accompanied by an induced potential which is bilinear in the SU(2) generators and arises as a result of eliminating the auxiliary field in the coordinate $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet.

The explicit form of the Hamiltonians and the supercharges were presented. The corresponding expressions respect $\mathcal{N} = 4$ *on-shell* supersymmetry for any self-dual or anti-self-dual, Abelian or non-Abelian gauge field in four-dimensions, not necessarily in the 't Hooft ansatz form. In three dimensions, an arbitrary BPS monopole background can be used in the non-Abelian case.

It is worthwhile to note that similar constraints (Bogomolny equations) on the external non-Abelian three-dimensional gauge field were found in [58], while considering an $\mathcal{N} = 4$ extension of Berry phase in quantum mechanics. However, no invariant actions and/or the explicit expressions for the Hamiltonian and $\mathcal{N} = 4$ supercharges were presented there.

The nonlinear counterpart of $q^{+\dot{\alpha}}$ multiplet is discussed in [46]. In this case, the bosonic target geometry is more general as compared to the conformally-flat geometry associated with the linear $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet.

Among the possible directions of further study, we mention the construction of higher $\mathcal{N}$ SQM models with non-Abelian gauge field backgrounds, e.g. $\mathcal{N} = 8$ ones, as well as studying various supersymmetry-preserving reductions of these models to lower-dimensional target bosonic manifolds by the gauging procedure of [56]. Actually, the method of the auxiliary "semi-dynamical" $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet with the Wess-Zumino type action, which was successfully applied in our construction here, could work with the equal efficiency for constructing a Lagrangian description of other supersymmetric quantum-mechanics problems involving the coupling to an external non-Abelian gauge field. Besides the obvious examples of quantum Hall effect (or Landau problem) in higher dimensions (see e.g.

the discussion in [8]), let us also mention supersymmetric Wilson loop functionals which can be interpreted in terms of a non-Abelian version of Chern-Simons (super)quantum mechanics [59], with the parameter along the loop as an evolution parameter. We hope that the quantized semi-dynamical variables could provide a new efficient tool to study this class of problems.

The 't Hooft type ansatz (5.3.5) and the choice of SU(2) as the gauge group are required for the existence of the *off-shell* superfield formulation of the discussed SQM systems. It is not known whether the most general system can be derived from some off-shell superfield formalism, with general instanton/monopole backgrounds obtained from the ADHM construction [60] or its three-dimensional reduction. Additionally, there remains a problem of extending the models to a generic SU(N) gauge group. Possibly, the above issues are related to the generalization of the interaction term (5.4.7) to

$$S_{\rm int} = \int dt\, du\, d\bar\theta^{+} d\theta^{+}\, K^{++}\left(q^{+\dot\alpha},\, u^{\pm}_{\beta},\, v^{+}\widetilde{v^{+}}\right).$$

It would be also interesting to study SQM models with nonlinear counterpart of the semi-dynamical multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ [56].

Annexe A
Text of reference [2]

ITEP-TH-52/09

Self-duality and supersymmetry

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We observe that the Hamiltonian $H = \not{D}^2$, where $\not{D}$ is the flat $4d$ Dirac operator in a self-dual gauge background, is supersymmetric, admitting 4 different real supercharges. A generalization of this model to the motion on a curved conformally flat $4d$ manifold exists. For an Abelian self-dual background, the corresponding Lagrangian can be derived from known harmonic superspace expressions.

A.1 Introduction

The main purpose of this paper is to present a simple supersymmetric quantum mechanical model which, surprisingly, did not attract much attention so far.

It was known since some time that one can treat the problem of the motion of a fermion on an even-dimensional manifold with an arbitrary gauge field background as a supersymmetric one such that, e.g., the Atiyah-Singer index of a Dirac operator can be interpreted as the Witten index of a certain supersymmetric Hamiltonian [1]. Our remark is that if the gauge field is self-dual and the $4d$ metric is flat, the system enjoys an extended supersymmetry with two pairs of supercharges. A similar $\mathcal{N} = 2$ supersymmetric [1] system can be written for conformally flat 4d manifolds, though supercharges in this case are not related to $\not{\mathcal{D}}$, and the Hamiltonian does not coincide with $\not{\mathcal{D}}^2$.

In Sect. A.2, we present the model. In Sect. A.3, we analyze in some more details its simplest version (flat metric and constant self-dual Abelian field density) and derive the spectrum. In Sect. A.4, we derive the component Lagrangian from a certain harmonic superspace (HSS) [2] action suggested in Ref. [3].

A.2 Fermions in $4d$ self-dual background

Consider the Dirac operator in flat $4d$ Euclidean space

$$\not{\mathcal{D}} \;=\; \sum_{\mu=0,1,2,3} \mathcal{D}_\mu \gamma_\mu \ , \tag{A.2.1}$$

where $\mathcal{D}_\mu = \partial_\mu - i\mathcal{A}_\mu$ and γ_μ are Euclidean anti-Hermitian gamma–matrices,

$$\gamma_\mu = \begin{pmatrix} 0 & -\sigma_\mu^\dagger \\ \sigma_\mu & 0 \end{pmatrix}, \qquad \{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu} \ , \tag{A.2.2}$$

with $(\sigma_\mu)_{\alpha\dot\beta} = \{i, \vec\sigma\}_{\alpha\dot\beta}$ and $(\sigma_\mu^\dagger)^{\dot\beta\alpha} = \{-i, \vec\sigma\}^{\dot\beta\alpha}$ ($\vec\sigma$ are ordinary Pauli matrices). The indices are raised and lowered, as usual, with antisymmetric Levi-Civita tensor $\varepsilon_{\alpha\beta} = \varepsilon_{\dot\alpha\dot\beta} = -\varepsilon^{\alpha\beta} = -\varepsilon^{\dot\alpha\dot\beta}$, $\varepsilon_{12} = 1$. (These are more or less the conventions of [4] rotated to Euclidean space.) The Hamiltonians we are going to construct enjoy SO(4) = SU(2) × SU(2) covariance such that the undotted spinor index refers to the first SU(2) factor, while the dotted one to the second. The matrices σ_μ, $\sigma_\mu^\dagger$ satisfy the identities

$$\begin{gathered} \sigma_\mu \sigma_\nu^\dagger + \sigma_\nu \sigma_\mu^\dagger = \sigma_\mu^\dagger \sigma_\nu + \sigma_\nu^\dagger \sigma_\mu = 2\delta_{\mu\nu}, \\ \sigma_\mu^\dagger \sigma_\nu - \sigma_\nu^\dagger \sigma_\mu = 2i\, \eta^a_{\mu\nu} \sigma_a, \\ \sigma_\mu \sigma_\nu^\dagger - \sigma_\nu \sigma_\mu^\dagger = 2i\, \bar\eta^a_{\mu\nu} \sigma_a, \end{gathered} \tag{A.2.3}$$

where $\eta^a_{\mu\nu}$, $\bar\eta^a_{\mu\nu}$ are the ’t Hooft symbols,

$$\eta^a_{ij} = \bar\eta^a_{ij} = \varepsilon_{aij}, \quad \eta^a_{i0} = -\eta^a_{0i} = \bar\eta^a_{0i} = -\bar\eta^a_{i0} = \delta_{ai} \tag{A.2.4}$$

[1] $\mathcal{N}$ counts the number of complex supercharges.

(σ_a – Pauli matrices, indices a, i, j run from 1 to 3). They are self-dual (anti-self-dual),

$$\eta^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}\eta^a_{\rho\lambda}, \qquad \bar{\eta}^a_{\mu\nu} = -\frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}\bar{\eta}^a_{\rho\lambda}, \tag{A.2.5}$$

with the convention $\varepsilon_{0123} = -1$. Another useful identity is

$$\sigma_2\sigma_\mu^T\sigma_2 = \ -\sigma_\mu^\dagger \ . \tag{A.2.6}$$

Consider the operator

$$H = \frac{1}{2}\not{\mathcal{D}}^2 \ = \ -\frac{1}{2}\mathcal{D}^2 - \frac{i}{4}\mathcal{F}_{\mu\nu}\gamma_\mu\gamma_\nu \ , \tag{A.2.7}$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i\,[\mathcal{A}_\mu, \mathcal{A}_\nu]$ is the field strength. It is well known that nonzero eigenvalues of the Euclidean Dirac operator come in pairs $(-\lambda, \lambda)$ and hence the spectrum of the Hamiltonian H is double-degenerate for all excited states. This means that, for any external field $\mathcal{A}_\mu$, this Hamiltonian is supersymmetric [1] admitting two different anticommuting real supercharges : $\not{\mathcal{D}}$ and $i\not{\mathcal{D}}\gamma_5$ ($\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$). Suppose now that the background field is self-dual,

$$\mathcal{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\delta}\mathcal{F}_{\rho\delta} \quad \longleftrightarrow \quad \mathcal{F}_{\mu\nu} = \eta^a_{\mu\nu}B_a \ . \tag{A.2.8}$$

One can be easily convinced that in this case the Hamiltonian admits *four* different Hermitian square roots S_A that satisfy the extended supersymmetry algebra

$$\{S_A, S_B\} = 4\delta_{AB}H \ . \tag{A.2.9}$$

One of the choices is

$$\begin{array}{l} S_1 = \not{\mathcal{D}} = \gamma_0\mathcal{D}_0 + \gamma_1\mathcal{D}_1 + \gamma_2\mathcal{D}_2 + \gamma_3\mathcal{D}_3 \ , \\ S_2 = \gamma_0\mathcal{D}_3 + \gamma_1\mathcal{D}_2 - \gamma_2\mathcal{D}_1 - \gamma_3\mathcal{D}_0 \ , \\ S_3 = \gamma_0\mathcal{D}_2 - \gamma_1\mathcal{D}_3 - \gamma_2\mathcal{D}_0 + \gamma_3\mathcal{D}_1 \ , \\ S_4 = \gamma_0\mathcal{D}_1 - \gamma_1\mathcal{D}_0 + \gamma_2\mathcal{D}_3 - \gamma_3\mathcal{D}_2 \ . \end{array} \tag{A.2.10}$$

Introducing the complex supercharges

$$\begin{array}{ll} Q_1 = (S_1 - iS_2)/2, & Q_2 = (S_3 - iS_4)/2, \\ \bar{Q}^1 = (S_1 + iS_2)/2, & \bar{Q}^2 = (S_3 + iS_4)/2, \end{array} \tag{A.2.11}$$

we obtain the standard $\mathcal{N} = 2$ supersymmetry algebra [2]

$$\{Q_\alpha, Q_\beta\} = 0, \qquad \left\{Q_\alpha, \bar{Q}^\beta\right\} = 2\delta^\beta_\alpha H. \tag{A.2.12}$$

Correspondingly, the excited spectrum of H is four-fold degenerate, while the spectrum of $\not{\mathcal{D}}$ consists of the quartets involving two degenerate positive and two degenerate negative eigenvalues.

[2]Note that, in contrast to $\not{\mathcal{D}}$, the operator $\not{\mathcal{D}}\gamma_5$ is not expressed into a linear combination of S_A. In other words, the $\mathcal{N} = 1$ supersymmetry algebra with the operators $\not{\mathcal{D}}(1 \pm \gamma_5)$ is not a subalgebra of the $\mathcal{N} = 2$ algebra (A.2.12).

The algebra (A.2.9) with supercharges (A.2.10) holds for any self-dual field, irrespectively of whether it is Abelian or non-Abelian. Thus, the additional 2-fold degeneracy of the spectrum of the Dirac operator mentioned above should be there for a generic self-dual field. One particular example of a non-Abelian self-dual field is the instanton solution, where this degeneracy was observed back in [5] [see Eqs. (4.15) there].

To make contact with the Lagrangian (and, especially, superfield) description, it is convenient to introduce holomorphic fermion variables, which satisfy the standard anticommutation relations

$$\{\psi_{\dot\alpha}, \psi_{\dot\beta}\} \;=\; \{\bar\psi^{\dot\alpha}, \bar\psi^{\dot\beta}\} \;=\; 0, \qquad \{\bar\psi^{\dot\alpha}, \psi_{\dot\beta}\} = \delta^{\dot\alpha}_{\dot\beta} \;. \tag{A.2.13}$$

One of the possible choices is

$$\begin{aligned} \psi_{\dot 1} &= \frac{-\gamma_0 + i\gamma_3}{2}, & \bar\psi^{\dot 1} &= \frac{\gamma_0 + i\gamma_3}{2}, \\ \psi_{\dot 2} &= \frac{\gamma_2 + i\gamma_1}{2}, & \bar\psi^{\dot 2} &= \frac{-\gamma_2 + i\gamma_1}{2}. \end{aligned} \tag{A.2.14}$$

Then two complex supercharges (A.2.11) are expressed in a very simple way,

$$\begin{aligned} Q_\alpha &= \left(\sigma_\mu \bar\psi\right)_\alpha (\hat p_\mu - \mathcal{A}_\mu)\,, \\ \bar Q^\alpha &= \left(\psi \sigma_\mu^\dagger\right)^\alpha (\hat p_\mu - \mathcal{A}_\mu)\,, \end{aligned} \tag{A.2.15}$$

with $\hat p_\mu = -i\partial_\mu$. The Hamiltonian (A.2.7) is expressed in these terms as

$$H = \frac{1}{2}\,(\hat p_\mu - \mathcal{A}_\mu)^2 + \frac{i}{4}\,\mathcal{F}_{\mu\nu}\,\psi\sigma_\mu^\dagger\sigma_\nu\bar\psi \;. \tag{A.2.16}$$

It is clear now why the spinor indices in Eq.(A.2.12) are undotted, while in Eq.(A.2.14) they are dotted. The supercharges are rotated by the first SU(2) and the variables $\psi_{\dot\alpha}$ by the second [3]. A careful distinction between two different SU(2) factors allows one to understand better the reason why the supercharges (A.2.15) satisfy the simple algebra (A.2.12) in a self-dual background. The self-dual field density $\mathcal{F}$ carries in the spinor notation only dotted indices. Therefore any expression involving $\mathcal{F}, \psi, \bar\psi$ is a scalar with respect to undotted SU(2). The only such scalar that can appear in the r.h.s. of the anticommutators of the supercharges $\{Q_\alpha, \bar Q^\beta\}$ is the structure which is proportional to δ_α^β, i.e. the Hamiltonian. No other operator is allowed.

In the *Abelian* case, the supercharges (A.2.15) and the Hamiltonian (A.2.16) are scalar operators not carrying matrix indices anymore. This allows one to derive the Lagrangian,

$$L = \frac{1}{2}\,\dot x_\mu \dot x_\mu + \mathcal{A}_\mu(x)\dot x_\mu + i\bar\psi^{\dot\alpha}\dot\psi_{\dot\alpha} - \frac{i}{4}\mathcal{F}_{\mu\nu}\,\psi\sigma_\mu^\dagger\sigma_\nu\bar\psi \;. \tag{A.2.17}$$

In the non-Abelian case, the expressions (A.2.15, A.2.16) still keep their color matrix structure, and one cannot derive the Lagrangian in a so straightforward way. One of the ways to handle the matrix structure is to introduce a set of color fermion variables (say, in

[3]Note that complex conjugation leaves the spinors in the same representation, the symmetry group here is SO(4) rather that SO(3, 1).

the fundamental representation of the group) and impose the extra constraint considering only the sector with unit fermion charge [1]. An alternative (non-Abelian) construction of the Lagrangian is presented in [6], but in this paper we consider Lagrangians only for Abelian fields.

As will be demonstrated explicitly in Sect. A.4, the component Lagrangian (A.2.17) follows from the superfield action written earlier by Ivanov and Lechtenfeld in the framework of harmonic superspace approach [3]. We will see that one can naturally derive in this way a σ-model type generalization of the Lagrangian (A.2.17) describing the motion over the manifold with nontrivial conformally flat metric $ds^2 = \{f(x)\}^{-2}\, dx_\mu dx_\mu$. It is written as follows

$$L = \frac{1}{2} f^{-2}\, \dot{x}_\mu \dot{x}_\mu + \mathcal{A}_\mu(x)\dot{x}^\mu + i\bar{\psi}^{\dot\alpha}\dot{\psi}_{\dot\alpha} - \frac{i}{4} f^2 \mathcal{F}_{\mu\nu}\, \psi\sigma^\dagger_\mu\sigma_\nu\bar{\psi} + \frac{1}{4}\left\{3\,(\partial_\mu f)^2 - f\partial^2 f\right\}\psi^4 + \frac{i}{2} f^{-1}\partial_\mu f\, \dot{x}_\nu\, \psi\sigma^\dagger_{[\mu}\sigma_{\nu]}\bar{\psi}\ . \quad \text{(A.2.18)}$$

The corresponding (quantum) Noether supercharges and the Hamiltonian are

$$\begin{aligned} Q_\alpha &= f\left(\sigma_\mu\bar{\psi}\right)_\alpha (\hat{p}_\mu - \mathcal{A}_\mu) - \psi_{\dot\gamma}\bar{\psi}^{\dot\gamma}\left(\sigma_\mu\bar{\psi}\right)_\alpha i\partial_\mu f, \\ \bar{Q}^\alpha &= \left(\psi\sigma^\dagger_\mu\right)^\alpha (\hat{p}_\mu - \mathcal{A}_\mu)\, f + i\partial_\mu f\left(\psi\sigma^\dagger_\mu\right)^\alpha \cdot \psi_{\dot\gamma}\bar{\psi}^{\dot\gamma}, \end{aligned} \quad \text{(A.2.19)}$$

$$H = \frac{1}{2} f\,(\hat{p}_\mu - \mathcal{A}_\mu)^2\, f + \frac{i}{4} f^2\, \mathcal{F}_{\mu\nu}\, \psi\sigma^\dagger_\mu\sigma_\nu\bar{\psi} - \frac{1}{2} f i\partial_\mu f\,(\hat{p}_\nu - \mathcal{A}_\nu)\,\psi\sigma^\dagger_{[\mu}\sigma_{\nu]}\bar{\psi} + f\partial^2 f\left\{\psi_{\dot\gamma}\bar{\psi}^{\dot\gamma} - \frac{1}{2}\left(\psi_{\dot\gamma}\bar{\psi}^{\dot\gamma}\right)^2\right\}. \quad \text{(A.2.20)}$$

On the other hand, one can explicitly calculate the anticommutators of the supercharges (A.2.19) for any self-dual [4] field $\mathcal{A}_\mu(x)$, Abelian or non-Abelian, and verify that the algebra (A.2.12) holds. While doing this, the use of the following Fierz identity

$$\left(\bar{\psi}\sigma^\dagger_\mu\right)^\beta \left(\sigma_\nu\psi\right)_\alpha - \left(\sigma_\mu\bar{\psi}\right)_\alpha \left(\psi\sigma^\dagger_\nu\right)^\beta = \delta^\beta_\alpha\, \bar{\psi}\sigma^\dagger_\mu\sigma_\nu\psi\ , \quad \text{(A.2.21)}$$

which can be proven using (A.2.6), is convenient.

Note that, with a nontrivial factor $f(x)$, the supercharges (A.2.19) have nothing to do with the Dirac operator $\not{\mathcal{D}}$ in a conformally flat background : the latter cannot be expressed as a linear combination of Q_α and $\bar{Q}^\alpha$. In addition, the Hamiltonian (A.2.20) does not coincide with $\not{\mathcal{D}}^2/2$.

The model (A.2.18-A.2.20) is a close relative to the model constructed in Ref. [7] (see Eqs. (30,31) there), which describes the motion on a *three*-dimensional conformally flat manifold in external magnetic field and a scalar potential. In fact, the latter model can be obtained from the former, if assuming that the metric and the vector potential

[4] Anti-self-duality conditions are obtained when one interchanges σ_μ and $\sigma^\dagger_\mu$ in all the formulas. This is equivalent to the interchange of two spinor representations of SO(4).

$\mathcal{A}_\mu \equiv (\Phi, \vec{\mathcal{A}})$ depend only on three spatial coordinates x_i. If assuming further that the metric is flat, one is led to the Hamiltonian [8]

$$H \;=\; \frac{1}{2}\left(\hat{\vec{p}} - \vec{\mathcal{A}}\right)^2 + \frac{1}{2}\Phi^2 + \vec{\nabla}\Phi\,\psi\vec{\sigma}\bar{\psi}, \tag{A.2.22}$$

which is supersymmetric under the condition $\mathcal{F}_{ij} = \varepsilon_{ijk}\partial_k\Phi$ (the $3d$ reduction of the $4d$ self-duality condition). It was noticed in Ref. [7] that the effective Hamiltonian of a chiral supersymmetric electrodynamics in finite spatial volume belongs to this class with $\Phi \propto 1/|\vec{A}|$. The vector potential $\vec{\mathcal{A}}(\vec{A})$ describes in this case a Dirac magnetic monopole such that the Berry phase appears. The three dynamical variables $\vec{A}$ (do not confuse with curly $\vec{\mathcal{A}}$!) have in this case the meaning of the zero Fourier harmonic of the vector potential in the original field theory. In the leading order, the metric is flat. When higher loop corrections are included, a (conformally flat!) metric on the moduli space $\{\vec{A}\}$ appears.

Performing the Hamiltonian reduction of Eq. (A.2.20) with non-Abelian $\mathcal{A}_\mu$, a non-Abelian generalization of Eq. (A.2.22) can easily be derived. It keeps the gauge structure of Eq. (A.2.22) with matrix-valued $\vec{\mathcal{A}}$ and Φ satisfying the condition $\mathcal{F}_{ij} = \varepsilon_{ijk}\mathcal{D}_k\Phi$. Note that such Hamiltonian does *not* coincide with the non-Abelian $3d$ Hamiltonian derived in Ref. [9].

A.3 Constant field

As an illustration, consider the system described by the Hamiltonian (A.2.16) in a constant self-dual Abelian background. The constant self-dual field strength $\mathcal{F}_{\mu\nu} = \eta^a_{\mu\nu}B_a$ is parametrized by three independent components. Let us direct B^a along the third axis, $B_a = (0, 0, B)$, and choose the gauge

$$\mathcal{A}_0 = Bx_3, \qquad \mathcal{A}_2 = Bx_1, \qquad \mathcal{A}_1 = \mathcal{A}_3 = 0. \tag{A.3.1}$$

The Hamiltonian (A.2.16) acquires the form

$$H = \left\{\frac{1}{2}\left(\hat{p}_0 - Bx_3\right)^2 + \frac{1}{2}\hat{p}_3^2 + B\left(\chi_1\bar{\chi}^1 - \frac{1}{2}\right)\right\} + \left\{\frac{1}{2}\left(\hat{p}_2 - Bx_1\right)^2 + \frac{1}{2}\hat{p}_1^2 + B\left(\chi_2\bar{\chi}^2 - \frac{1}{2}\right)\right\}. \tag{A.3.2}$$

For convenience, we have introduced notations $\chi_1 = \bar{\psi}^{\dot{1}}$, $\bar{\chi}^1 = \psi_{\dot{1}}$, $\chi_2 = \psi_{\dot{2}}$, $\bar{\chi}^2 = \bar{\psi}^{\dot{2}}$. The Hamiltonian is thus reduced to the sum $H_1 + H_2$ of two independent (acting in different Hilbert spaces) supersymmetric Hamiltonians, each describing the 2-dimensional motion of an electron in homogeneous orthogonal to the plane magnetic field $\vec{B}$. The bosonic sector of each such Hamiltonian corresponds to the spin projection $\vec{s}\vec{B}/|\vec{B}| = -1/2$, and the fermionic sector to the spin projection $\vec{s}\vec{B}/|\vec{B}| = 1/2$. This is the first and the simplest supersymmetric quantum problem ever considered [10]. The energy levels for each Hamiltonian are $\varepsilon_i = B\left(n_i + \frac{1}{2} + s_i\right)$, $n_i \geq 0$ – integers, $s_i = \pm\frac{1}{2}$. Each level of H_i is doubly degenerate. Besides, there is an infinite degeneracy associated with the positions of the center of the orbit along the axes 1 and 3 that are proportional to the integrals of motion p_2 and p_0. The full spectrum

$$E = B\,(n_1 + n_2 + 1 + s_1 + s_2) \tag{A.3.3}$$

is thus 4-fold degenerate at each level (except for the state with $E = 0$).

It might be instructive to explicitly associate this degeneracy with the action of supercharges (A.2.15). Let us assume for definiteness $B > 0$. One can represent Q_α as

$$Q_1 = \sqrt{2B}\left(b\chi_1 + a^\dagger \bar{\chi}^2\right), \qquad Q_2 = \sqrt{2B}\left(a\chi_1 - b^\dagger \bar{\chi}^2\right)\ , \tag{A.3.4}$$

where $a^\dagger$, $b^\dagger$ and a, b are the creation and annihilation operators,

$$a = \frac{1}{\sqrt{2B}}\,(\hat{p}_1 - iBx_1\ + ip_2)\,, \qquad b = \frac{1}{\sqrt{2B}}\,(\hat{p}_3 - iBx_3\ + ip_0)\ , \tag{A.3.5}$$

$$\left[a, a^\dagger\right] = 1, \qquad \left[b, b^\dagger\right] = 1. \tag{A.3.6}$$

In these notations, the Hamiltonian (A.3.2) takes a very simple form

$$H = B\left\{a^\dagger a + b^\dagger b + \chi_1\bar{\chi}^1 + \chi_2\bar{\chi}^2\right\}. \tag{A.3.7}$$

Obviously, the energy levels of the Hamiltonian (A.3.2) are defined by two integrals of motion $p_{2,0}$, two oscillator excitation numbers $n_{1,2}$ and two spins $s_{1,2}$, as in Eq. (A.3.3). For each p_2, p_0, there is a unique ground zero energy state $|0\rangle$ annihilated by all supercharges. A quartet of excited states can be represented as

$$|n_1, n_2\rangle\,, \quad Q_1^\dagger\,|n_1, n_2\rangle\,, \quad Q_2^\dagger\,|n_1, n_2\rangle\,, \quad Q_1^\dagger Q_2^\dagger\,|n_1, n_2\rangle\ , \tag{A.3.8}$$

where the state

$$|n_1, n_2\rangle \equiv \chi_1 \cdot \left(a^\dagger\right)^{n_1}\left(b^\dagger\right)^{n_2}|0\rangle$$

of energy $E = B(n_1 + n_2 + 1)$ is annihilated by both Q_1 and Q_2.

For each p_2, p_0, there are N such quartets at the energy level $E = BN$.

A.4 From harmonic superspace to components

In this section, we derive the Hamiltonian (A.2.20) in the HSS approach. To make the paper self-consistent, we present in the Appendix its salient features and definitions in application to quantum mechanical problems. The relevant superfield action was written in [3], and we show here that the corresponding component Lagrangian coincides with (A.2.18). The corresponding supercharges (A.2.19) and the Hamiltonian (A.2.20) involve an Abelian self-dual gauge field $\mathcal{A}_\mu(x)$. The non-Abelian case is treated in a separate publication [6].

Let us introduce a doublet of superfields $q^{+\dot\alpha}$ with charge $+1$ ($D^0 q^+ = q^+$) satisfying the constraints (A.5.11). The index $\dot\alpha$ is the fundamental representation index of an additional external group SU(2). The solution for these constraints in the analytical basis is [see Eq.(A.5.12)]

$$q^{+\dot\alpha} = x^{\alpha\dot\alpha}(t_{\rm A})u^+_\alpha - 2\theta^+\chi^{\dot\alpha}(t_{\rm A}) - 2\bar{\theta}^+\bar{\chi}'^{\dot\alpha}(t_{\rm A}) - 2i\theta^+\bar{\theta}^+\partial_{\rm A}x^{\alpha\dot\alpha}(t_{\rm A})u^-_\alpha\ . \tag{A.4.1}$$

We impose now the additional pseudoreality condition

$$q^{+\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\widetilde{q^{+\dot\beta}} \ , \tag{A.4.2}$$

the field $\tilde q^{+}$ being defined in Eq.(A.5.15). It implies

$$x^{\alpha\dot\alpha} = -\left(x_{\alpha\dot\alpha}\right)^* , \qquad \bar\chi'^{\dot\alpha} = \left(\chi_{\dot\alpha}\right)^* \equiv \bar\chi^{\dot\alpha} \ . \tag{A.4.3}$$

Let us go back now to the central basis $\left\{t, \theta_\alpha, \bar\theta^\beta, u^\pm_\gamma\right\}$. The solution can be presented as $q^{+\dot\alpha} = u^+_\alpha q^{\alpha\dot\alpha}$ where $q^{\alpha\dot\alpha}$ does not depend on $u^\pm_\alpha$ (the latter follows from the constraint $D^{++}q^{+\dot\alpha} = 0$ and the definition $D^{++} = u^+_\alpha \frac{\partial}{\partial u^-_\alpha}$). It is convenient to go over to the $4d$ vector notation, introducing

$$q_\mu = -\frac{1}{2}\left(\sigma_\mu\right)_{\alpha\dot\alpha} q^{\alpha\dot\alpha}, \qquad q^{+\dot\alpha} = -q_\mu \left(\sigma^\dagger_\mu\right)^{\dot\alpha\alpha} u^+_\alpha \ . \tag{A.4.4}$$

Now, q_μ is a vector with respect to the group SO(4) = SU(2) × SU(2), with the first factor representing the $\mathcal{N} = 2$ R-symmetry group and the second one being the extra global SU(2) group rotating the dotted "flavor" indices.

Pseudoreality condition (A.4.2) implies that the superfield q_μ is real. The latter is expressed in components as follows,

$$q_\mu = x_\mu + \theta\sigma_\mu\chi + \bar\theta\sigma_\mu\bar\chi - \frac{i}{2}\dot x_\nu\,\bar\theta\sigma_{[\mu}\sigma^\dagger_{\nu]}\theta + \frac{i}{2}\bar\theta\sigma_\mu\dot\chi\,\theta^2 - \frac{i}{2}\theta\sigma_\mu\dot{\bar\chi}\,\bar\theta^2 - \frac{1}{4}\ddot x_\mu\,\theta^4 \ , \tag{A.4.5}$$

where $\theta^2 \equiv \theta^\alpha\theta_\alpha$, $\bar\theta^2 \equiv \bar\theta^\alpha\bar\theta_\alpha$, $\theta^4 \equiv \theta^2\bar\theta^2$.

The classical $\mathcal{N} = 2$ SUSY invariant action for the superfield q_μ can now be written. It consists of two parts, $S = S_{\rm kin} + S_{\rm int}$. The kinetic part,

$$S_{\rm kin} = \int dt\, d^4\theta du\, R'_{\rm kin}(q^{+\dot\alpha}, q^{-\dot\beta}, u^\pm_\gamma) = \int dt\, d^4\theta\, R_{\rm kin}(q_\mu), \tag{A.4.6}$$

depends on an arbitrary function $R_{\rm kin}(q_\mu)$. Plugging (A.4.5) into (A.4.6) and adding/subtracting proper total derivatives, we obtain

$$S_{\rm kin} = \int dt \left\{\frac{1}{2}g(x)\,\dot x_\mu\dot x_\mu + \frac{i}{2}g(x)\left(\bar\chi^{\dot\alpha}\dot\chi_{\dot\alpha} - \dot{\bar\chi}^{\dot\alpha}\chi_{\dot\alpha}\right) + \frac{1}{8}\partial^2 g(x)\,\chi^4 - \frac{i}{4}\partial_\mu g(x)\,\dot x_\nu\,\chi\sigma^\dagger_{[\mu}\sigma_{\nu]}\bar\chi\right\}, \tag{A.4.7}$$

where $g(x) = \frac{1}{2}\partial^2_x R_{\rm kin}(x)$ and $\chi^4 = \chi^{\dot\alpha}\chi_{\dot\alpha}\bar\chi^{\dot\beta}\bar\chi_{\dot\beta}$.

To couple x_μ to an external gauge field, one should add the interaction term $S_{\rm int}$ that represents an integral over *analytic* superspace,

$$S_{\rm int} = \int dt\, du\, d\bar\theta^+ d\theta^+ R^{++}_{\rm int}\left(q^{+\dot\alpha}(t_{\rm A}, \theta^+, \bar\theta^+), u^\pm_\gamma\right) \ . \tag{A.4.8}$$

We choose $R^{++}_{\rm int}$ (it carries the charge 2) satisfying the condition $\widetilde{R}^{++}_{\rm int} = -R^{++}_{\rm int}$ [the involution operation $\widetilde{X}$ being defined in Eqs. (A.5.13), (A.5.14)] such that the action (A.4.8) is real.

To do the integral over θ^+ and $\bar\theta^+$, introduce $x^{+\dot\alpha} = -x_\mu \left(\sigma^\dagger_\mu\right)^{\dot\alpha\alpha} u^+_\alpha \equiv x^{\alpha\dot\alpha}u^+_\alpha$ [see Eq.(A.4.4)]. Then

$$S_{\rm int} = \int dt\, du \left\{ 2i \left(\sigma^\dagger_\mu\right)^{\dot\alpha\alpha} \partial_{+\dot\alpha} R^{++}_{\rm int}\, u^-_\alpha \cdot \dot x_\mu - 4\chi^{\dot\alpha}\bar\chi^{\dot\beta}\, \partial_{+\dot\alpha}\partial_{+\dot\beta} R^{++}_{\rm int} \right\} \tag{A.4.9}$$

with

$$\partial_{+\dot\alpha} R^{++}_{\rm int}(x,u) \equiv \frac{\partial R^{++}_{\rm int}(x^{+\dot\gamma}, u^\pm_\gamma)}{\partial x^{+\dot\alpha}}. \tag{A.4.10}$$

Now, define the gauge field,

$$\mathcal{A}_\mu(x) \equiv \int du \left\{ 2i \left(\sigma^\dagger_\mu\right)^{\dot\alpha\alpha} \partial_{+\dot\alpha} R^{++}_{\rm int}\, u^-_\alpha \right\}. \tag{A.4.11}$$

As the action (A.4.9) is real, the field $\mathcal{A}_\mu(x)$ is also real. It has zero divergence, $\partial_\mu \mathcal{A}_\mu = 0$.

The field strength is expressed as

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu = -2\eta^a_{\mu\nu} \int du\, \partial_{+\dot\alpha}\partial_{+\dot\beta} R^{++}_{\rm int}\, \varepsilon^{\dot\alpha\dot\gamma} \left(\sigma_a\right)^{\dot\beta}{}_{\dot\gamma} \tag{A.4.12}$$

(the identities (A.2.3) were used). It is obviously self-dual. With the definitions (A.4.11) and (A.4.12) in hand, one can represent the interaction term (A.4.9) as

$$S_{\rm int} = \int dt \left\{ \mathcal{A}_\mu(x)\dot x_\mu - \frac{i}{4}\mathcal{F}_{\mu\nu}\, \chi\sigma^\dagger_\mu\sigma_\nu\bar\chi \right\}. \tag{A.4.13}$$

Adding this to the kinetic term in (A.4.7) [where one can get rid of the factor $g(x)$ in the fermion kinetic term by introducing canonically conjugated $\psi_{\dot\alpha} = f^{-1}(x)\chi_{\dot\alpha}$, $\bar\psi^{\dot\alpha} = f^{-1}(x)\bar\chi^{\dot\alpha}$ with $f(x) = g^{-1/2}(x)$], one can explicitly check that the Lagrangian $L = L_{\rm kin} + L_{\rm int}$ coincides, up to a total derivative, with (A.2.18). The action is invariant under supersymmetry transformations,

$$\begin{aligned} x_\mu &\to x_\mu + f\epsilon\sigma_\mu\psi + f\bar\epsilon\sigma_\mu\bar\psi, \\ f\psi_{\dot\alpha} &\to f\psi_{\dot\alpha} + i\dot x_\mu \left(\bar\epsilon\sigma_\mu\right)_{\dot\alpha}, \\ f\bar\psi^{\dot\alpha} &\to f\bar\psi^{\dot\alpha} - i\dot x_\mu \left(\sigma^\dagger_\mu\epsilon\right)^{\dot\alpha}. \end{aligned} \tag{A.4.14}$$

The Noether classical supercharges expressed in terms of $\psi_{\dot\alpha}, \bar\psi^{\dot\alpha}$, x_μ and their canonical momenta,

$$p_\mu \;=\; f^{-2}\dot x_\mu + \mathcal{A}_\mu - \frac{i}{2} f^{-1}\partial_\nu f\, \psi\sigma^\dagger_{[\mu}\sigma_{\nu]}\bar\psi\ , \tag{A.4.15}$$

are

$$\begin{aligned} Q_\alpha &= f\left(\sigma_\mu\bar\psi\right)_\alpha \left(p_\mu - \mathcal{A}_\mu\right) - i\partial_\mu f\, \psi_{\dot\gamma}\bar\psi^{\dot\gamma} \left(\sigma_\mu\bar\psi\right)_\alpha, \\ \bar Q^\alpha &= [\text{complex conjugate}]\ . \end{aligned}$$

The quantum supercharges are obtained from the classical ones by Weyl ordering procedure [11]. This gives (A.2.19). The anticommutator $\{Q_\alpha, \bar Q^\alpha\}$ gives the quantum Hamiltonian (A.2.20).

As was noticed, the field A_μ naturally obtained in the HSS framework satisfies the constraint $\partial_\mu \mathcal{A}_\mu = 0$ [3]. This does not really impose a restriction, however, because gauge transformations of A_μ that shift it by the gradient of an arbitrary function amount to adding a total derivative in the Lagrangian (A.4.13).

Acknowledgments

We are indebted to Evgeny Ivanov for illuminating discussions. The work of MK was supported in part by grants RFBR-07-02-01161 and grant of leading scientific schools NSH-3036.2008.2.

A.5 Appendix : Harmonic superspace in quantum mechanics

In this appendix, we introduce some basic HSS notations and definitions (see Ref. [2] for detailed explanations) in application to quantum mechanical systems.

Consider the ordinary $\mathcal{N}=2$ superspace

$$\mathbb{R}^{1|4} = \left\{t, \theta_\alpha, \bar\theta^\beta\right\} \ , \tag{A.5.1}$$

with θ_α and $\bar\theta^\beta = \varepsilon^{\beta\gamma}\bar\theta_\gamma = (\theta_\beta)^\dagger$ belonging to the fundamental representation of SU(2).

Introduce the supercharges [5]

$$Q^\alpha = \frac{\partial}{\partial\theta_\alpha} + i\bar\theta^\alpha\frac{\partial}{\partial t}, \qquad \bar Q_\alpha = \frac{\partial}{\partial\bar\theta^\alpha} + i\theta_\alpha\frac{\partial}{\partial t} \tag{A.5.2}$$

and superderivatives

$$D^\alpha = \frac{\partial}{\partial\theta_\alpha} - i\bar\theta^\alpha\frac{\partial}{\partial t}, \qquad \bar D_\alpha = \frac{\partial}{\partial\bar\theta^\alpha} - i\theta_\alpha\frac{\partial}{\partial t}. \tag{A.5.3}$$

The supercharges form the $\mathcal{N}=2$ SUSY algebra, while the superderivatives anticommute with Q^α and $\bar Q_\beta$,

$$\left\{Q^\alpha, \bar Q_\beta\right\} = 2\delta^\alpha_\beta\, i\partial_t, \qquad \left\{D^\alpha, \bar D_\beta\right\} = -2\delta^\alpha_\beta\, i\partial_t. \tag{A.5.4}$$

To proceed to harmonic superspace $\mathbb{HR}^{1+2|4} = \mathbb{R}^{1|4}\times \mathrm{S}^2$, we introduce a set of two complex coordinates $u^{+\alpha}$. Introduce also $u^-_\alpha = (u^{+\alpha})^*$ and impose the condition

$$u^{+\alpha}u^-_\alpha = 1 \ . \tag{A.5.5}$$

Then $u^{+\alpha}$ parametrize the R-symmetry group SU(2). The differential operators

$$D^{++} = u^+_\alpha\frac{\partial}{\partial u^-_\alpha}, \qquad D^{--} = u^-_\alpha\frac{\partial}{\partial u^+_\alpha} \ , \qquad D^0 = u^+_\alpha\frac{\partial}{\partial u^+_\alpha} - u^-_\alpha\frac{\partial}{\partial u^-_\alpha} \tag{A.5.6}$$

are called *harmonic derivatives.* The U(1) charge operator D^0 plays a special role. The functions of zero U(1) charge live on the coset $\mathrm{S}^2 = \mathrm{SU(2)/U(1)}$. The coordinates u^+_α have charge 1, the coordinates u^-_α have charge -1, etc.

[5] Our convention follows the convention in Ref. [12], but differs from the convention of Ref. [3] by the change of time direction $t\to -t$. With this, we reproduce the correct sign in the kinetic term for the spinor field in Eq. (A.4.7).

One can define now harmonic projections $D^\pm = u^\pm_\alpha D^\alpha$, $\bar{D}^\pm = u^\pm_\alpha \bar{D}^\alpha$. It is convenient to go over in the *analytic basis* in HSS,

$$\mathbb{HR}^{1+2|4} = \left\{ t_{\rm A}, \theta^\pm, \bar{\theta}^\pm, u^\pm_\alpha \right\}, \tag{A.5.7}$$

where

$$t_{\rm A} = t + i\left(\theta^+\bar{\theta}^- + \theta^-\bar{\theta}^+\right), \qquad \theta^\pm = u^\pm_\alpha \theta^\alpha, \qquad \bar{\theta}^\pm = u^\pm_\alpha \bar{\theta}^\alpha. \tag{A.5.8}$$

In this basis, the covariant spinor derivatives D^+, $\bar{D}^+$ are just

$$D^+ = \frac{\partial}{\partial \theta^-}, \qquad \bar{D}^+ = -\frac{\partial}{\partial \bar{\theta}^-}\ , \tag{A.5.9}$$

while the operator D^{++} acquires the form

$$D^{++} = u^+_\alpha \frac{\partial}{\partial u^-_\alpha} + \theta^+ \frac{\partial}{\partial \theta^-} + \bar{\theta}^+ \frac{\partial}{\partial \bar{\theta}^-} + 2i\theta^+\bar{\theta}^+ \frac{\partial}{\partial t_{\rm A}}\ . \tag{A.5.10}$$

The derivative operators D^+, $\bar{D}^+$, D^{++} (anti)commute with each other and with supercharges. Because of this, it is possible to consider a superfield q^+ with U(1) charge $+1$ satisfying

$$D^+ q^+ = 0, \qquad \bar{D}^+ q^+ = 0, \qquad D^{++} q^+ = 0. \tag{A.5.11}$$

In the analytic superspace coordinates, the first and the second equations mean that q^+ depend only on θ^+ and $\bar{\theta}^+$, but not on θ^- and $\bar{\theta}^-$. This is the so-called *superfield analyticity condition*. When expanding the field $q^+(t_A, \theta^+, \bar{\theta}^+, u^\pm_\alpha)$ over spinor coordinates and the harmonics, one obtains an infinite set of physical fields $\Phi(t_A)$. However, imposing also the condition $D^{++}q^+ = 0$ drastically reduces the number of such fields, making it finite. In the analytic basis, the solution of the constraints (A.5.11) reads

$$q^+ = x^\alpha(t_{\rm A}) u^+_\alpha - 2\theta^+ \chi(t_{\rm A}) - 2\bar{\theta}^+ \bar{\chi}'(t_{\rm A}) - 2i\theta^+\bar{\theta}^+ \partial_{\rm A} x^\alpha(t_{\rm A}) u^-_\alpha \tag{A.5.12}$$

with the factors -2 introduced for convenience.

The constraints (A.5.11) admit an involution symmetry $q^+ \to \widetilde{q^+}$ which commutes with SUSY transformations [3, 2]. This involution acts just as the ordinary complex conjugation *except* its action on the harmonics $u^\pm_\alpha$, which is

$$\widetilde{u^\pm_\alpha} = u^{\pm\alpha}, \qquad \widetilde{u^{\pm\alpha}} = -u^\pm_\alpha\ . \tag{A.5.13}$$

This gives

$$\widetilde{t_{\rm A}} = t_{\rm A}, \qquad \widetilde{\theta^\pm} = \bar{\theta}^\pm, \qquad \widetilde{\bar{\theta}^\pm} = -\theta^\pm, \tag{A.5.14}$$

and hence

$$\widetilde{q^+} = \left[x_\alpha(t_{\rm A})\right]^* u^+_\alpha - 2\theta^+ \bar{\chi}'^*(t_{\rm A}) + 2\bar{\theta}^+ \chi^*(t_{\rm A}) - 2i\theta^+\bar{\theta}^+ \partial_{\rm A} \left[x_\alpha(t_{\rm A})\right]^* u^-_\alpha. \tag{A.5.15}$$

It is straightforward to see that the field $\widetilde{q^+}$ satisfies the same constraints (A.5.11) as the field q^+. The involution symmetry was used in the main text to impose the pseudoreality condition (A.4.2) on the field $q^{+\dot{\alpha}}$.

The invariant actions involve the harmonic integral $\int du$. To find such integral of any function $f(u^{\pm}_{\alpha})$, one should expand f in the harmonic Taylor series and, for each term, do the integrals using the rules

$$\int du\, 1 = 1, \qquad \int du\, u^{+}_{\{\alpha_1} \dots u^{+}_{\alpha_k} u^{-}_{\alpha_{k+1}} \dots u^{-}_{\alpha_{k+\ell}\}} = 0\ , \tag{A.5.16}$$

where the integrand is symmetrized over all indices. The values of the integrals of all other harmonic monoms (for example, $\int du\, u^{+}_{\alpha} u^{-}_{\beta} = \frac{1}{2}\varepsilon_{\alpha\beta}$) follow from (A.5.16) and the definition (A.5.5).

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Annexe B Text of reference [3]

ITEP-TH-81/09

SQM with non-Abelian self-dual fields : harmonic superspace description

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We present a Lagrangian formulation for $\mathcal{N} = 4$ supersymmetric quantum-mechanical systems describing the motion in external non-Abelian self-dual gauge fields. For any such system, one can write a component supersymmetric Lagrangian by introducing extra bosonic variables with topological Chern-Simons type interaction. For a special class of such system when the fields are expressed in the 't Hooft ansatz form, it is possible to give a superfield description using harmonic superspace formalism. As a new explicit example, the $\mathcal{N} = 4$ mechanics with Yang monopole is constructed.

B.1 Introduction

Supersymmetric quantum mechanics (SQM) provides a proper venue for exploring and modelling salient features of supersymmetric field theories in diverse dimensions [1]. Some SQM models are, in turn, $d = 1$ reduction of higher-dimensional supersymmetric theories. At the same time, many interesting models of this kind can be constructed directly in (0+1) dimensions, without any reference to the dimensional reduction procedure. They exhibit some surprising properties related to the peculiarities of $d = 1$ supersymmetry. For any SQM model (like for any supersymmetric field theory), it is desirable, besides the component formulation, to find out the appropriate superfield Lagrangians. They make supersymmetry manifest, prompt possible generalizations of the model, and allow one to reveal relationships with other cognate theories. The basic aim of the present paper is to construct such a Lagrangian formulation for a wide class of $\mathcal{N} = 4$ SQM models,[1] with self-dual non-Abelian gauge field backgrounds. The natural and necessary device for this formulation proves to be the harmonic superspace (HSS) approach [2] adapted to the one-dimensional case in [3].

The SQM models considered in this paper represent a subclass of the wider well-known class of system that describes the motion of a fermion on an even-dimensional manifold with an arbitrary gauge background. It was observed many years ago that one can treat this system as a supersymmetric one [4]. The corresponding supercharges and the Hamiltonian are

$$Q = \not{\mathcal{D}}(1+\gamma_5), \qquad \bar{Q} = \not{\mathcal{D}}(1-\gamma_5), \qquad H = \not{\mathcal{D}}^2. \tag{B.1.1}$$

Indeed, for any eigenstate Ψ of the massless Dirac operator $\not{\mathcal{D}}$ with a nonzero eigenvalue λ, the state $\gamma^5\Psi$ is also an eigenstate of $\not{\mathcal{D}}$ with the eigenvalue $-\lambda$. Thus, all excited states of H are doubly degenerate.

For a four-dimensional flat manifold and self-dual Abelian or non-Abelian gauge fields, the spectrum of H is 4-fold degenerate implying extended $\mathcal{N} = 4$ supersymmetry. For a flat Dirac operator in the instanton background, this can be traced back to Ref. [5]. In Ref. [3], a $\mathcal{N} = 4$ supersymmetric generalization of this system describing the motion on a conformally flat $4D$ manifold with an Abelian self-dual background was found and its off-shell Lagrangian formulation in the $d = 1$ harmonic superspace was presented. In Ref. [6], it was noticed that a similar generalization exists for non-Abelian fields. The corresponding supercharges and the Hamiltonian have the form

$$\begin{aligned} Q_\alpha &= f\left(\sigma_\mu\bar{\psi}\right)_\alpha(\hat{p}_\mu - \mathcal{A}_\mu) - \psi_{\dot\gamma}\bar{\psi}^{\dot\gamma}\left(\sigma_\mu\bar{\psi}\right)_\alpha i\partial_\mu f, \\ \bar{Q}^\alpha &= \left(\psi\sigma^\dagger_\mu\right)^\alpha(\hat{p}_\mu - \mathcal{A}_\mu)\,f + i\partial_\mu f\left(\psi\sigma^\dagger_\mu\right)^\alpha\psi_{\dot\gamma}\bar{\psi}^{\dot\gamma}, \end{aligned} \tag{B.1.2}$$

$$\begin{aligned} H = {} & \frac{1}{2}f\left(\hat{p}_\mu - \mathcal{A}_\mu\right)^2 f + \frac{i}{4}f^2\,\mathcal{F}_{\mu\nu}\,\psi\sigma^\dagger_\mu\sigma_\nu\bar{\psi} \\ & - \frac{1}{2}fi\partial_\mu f\left(\hat{p}_\nu - \mathcal{A}_\nu\right)\psi\sigma^\dagger_{[\mu}\sigma_{\nu]}\bar{\psi} + f\partial^2 f\left\{\psi_{\dot\gamma}\bar{\psi}^{\dot\gamma} - \frac{1}{2}\left(\psi_{\dot\gamma}\bar{\psi}^{\dot\gamma}\right)^2\right\} \end{aligned} \tag{B.1.3}$$

[1] Hereafter, $\mathcal{N}$ counts the number of *real* supercharges.

with self-dual gauge field $\mathcal{A}_\mu$ (Abelian or non-Abelian), $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i\,[\mathcal{A}_\mu, \mathcal{A}_\nu]$. Complex fermion variables $\psi_{\dot\alpha}$ have two components,[2] $\hat{p}_\mu = -i\partial/\partial x_\mu$ and $f(x)$ is an arbitrary scalar function determining the conformally flat metric, $ds^2 = \{f(x)\}^{-2}\,dx_\mu^2$. Note that the supercharges and the Hamiltonian written above are SO(4) = SU(2)×SU(2) covariant ; the undotted indices come from the first SU(2) (R-symmetry group), while the dotted ones – from the second SU(2) commuting with supersymmetry.

When the gauge field $\mathcal{A}_\mu$ is Abelian, the corresponding Lagrangian can be easily written,

$$
\begin{aligned}
L \;=\; & \frac{1}{2} f^{-2}\,\dot{x}_\mu \dot{x}_\mu + i\bar{\psi}^{\dot\alpha}\dot{\psi}_{\dot\alpha} + \frac{1}{4}\left\{3\,(\partial_\mu f)^2 - f\partial^2 f\right\}\psi^4 + \frac{i}{2} f^{-1}\partial_\mu f\,\dot{x}_\nu\,\psi\sigma^\dagger_{[\mu}\sigma_{\nu]}\bar{\psi} \\
& + \mathcal{A}_\mu(x)\dot{x}^\mu - \frac{i}{4} f^2 \mathcal{F}_{\mu\nu}\,\psi\sigma^\dagger_\mu\sigma_\nu\bar{\psi}, \quad \text{(B.1.4)}
\end{aligned}
$$

where the second line represents the interaction term with the Abelian gauge field. This Lagrangian admits a superfield formulation [3, 6] in the framework of the harmonic superspace (HSS) approach [2].

For a matrix-valued non-Abelian self-dual field $\mathcal{A}_\mu$, the (scalar) Lagrangian cannot be straightforwardly derived from (B.1.3). We will show that this can be done by introducing extra "semi-dynamical" fields φ_i in the fundamental representation of SU(N) and the auxiliary U(1) gauge field $B(t)$. The second line in (B.1.4) is then generalized to

$$
L^{\mathrm{SU}(N)}_{\mathrm{int}} = i\bar{\varphi}^i\,(\dot{\varphi}_i + iB\varphi_i) + kB + \mathcal{A}^a_\mu T^a\,\dot{x}_\mu - \frac{i}{4} f^2 \mathcal{F}^a_{\mu\nu} T^a\,\psi\sigma^\dagger_\mu\sigma_\nu\bar{\psi} \quad \text{(B.1.5)}
$$

with k integer and

$$
T^a = \bar{\varphi}^i\,(t^a)_i^{\;j}\,\varphi_j, \quad \text{(B.1.6)}
$$

t^a being standard SU(N) algebra generators. The interaction Lagrangian (B.1.5) is $\mathcal{N} = 4$ supersymmetric. The corresponding supersymmetry transformations are written in Eqs. (B.3.13). It is easy to check that it is covariant with respect to the target space non-Abelian gauge transformations

$$
\begin{aligned}
\mathcal{A}^a_\mu\,t^a &\to U^\dagger \mathcal{A}^a_\mu\,t^a U + iU^\dagger\partial_\mu U \\
\varphi_i &\to \left(U^\dagger\varphi\right)_i, \quad \bar{\varphi}^i \to (\bar{\varphi}U)^i
\end{aligned} \quad \text{(B.1.7)}
$$

with $U(x) \in \mathrm{SU}(N)$.

It is not immediately clear how to extend the Abelian superfield description to the general non-Abelian case, i.e. to the gauge group SU(N) . In this paper, we construct such a description for the particular case of SU(2) self-dual fields expressed in the form

$$
\text{(a)}\;\; \mathcal{A}^a_\mu \;=\; -\bar{\eta}^a_{\mu\nu}\partial_\nu \ln h(x) \qquad \text{or} \qquad \text{(b)}\;\; \mathcal{A}^a_\mu \;=\; -\eta^a_{\mu\nu}\partial_\nu \ln h(x)\,, \quad \text{(B.1.8)}
$$

[2]We use the following SO(4) spinor notation :

$$
(\sigma_\mu)_{\alpha\dot\beta} = \{i, \sigma_a\}_{\alpha\dot\beta}\,,\; \left(\sigma^\dagger_\mu\right)^{\dot\beta\alpha} = \{-i, \sigma_a\}^{\dot\beta\alpha} = -\varepsilon^{\dot\beta\dot\gamma}\varepsilon^{\alpha\gamma}(\sigma_\mu)_{\gamma\dot\gamma}\,, \quad \varepsilon_{\alpha\beta} = -\varepsilon^{\alpha\beta}, \varepsilon_{\dot\alpha\dot\beta} = -\varepsilon^{\dot\alpha\dot\beta}, \varepsilon_{12} = 1\,,
$$

$$
v_{\alpha\dot\alpha} = v_\mu(\sigma_\mu)_{\alpha\dot\alpha}\,, \quad v_\mu = \frac{1}{2} v_{\alpha\dot\alpha}(\sigma^\dagger_\mu)^{\dot\alpha\alpha} \;=\; -\frac{1}{2} v^{\alpha\dot\alpha}(\sigma_\mu)_{\alpha\dot\alpha}\,,\; v^{\alpha\dot\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta} v_{\beta\dot\beta} \;=\; -v_\mu(\sigma^\dagger_\mu)^{\dot\alpha\alpha}\,.
$$

with harmonic function $h(x)$,

$$\partial^2_\mu h(x) = \text{a finite sum of delta functions}$$

(the expressions (B.1.8a) and (B.1.8b) correspond, respectively, to self-dual and anti-self-dual fields). This is the so called 't Hooft ansatz for a multi-instanton SU(2) solution [7], the symbols $\eta^a_{\mu\nu}$ being defined as

$$\eta^a_{ij} = \bar{\eta}^a_{ij} = \varepsilon_{aij}, \quad \eta^a_{i0} = -\eta^a_{0i} = \bar{\eta}^a_{0i} = -\bar{\eta}^a_{i0} = \delta_{ai} \qquad (i,j,a = 1,2,3)\,. \tag{B.1.9}$$

For generic self-dual ADHM [8] configurations, the problem of finding a superfield Lagrangian is more complicated. This problem is under study now.

B.2 Derivation

In the $\mathcal{N}=4$, $d=1$ HSS approach [3], the superfields depend on bosonic variables t, $u^{\pm\alpha}$ (the harmonics $u^{+\alpha}, u^-_\alpha = (u^{+\alpha})^*$ satisfying the constraint $u^{+\alpha}u^-_\alpha = 1$ parametrize the R-symmetry group SU(2) of the $\mathcal{N}=4$ superalgebra) and on fermionic variables $\theta^\pm = u^\pm_\alpha \theta^\alpha$, $\bar{\theta}^\pm = u^\pm_\alpha \bar{\theta}^\alpha$. The most striking feature of HSS is the presence of an *analytic superspace* $\left\{t_{\rm A}, \theta^+, \bar{\theta}^+, u^{\pm\alpha}\right\}$ in it (an analog of $\mathcal{N}=2$ chiral superspace) involving the "analytic time" $t_A = t + i(\theta^+\bar{\theta}^- + \theta^-\bar{\theta}^+)$ and containing twice as less fermionic coordinates. Our definitions and conventions are the same as in [6] (and similar to [3]; see [2] for more details) and we mostly will not repeat them here.

To construct the action, introduce, following [3, 6], a doublet of superfields $q^{+\dot{\alpha}}$ with charge +1 ($D^0 q^+ = q^+$) satisfying the constraints

$$D^+ q^{+\dot{\alpha}} = 0, \qquad \bar{D}^+ q^{+\dot{\alpha}} = 0, \qquad D^{++} q^{+\dot{\alpha}} = 0\,, \tag{B.2.1}$$

where D^+, $\bar{D}^+$ and D^{++} are spinor and harmonic derivatives.[3] We impose also the pseudoreality condition

$$\widetilde{q^{+\dot{\alpha}}} = q^+_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}} q^{+\dot{\beta}}\ , \qquad \widetilde{\widetilde{q^{+\dot{\beta}}}} = -q^{+\dot{\beta}}\,, \tag{B.2.2}$$

where the field $\widetilde{q^+}$ is obtained from q^+ by an involution transformation. *On top of that*, we introduce an analytic gauge superfield V^{++} of charge +2 satisfying the constraints

$$D^+ V^{++} = \bar{D}^+ V^{++} = 0\ , \qquad V^{++} = \widetilde{V^{++}} \tag{B.2.3}$$

and the "matter" superfield v^+ of charge +1. The constraints it satisfies,

$$D^+ v^+ = 0, \qquad \bar{D}^+ v^+ = 0, \qquad (D^{++} + iV^{++})v^+ = 0\,, \tag{B.2.4}$$

[3]The constraints $D^+q^+ = \bar{D}^+q^+ = 0$ are akin to widely known chirality constraints like $D_\alpha\Phi = 0$ in $\mathcal{N}=1$, $d=4$ supersymmetric theories. In the analytic basis, they simply mean that q^+ does not depend on $\theta^-, \bar{\theta}^-$. Such constraints appear naturally in the HSS formalism and are common also in $d=4$ theories. A possibility to impose the extra constraint $D^{++}q^+ = 0$ is specific for the (0+1)-dimensional case, where it has a pure kinematic nature. In $\mathcal{N}=2$, $d=4$ theories, the relation $D^{++}q^+ = 0$ is not a kinematic constraint, it is the equation of motion for the *free* hypermultiplet derived from the action $S = \int d^4x\, du\, d^4\theta^+\, \widetilde{q^+} D^{++} q^+$ [2].

differ from (B.2.1) by the presence of the covariant harmonic derivative $\mathcal{D}^{++} = D^{++} + iV^{++}$ [9]. The constraint $\mathcal{D}^{++}v^+ = 0$ is covariant with respect to gauge transformations

$$V^{++} \;\to\; V^{++} + D^{++}\Lambda, \qquad v^+ \;\to\; e^{-i\Lambda}v^+, \qquad D^+\Lambda = \bar{D}^+\Lambda = 0 \; . \tag{B.2.5}$$

The constraints (B.2.1), (B.2.4) drastically reduce the number of the physical component fields in the superfields $q^{+\dot\alpha}$ and v^+. The solution of (B.2.1), (B.2.2) for $q^{+\dot\alpha}$ in the analytic basis, with

$$D^{++} = \partial^{++} + 2i\theta^+\bar\theta^+\frac{\partial}{\partial t_{\rm A}}\,, \qquad \partial^{++} = u^+_\alpha\frac{\partial}{\partial u^-_\alpha}\,, \tag{B.2.6}$$

is

$$q^{+\dot\alpha} = x^{\alpha\dot\alpha}(t_{\rm A})u^+_\alpha - 2\theta^+\chi^{\dot\alpha}(t_{\rm A}) - 2\bar\theta^+\bar\chi^{\dot\alpha}(t_{\rm A}) - 2i\theta^+\bar\theta^+\dot{x}^{\alpha\dot\alpha}u^-_\alpha \;, \tag{B.2.7}$$

where

$$x^{\alpha\dot\alpha} = -\left(x_{\alpha\dot\alpha}\right)^*, \qquad \bar\chi^{\dot\alpha} = \left(\chi_{\dot\alpha}\right)^* \;. \tag{B.2.8}$$

The first condition in the latter equation means that $x_\mu = -\frac{1}{2}x^{\alpha\dot\alpha}(\sigma_\mu)_{\alpha\dot\alpha}$ is real, and we are left with four dynamic bosonic variables.

We can use the gauge freedom (B.2.5) to eliminate almost all components from V^{++} and to present it as

$$V^{++} \;=\; 2i\,\theta^+\bar\theta^+B, \tag{B.2.9}$$

where the gauge field $B(t)$ is real. This is a $d=1$ counterpart of the familiar Wess-Zumino gauge in $d=4$ theories. Then the superfield v^+ is expressed in the analytical basis as

$$v^+ = \phi^\alpha u^+_\alpha - 2\theta^+\omega_1 - 2\bar\theta^+\bar\omega_2 - 2i\theta^+\bar\theta^+(\dot\phi^\alpha + iB\phi^\alpha)u^-_\alpha \;, \tag{B.2.10}$$

from which it follows that

$$\widetilde{v^+} = \bar\phi^\alpha u^+_\alpha - 2\theta^+\omega_2 + 2\bar\theta^+\bar\omega_1 - 2i\theta^+\bar\theta^+(\dot{\bar\phi}^\alpha - iB\bar\phi^\alpha)u^-_\alpha \tag{B.2.11}$$

with $\bar\phi^\alpha = (\phi_\alpha)^*$. Thus, the fields ϕ_α and $\bar\phi^\alpha$ are charged under U(1) gauge field B and have opposite charges.

The $\mathcal{N}=4$ SUSY invariant action that we are going to write consists of three parts, $S = S_{\rm kin} + S_{\rm int} + S_{\rm FI}$. The kinetic part is more convenient to express in the central basis $\{t, \theta_\alpha, \bar\theta^\beta\}$. It has the same form as in [3, 6],

$$S_{\rm kin} = \int dt\, d^4\theta\, du\, R'_{\rm kin}(q^{+\dot\alpha}, q^{-\dot\beta}, u^\pm_\gamma) = \int dt\, d^4\theta\, R_{\rm kin}(q_\mu), \tag{B.2.12}$$

where $q_\mu = -\frac{1}{2}q^{\alpha\dot\alpha}\left(\sigma_\mu\right)_{\alpha\dot\alpha}$ (the equation $D^{++}q^{+\dot\alpha} = 0$ ensures that in the central basis $q^{+\dot\alpha}$ depends on $u^\pm_\beta$ linearly, i.e. $q^{+\dot\alpha} = q^{\alpha\dot\alpha}u^+_\alpha$; see also Eq. (35) in [6]), with an arbitrary function $R_{\rm kin}(q_\mu)$ of the real superfield q_μ. The component expansion of (B.2.12) coincides with the first line in Eq. (B.1.4), where $f(x) = \left[\frac{1}{2}\partial^2_\mu R_{\rm kin}\right]^{-1/2}$ and $\psi_{\dot\alpha} = f^{-1}\chi_{\dot\alpha}$ [6].

The interaction part is chosen as

$$S_{\rm int} = -\frac{1}{2}\int dt\, du\, d\bar\theta^+ d\theta^+\, K\left(q^{+\dot\alpha}, u^\pm_\beta\right) v^+\widetilde{v^+} \;, \tag{B.2.13}$$

where the condition $\widetilde{K} = K$ is imposed to ensure the action to be real. Finally, we add the Fayet-Illiopoulos term

$$S_{\rm FI} \;=\; -\frac{ik}{2}\int dt\, du\, d\bar\theta^+ d\theta^+\, V^{++} \;=\; k\int dt\, B\;, \tag{B.2.14}$$

which is invariant under gauge transformations (B.2.5). At the classical level, k is an arbitrary real number. As we will shortly see, a benign quantum theory can only be defined if the requirement

$$k \;=\; \text{integer} \tag{B.2.15}$$

is fulfilled.

Let us concentrate on the interaction part. It is convenient to introduce new variables

$$\varphi_\alpha \;=\; \phi_\alpha\sqrt{h(x)}\;, \tag{B.2.16}$$

where

$$h(x) = \int du\, K(x^{+\dot\alpha}, u^\pm_\beta), \qquad x^{+\dot\alpha} = x^{\alpha\dot\alpha}u^+_\alpha, \tag{B.2.17}$$

is a harmonic function.[4] Indeed,

$$\partial^2_\mu h(x) = 4\varepsilon^{\dot\alpha\dot\beta}\int du\, \partial_{+\dot\alpha}\partial_{-\dot\beta}K(x^{+\dot\gamma}, u^\pm_\beta) = 0\,.$$

Substituting (B.2.7), (B.2.10) and (B.2.11) into (B.2.13) and eliminating the auxiliary fermionic degrees of freedom $\omega_{1,2}$, $\bar\omega_{1,2}$ by their algebraic equations of motion, we derive after some algebra

$$L_{\rm int} \;=\; i\bar\varphi^\alpha(\dot\varphi_\alpha + iB\varphi_\alpha) - \frac{1}{2}\bar\varphi^\beta\varphi_\gamma\,(\mathcal{A}_{\alpha\dot\alpha})_\beta{}^\gamma\,\dot x^{\alpha\dot\alpha} - \frac{i}{4}\left(\mathcal{F}_{\dot\alpha\dot\beta}\right)_\beta{}^\gamma\,\chi^{\dot\alpha}\bar\chi^{\dot\beta}\bar\varphi^\beta\varphi_\gamma\,. \tag{B.2.18}$$

Here

$$(\mathcal{A}_{\alpha\dot\alpha})_\beta{}^\gamma = -\frac{2i}{\int du\, K}\int du\,\partial_{+\dot\alpha}K\left(u^{+\gamma}\varepsilon_{\alpha\beta} - \frac{1}{2}u^+_\alpha\delta^\gamma_\beta\right) = \frac{i}{h}\left(\varepsilon_{\alpha\beta}\,\partial^\gamma{}_{\dot\alpha}h - \frac{1}{2}\delta^\gamma_\beta\,\partial_{\alpha\dot\alpha}h\right) \tag{B.2.19}$$

($\partial_{\alpha\dot\alpha} \equiv (\sigma_\mu)_{\alpha\dot\alpha}\,\partial_\mu = -2\partial/\partial x^{\alpha\dot\alpha}$) is a Hermitean traceless matrix, the gauge field, and

$$\left(\mathcal{F}_{\dot\alpha\dot\beta}\right)_\beta{}^\gamma = (\mathcal{F}_{\mu\nu})_\beta{}^\gamma\left(\sigma^\dagger_\mu\sigma_\nu\right)_{\dot\alpha\dot\beta} = \partial_{\delta\dot\alpha}(\mathcal{A}^\delta{}_{\dot\beta})_\beta{}^\gamma - i(\mathcal{A}_{\delta\dot\alpha})_\beta{}^\lambda(\mathcal{A}^\delta{}_{\dot\beta})_\lambda{}^\gamma \;+ (\dot\alpha\leftrightarrow\dot\beta) \tag{B.2.20}$$

is its self-dual part. It is easy to check explicitly, that the anti-self-dual part of the gauge field $\mathcal{A}_\mu$ vanishes,

$$(\mathcal{F}_{\alpha\beta})_\gamma{}^\delta = (\mathcal{F}_{\mu\nu})_\gamma{}^\delta\left(\sigma_\mu\sigma^\dagger_\nu\right)_{\alpha\beta} = -\partial_{\alpha\dot\alpha}(\mathcal{A}_\beta{}^{\dot\alpha})_\gamma{}^\delta + i(\mathcal{A}_{\alpha\dot\alpha})_\gamma{}^\lambda(\mathcal{A}_\beta{}^{\dot\alpha})_\lambda{}^\delta + (\alpha\leftrightarrow\beta) = 0. \tag{B.2.21}$$

Thus, the field strength $\mathcal{F}^a_{\mu\nu}$ is self-dual and belongs to the representation $(0,1)$ of SO(4). Passing to $\mathcal{A}^a_\mu$ as $(\mathcal{A}_\mu)_\beta{}^\gamma = \mathcal{A}^a_\mu\,(\sigma_a)_\beta{}^\gamma/2$, we find that the representation (B.2.19) precisely amounts to the self-dual 't Hooft ansatz (B.1.8a). The anti-self-dual expression

[4] We assumed here that $h(x) > 0$.

(B.1.8b) arises if one interchanges altogether dotted and undotted indices, i.e. effectively interchanges σ_μ and $\sigma_\mu^\dagger$. This also implies passing to the harmonics $u^\pm_{\dot\alpha}$ and in fact to another $\mathcal{N}=4$ supersymmetry, with the second SU(2) (acting on dotted indices) as the R-symmetry group.

Finally, substituting $\bar\varphi^\beta\varphi_\gamma = T^a\,(\sigma_a)_\gamma^{\ \beta}$ and $\chi_{\dot\alpha} = f\psi_{\dot\alpha}$ into (B.2.18), where T^a is defined in (B.1.6) with $t^a = \frac{1}{2}\sigma_a$, we convince ourselves that the interaction term together with the FI term (B.2.14) yields just (B.1.5) for the SU(2) case. The canonical Hamiltonian derived from the Lagrangian $L_{\rm kin} + L_{\rm int} + L_{\rm FI}$ has the form (B.1.3) with $\mathcal{A}_\mu \equiv \mathcal{A}_\mu^a T^a$ and $\mathcal{F}_{\mu\nu} \equiv \mathcal{F}^a_{\mu\nu} T^a$.

Observe that the variables φ_α enter the Lagrangian with only one time derivative. Thus, they are not full-fledged dynamic variables (like x_μ) and not auxiliary fields (like $\omega_{1,2}$). They have a kind of intermediate nature.[5] To understand it better, perform the quantization. To begin with, it is sufficient to restrict oneself by the first term in (B.2.18) with addition of the Fayet-Illiopoulos term (B.2.14). The action

$$S \;=\; \int dt\left[i\bar\varphi^\alpha(\dot\varphi_\alpha + iB\varphi_\alpha) \;+\; kB\right] \tag{B.2.22}$$

much resembles the $3D$ Chern-Simons action,

$$S_{\rm CS} = \kappa\int\left(A\wedge dA - \frac{2i}{3}A\wedge A\wedge A\right)\ . \tag{B.2.23}$$

In both systems, the canonical Hamiltonian is zero, the canonical momenta are algebraically expressed through coordinates, and the quantization consists in imposing certain second class constraints (for a nice review of the classical and quantum aspects of the Chern-Simons theory, see [13]). Another well-known feature of CS theory is the quantization of the coupling, $k_{\rm CS} = 4\pi\kappa =$ integer. This follows from the requirement for the Euclidean path integral to be invariant with respect to large (topologically nontrivial) gauge transformations. As was mentioned above, in our case the coefficient k is also quantized. This can be derived following a similar reasoning.

Notice first that the action (B.2.22) is invariant with respect to gauge transformations,

$$B(t)\to\ B(t) + \frac{d\alpha(t)}{dt}, \qquad\qquad \varphi(t)\to e^{-i\alpha(t)}\varphi(t)\ , \tag{B.2.24}$$

which, in the Euclidean version of the theory, become

$$B(\tau)\to\ B(\tau) + i\frac{d\alpha(\tau)}{d\tau}, \qquad\qquad \varphi(\tau)\to e^{-i\alpha(\tau)}\varphi(\tau)\ . \tag{B.2.25}$$

This is a remnant of gauge transformations (B.2.5), which survives in the Wess-Zumino gauge (B.2.9). To discover topologically nontrivial gauge transformation, consider the Euclidean version of this theory and regularize it in the infrared by putting it on a finite Euclidean interval $\tau\in(0,\beta)$ and imposing the periodic boundary conditions $B(\beta) =$

[5] In the context of $\mathcal{N}=4$ SQM models, such variables (together with their analytic superfield carriers $v^+, \widetilde{v^+}$) were introduced in [10, 11] (for a recent application, see also [12]).

$B(0),\ \varphi(\beta) = \varphi(0)$.[6] Then the only admissible gauge transformations (B.2.25) are those which do not break these periodicity conditions. We see that the transformation with $\alpha(\tau) = 2\pi\tau/\beta$ is topologically nontrivial, it cannot be reduced to a chain of infinitesimal transformations. This transformation shifts the Euclidean Fayet-Illiopoulos action by an *imaginary* constant, $\Delta S_{\rm FI} = -2\pi i k$. The requirement that the Euclidean path integrals (involving the factor $e^{-S_{\rm FI}}$) are not changed leads [14] to the quantization condition (B.2.15).

The fact that k must be integer leads to the finite representations of the operator algebra φ_α, $\bar\varphi^\alpha$. Indeed, consider the integer k to be positive as required in the classical case due to the constraint

$$\bar\varphi^\alpha \varphi_\alpha = k\,, \qquad \text{(B.2.26)}$$

which follows from (B.2.22) by varying with respect to B (at the quantum level, negative k can be equally chosen). The canonical commutation relations following from the same action (B.2.22) through the standard Dirac prescription are

$$[\varphi_\alpha, \bar\varphi^\beta] = \delta_\alpha^\beta\ , \qquad [\varphi_\alpha, \varphi_\beta] = [\bar\varphi^\alpha, \bar\varphi^\beta] = 0\ . \qquad \text{(B.2.27)}$$

In quantum theory, one can choose $\varphi_\alpha \equiv \partial/\partial\bar\varphi^\alpha$ and impose (B.2.26) on the wave functions :

$$\bar\varphi^\alpha \varphi_\alpha \Psi = \bar\varphi^\alpha \frac{\partial}{\partial\bar\varphi^\alpha}\Psi = k\Psi\ . \qquad \text{(B.2.28)}$$

In other words, the wave functions represent homogeneous polynomials of $\bar\varphi^\alpha$ of (an integer) degree k .[7] The number of such (linearly independent) polynomials is $k+1$. Moreover, it is also easy to see that the operators (B.1.6) satisfy the following algebra

$$[T^a, T^b] \ = \ i\varepsilon^{abc} T^c\ . \qquad \text{(B.2.29)}$$

In addition, taking into account (B.2.28), we derive

$$T^a T^a \ = \ \frac{1}{4}\left[(\bar\varphi^\alpha\varphi_\alpha)^2 + 2(\bar\varphi^\alpha\varphi_\alpha)\right] \ = \ \frac{k}{2}\left(\frac{k}{2}+1\right). \qquad \text{(B.2.30)}$$

In other words, T^a can be treated as the generators of SU(2) in the representation of spin $k/2$.[8] The nice feature is that this gauge SU(2) is in fact R-symmetry group of $\mathcal{N} = 4$ supersymmetry algebra.

B.3 Discussion and outlook

Our main result is the HSS superfield action for the $\mathcal{N} = 4$ SQM corresponding to the Hamiltonian (B.1.3) with a non-Abelian SU(2) gauge field $\mathcal{A}_\mu$ which lives on a conformally flat 4-manifold and is representable in the 't Hooft ansatz form (B.1.8).

[6] This is of course equivalent to introducing a finite temperature $T = 1/\beta$.

[7] In the case $k < 0$ the algebra (B.2.27) is the same. One must choose $\bar\varphi^\alpha = -\partial/\partial\varphi_\alpha$ and consider polynomials of φ_α of degree $|k|$.

[8] This way of quantizing semi-dynamical variables $\varphi_\alpha, \bar\varphi^\alpha$ was employed in Ref. [11]. Alternatively, one could interpret $\varphi_\alpha, \bar\varphi^\alpha$ with the constraint (B.2.26) as a kind of the target harmonic variables representing a sphere S^2, solve (B.2.26) in terms of the stereographic projection coordinates $z(t)$ and $\bar z(t)$, and quantize the system by the Gupta-Bleuler method as in Ref. [15].

As an example of such a field, we can quote the instanton solution on S^4. Generically, it depends on the radius R of the sphere and the instanton size ρ. The configurations of maximal size, $\rho = R$, present a particular interest. In the stereographic coordinates on S^4,

$$ds^2 \;=\; \frac{4R^4 dx_\mu^2}{(x^2+R^2)^2}\,, \tag{B.3.1}$$

they are expressed by the same formulae as flat instantons in singular gauge,

$$\mathcal{A}_\mu^a \;=\; \frac{2R^2\bar\eta^a_{\mu\nu}x_\nu}{x^2(x^2+R^2)} \quad \text{or} \quad (\mathcal{A}_{\alpha\dot\alpha})_\beta{}^\gamma = -\frac{2i\,R^2}{x^2(x^2+R^2)}\left(\varepsilon_{\alpha\beta}x^\gamma_{\dot\alpha} - \frac{1}{2}\,\delta^\gamma_\beta\, x_{\alpha\dot\alpha}\right), \tag{B.3.2}$$

and

$$(\mathcal{F}_{\dot\alpha\dot\beta})_\beta{}^\gamma = \frac{8i\,R^2}{x^2(x^2+R^2)^2}\left(x^\gamma_{\dot\beta}x_{\beta\dot\alpha} + x^\gamma_{\dot\alpha}x_{\beta\dot\beta}\right). \tag{B.3.3}$$

The corresponding functions in Eq. (B.2.17) are taken in the form

$$K(x^{+\dot\alpha},u^\pm_\beta) = 1 + \frac{1}{\left(c^-_{\dot\alpha}x^{+\dot\alpha}\right)^2}\,, \qquad h(x) \equiv \int du\, K(x^{+\dot\alpha},u^\pm_\beta) = 1 + \frac{R^2}{x^2_\mu}, \tag{B.3.4}$$

where $c^-_{\dot\alpha} = c^\alpha{}_{\dot\alpha}u^-_\alpha$, $c^{\alpha\dot\alpha}$ – constant vector and $R^2 = 1/c^2_\mu$.[9] The field $\mathcal{A}^a_\mu$ can be brought to nonsingular gauge

$$\mathcal{A}_\mu^a = \frac{2\eta^a_{\mu\nu}x_\nu}{x^2+R^2}\,, \qquad \mathcal{F}^a_{\mu\nu} \;=\; -\frac{4R^2\eta^a_{\mu\nu}}{(x^2+R^2)^2}\,, \tag{B.3.5}$$

by the gauge transformations (B.1.7) with $U(x) = -i\sigma_\mu x_\mu/\sqrt{x^2}$ (this $U(x)$ is prompted by the form of the field strength (B.3.3)). The action density $\sim \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ is the same in this case at all points of S^4. It is worth noting that the singular gauge transformation converts the undotted gauge group indices into the dotted ones : the self-dual gauge potential and the field strength in the spinorial notation become

$$(\mathcal{A}_{\alpha\dot\alpha})^{\dot\gamma}{}_{\dot\beta} = \frac{2i}{x^2+R^2}\left(\varepsilon_{\dot\alpha\dot\beta}x^{\dot\gamma}_\alpha - \frac{1}{2}\,\delta^{\dot\gamma}_{\dot\beta}\, x_{\alpha\dot\alpha}\right), \quad (\mathcal{F}_{\dot\alpha\dot\beta})^{\dot\gamma}{}_{\dot\delta} = -\frac{8i\,R^2}{(x^2+R^2)^2}\left(\varepsilon_{\dot\alpha\dot\delta}\delta^{\dot\gamma}_{\dot\beta} + \varepsilon_{\dot\beta\dot\delta}\delta^{\dot\gamma}_{\dot\alpha}\right) \tag{B.3.6}$$

and, also, $\varphi_\alpha \to \varphi^{\dot\alpha} = -i\varphi_\alpha\, x^{\alpha\dot\alpha}/\sqrt{x^2}$, $\bar\varphi^\alpha \to \bar\varphi_{\dot\alpha} = -i\bar\varphi^\alpha\, x_{\alpha\dot\alpha}/\sqrt{x^2}$.

Actually, the field (B.3.2), (B.3.5) describes Yang monopole living in $\mathbb{R}^5$ [16]. The potential (B.3.5) has a nice group-theoretical meaning as one of the two SU(2) connections on the coset manifold SO(5)/[SU(2)×SU(2)] $\sim S^4$ (see e.g. [17]). It coincides with the flat self-dual instanton only in the conformally flat parametrization of S^4 as in (B.3.1). When coupled to the world-line through our semi-dynamical variables $\varphi_\alpha, \bar\varphi^\alpha$, the 5-dimensional Yang monopole is reduced to this SU(2) connection defined on S^4.

Let us elaborate on this point in more detail, choosing, without loss of generality, $R = 1$ in the above formulas. Consider the following $d = 1$ bosonic Lagrangian with the $\mathbb{R}^5$ target space and an additional coupling to Yang monopole

$$L_{\mathbb{R}^5} = \frac{1}{2}\left(\dot y_5\dot y_5 + \dot y_\mu\dot y_\mu\right) + \mathcal{B}^a_\mu(y)T^a\,\dot y_\mu\,. \tag{B.3.7}$$

[9] The integral on the right hand side of Eq. (B.3.4) can be calculated as the power series in $c^-_{\dot\alpha}c^{+\dot\alpha} = -c^2_\mu$ or directly after noting that this integral is SO(4) covariant and putting $c_\mu = (c,0,0,0)$, $x_\mu = (x_1,x_2,0,0)$.

Here, $\mathcal{B}^a_\mu$ is the standard form of the Yang monopole in the $\mathbb{R}^5$ coordinates,

$$\mathcal{B}^a_\mu = \frac{\eta^a_{\mu\nu} y_\nu}{r(r+y_5)}\,, \quad r = \sqrt{y_5^2 + y_\mu^2}\,, \tag{B.3.8}$$

T^a are defined as in (B.1.6) with $t^a = \frac{1}{2}\sigma^a$, and we omitted the action for the semi-dynamical variables $\varphi_\alpha, \bar\varphi^\alpha$. Now we pass to the polar decomposition of $\mathbb{R}^5$ into a radius r and the angular part S^4, $(y_5, y_\mu) \;\to\; (r, \tilde y_5, \tilde y_\mu)$, $\tilde y_5 = \sqrt{1-\tilde y_\mu^2}$, and rewrite (B.3.7) as

$$L_{\mathbb{R}^5} = \frac{1}{2}\dot r^2 + \frac{1}{2} r^2 \left(\dot{\tilde y}_5 \dot{\tilde y}_5 + \dot{\tilde y}_\mu \dot{\tilde y}_\mu\right) + \frac{\eta^a_{\mu\nu}\tilde y_\nu \dot{\tilde y}_\mu\, T^a}{1 + \sqrt{1-\tilde y_\mu^2}}\,. \tag{B.3.9}$$

The coordinates $\tilde y_\mu$ give a particular parametrization of S^4. Passing to the stereographic coordinates is accomplished by the redefinition

$$\tilde y_\mu = 2\,\frac{x_\mu}{1+x^2}\,,$$

which casts (B.3.9) into the form

$$L_{\mathbb{R}^5} = \frac{1}{2}\left\{\dot r^2 + 4r^2 \frac{\dot x_\mu \dot x_\mu}{(1+x^2)^2}\right\} + \frac{2\eta^a_{\mu\nu} x_\nu \dot x_\mu\, T^a}{1+x^2}\,. \tag{B.3.10}$$

We see that the S^4 metric (B.3.1) (with $R=1$) and the instanton vector potential (B.3.5) appear.

Thus, our approach, as a by-product, provides a solution to the long-standing problem of constructing $\mathcal{N}=4$ SQM with Yang monopole (see e.g. [18] and references therein). Obviously, the component Lagrangian (B.1.4) (with the relevant function $f(x)$) is just the S^4 part of the Lagrangian (B.3.10) with the "frozen" radial variable $r=1$. Presumably, one can restore the full 5-dimensional kinetic part in (B.3.10) by adding a coupling to the appropriately constrained scalar $\mathcal{N}=4$ zero-charge superfield $X(t,\theta,\bar\theta)$ which describes an off-shell multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ with one physical bosonic field [19], such that $X|_{\theta=\bar\theta=0} = r$.

The problem of finding a superfield formulation for a generic $\mathrm{SU}(N)$ self-dual field is more complicated and is not solved yet. However, by introducing extra variables φ_i, it is always possible to write a *component* Lagrangian (B.1.5) (together with the first line in (B.1.4)) corresponding to the matrix Hamiltonian (B.1.3).

This observation has actually nothing to do with supersymmetry. It boils down to the following. Consider the eigenvalue problem for a usual Hermitean matrix H_{jk}. It can be treated as a Schrödinger problem $\hat H \Psi(\varphi_j) = \lambda \Psi(\varphi_j)$ with the constraint $\hat G \Psi = 0$, where

$$\hat H \;=\; \varphi_j H_{jk} \frac{\partial}{\partial \varphi_k}\,, \qquad \hat G = \varphi_j \frac{\partial}{\partial \varphi_j} - 1\,. \tag{B.3.11}$$

The corresponding Lagrangian is

$$L \;=\; i\bar\varphi_j \dot\varphi_j - B(\bar\varphi_j \varphi_j - 1) - \bar\varphi_j H_{jk} \varphi_k\,, \tag{B.3.12}$$

where $\bar\varphi_j = (\varphi_j)^*$. This easily generalizes to the case where H is an operator depending on a set of canonically conjugated variables $\{p_\mu, x_\mu\}$. The only difference is that

$-H_{jk}$ is now replaced by the matrix L_{jk} obtained from H_{jk} by the appropriate Legendre transformation.[10]

Our initial goal was to find a Lagrangian representation for the Hamiltonian (B.1.3) with matrix-valued $\mathcal{A}_\mu$, $\mathcal{F}_{\mu\nu}$. The construction just described, with φ_i in the fundamental representation of SU(N), leads to the $N \times N$ matrix Hamiltonian. The Lagrangian (B.3.12) coincides in this case with the Lagrangian (B.1.5) with the choice $k = 1$, to which the first line from Eq. (B.1.4) is also added.

Obviously, one can describe the Hamiltonians in higher representations of SU(N) in a similar way, by choosing the number of components φ_i equal to the dimension of the representation. We have seen, however, that in the SU(2) case one can be more economic, introducing only a couple of dynamic variables $\bar{\varphi}^\alpha$, but multiplying the term $\sim B$ in the Lagrangian by an arbitrary integer k. This leads to the Hamiltonian in the representation of spin $|k|/2$. Certain SU(N) representations (namely, the symmetric products of $|k|$ fundamental or $|k|$ antifundamental representations) can also be attained in this way.

The Lagrangians (B.1.4), (B.1.5) are invariant, up to a total derivative, with respect to the following infinitesimal $\mathcal{N} = 4$ supersymmetry transformations :

$$\begin{aligned}
x_\mu &\to x_\mu + f\epsilon\sigma_\mu\psi + f\bar{\epsilon}\sigma_\mu\bar{\psi}, \\
f\psi_{\dot{\alpha}} &\to f\psi_{\dot{\alpha}} + i\dot{x}_\mu \left(\bar{\epsilon}\sigma_\mu\right)_{\dot{\alpha}}, \\
f\bar{\psi}^{\dot{\alpha}} &\to f\bar{\psi}^{\dot{\alpha}} - i\dot{x}_\mu \left(\sigma_\mu^\dagger\epsilon\right)^{\dot{\alpha}}, \\
\varphi_i &\to \varphi_i + if\left(t^a\varphi\right)_i \mathcal{A}_\mu^a \left(\epsilon\sigma_\mu\psi + \bar{\epsilon}\sigma_\mu\bar{\psi}\right), \\
\bar{\varphi}^i &\to \bar{\varphi}^i - if\left(\bar{\varphi}t^a\right)^i \mathcal{A}_\mu^a \left(\epsilon\sigma_\mu\psi + \bar{\epsilon}\sigma_\mu\bar{\psi}\right).
\end{aligned} \tag{B.3.13}$$

Obviously, one can also construct in this way a $\mathcal{N} = 2$ supersymmetric Lagrangian for the Hamiltonian (B.1.3) with generic (not necessarily self-dual) $\mathcal{A}_\mu$. A similar construction (but with extra fermionic rather than bosonic variables) was in fact discussed in Ref. [4]. A beauty of the HSS approach explored in this paper is, however, that such extra variables and the constraint (B.2.28) are not introduced by hand, but arise naturally from the manifestly off-shell supersymmetric superfield actions.

Among the directions of further study, it is worthwhile to mention the construction of higher $\mathcal{N}$ SQM models with non-Abelian gauge field backgrounds, e.g. $\mathcal{N} = 8$ ones, making use of a nonlinear counterpart of q^+ [9] to describe the basic 4-manifold (in this case, the bosonic geometry is not conformally-flat), as well as studying various supersymmetry-preserving reductions of these models to lower-dimensional target bosonic manifolds by the gauging procedure of [9]. Actually, the method of the auxiliary "semi-dynamical" $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet with the Wess-Zumino type action, which we successfully applied in our construction here, could work with the equal efficiency for constructing a Lagrangian description of other supersymmetric quantum-mechanics problems involving the coupling to an external non-Abelian gauge field. Besides the obvious examples of quantum Hall

[10] This elementary observation should be well known, for example, in matrix models. Surprisingly, we have not found it in such a "chemically pure" form in the literature, but similar constructions were discussed, e.g., in Refs. [14, 20].

effect (or Landau problem) in higher dimensions (see e.g. the discussion in [18]), we would like to mention supersymmetric Wilson loop functionals which can be interpreted in terms of a non-Abelian version of Chern-Simons (super)quantum mechanics [21], with the parameter along the loop as an evolution parameter. We hope that the quantized semi-dynamical variables could provide a new efficient tool to study this class of problems.

Acknowledgments

E.I. thanks SUBATECH (Nantes) for the warm hospitality extended to him during the course of this work. He acknowledges a partial support from the RFBR grants 08-02-90490, 09-02-01209 and 09-01-93107. E.I. and M.K. thank the University of Lyon, where a part of this work was fulfilled, for hospitality. The authors are grateful to F. Delduc and A. Gorsky for interest in the work and useful discussions. M.K. thanks O. Driga for encouraging during this work. The work of M.K. was supported in part by the RFBR grant 07-02-01161 and a grant for the leading scientific schools NSH-3036.2008.2.

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Annexe C

Text of reference [4]

ITEP-TH-19/10

$\mathcal{N}=4$, $3D$ Supersymmetric Quantum Mechanics in Non-Abelian Monopole Background

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Using the harmonic superspace approach, we construct the three-dimensional $\mathcal{N}=4$ supersymmetric quantum mechanics of the supermultiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ coupled to an external SU(2) gauge field. The off-shell $\mathcal{N}=4$ supersymmetry requires the gauge field to be a static form of the 't Hooft ansatz for the $4D$ self-dual SU(2) gauge fields, that is a particular solution of Bogomolny equations for BPS monopoles. We present the explicit form of the corresponding superfield and component actions, as well as of the quantum Hamiltonian and $\mathcal{N}=4$ supercharges. The latter can be used to describe a more general $\mathcal{N}=4$ mechanics system, with an arbitrary BPS monopole background and on-shell $\mathcal{N}=4$ supersymmetry. The essential feature of our construction is the use of semi-dynamical spin $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet with the Wess-Zumino type action.

C.1 Introduction

The models of supersymmetric quantum mechanics (SQM) with background gauge fields are of obvious interest for a few reasons. One reason is the close relation of these systems to the renowned Landau problem and its generalizations (see e.g. [1]). The Landau-type models constitute a basis of the theoretical description of quantum Hall effect (QHE), and it is natural to expect that their supersymmetric extensions, with extra fermionic variables added, may be relevant to spin versions of QHE. Also, these systems can provide quantum-mechanical realizations of various Hopf maps closely related to higher-dimensional QHE (see e.g. [2] and references therein). At last, they exhibit $d = 1$ prototypes of couplings to higher-p forms in superbranes and so offer a simplified framework to study these couplings.

$\mathcal{N} = 4$ SQM models with the background Abelian gauge fields were treated in the pioneer papers [3, 4] and, more recently, e.g. in [5, 6, 7, 8]. In particular, in [6] an off-shell Lagrangian superfield formulation of the general models associated with the multiplets $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ and $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ was given in the $\mathcal{N} = 4, d = 1$ harmonic superspace.[1] It was found that $\mathcal{N} = 4, d = 1$ supersymmetry requires the gauge field to be self-dual in the four-dimensional $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ case, or to obey a "static" version of the self-duality condition in the three-dimensional $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ case. In the papers [7, 8] it was observed (in a Hamiltonian approach) that the Abelian $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ $\mathcal{N} = 4$ SQM admits a simple generalization to arbitrary self-dual non-Abelian background.[2] In [11] an off-shell Lagrangian formulation was shown to exist for a particular class of such non-Abelian $\mathcal{N} = 4$ SQM models, with SU(2) gauge group and 't Hooft ansatz [12] for the self-dual SU(2) gauge field (see also [13]). As in the Abelian case, it was the use of $\mathcal{N} = 4, d = 1$ harmonic superspace that allowed us to construct such an off-shell formulation. A new non-trivial feature of the construction of [11] is the involvement of an auxiliary "semi-dynamical" $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet with the Wess-Zumino type action possessing an extra gauged U(1) symmetry. After quantization, the corresponding bosonic $d = 1$ fields become a sort of spin SU(2) variables to which the background gauge field naturally couples.[3]

In the present paper, we exploit a similar method to construct $\mathcal{N} = 4$ supersymmetric coupling of the multiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ to an external non-Abelian gauge field. Like in the $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ case, it is the $d = 1$ harmonic superspace which makes it possible to perform such a construction in a general form. Off-shell $\mathcal{N} = 4$ supersymmetry is shown to restrict the external gauge field to a "static" version of the 't Hooft ansatz for four-dimensional self-dual SU(2) gauge field, that is to a particular solution of the general monopole Bogomolny equations [18].[4] A new feature of the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ case is the appearance of "induced" potential term in the on-shell action as a result of eliminating the auxiliary field of the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet. This term is bilinear in the SU(2) gauge group generators. As a particular "spherically symmetric" case of our construction (with the exact SU(2) R-symmetry) we

[1]The first superfield formulation of general $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ SQM (without background gauge field couplings) was given in [9].

[2]The presence of $\mathcal{N} = 4$ supersymmetry in the Dirac operator with a self-dual gauge field was established first in [10], though in an implicit way.

[3]The use of such auxiliary bosonic variables for setting up coupling of a particle to Yang-Mills fields can be traced back to [14]. In the context of $\mathcal{N} = 4$ SQM, they were employed in [15, 16] and [2, 17].

[4]Some BPS monopole backgrounds in the framework of $\mathcal{N} = 2$ SQM were considered, e.g., in [19].

recover, up to an essentially different treatment of the spin variables, the $\mathcal{N}=4$ mechanics with Wu-Yang monopole [20] recently considered in [17].

C.2 Superfield Formulation

In the $\mathcal{N}=4$, $d=1$ harmonic superspace (HSS) approach [6], the superfields depend on bosonic variables $t,\ u^{\pm\alpha}$, where the harmonics $u^{+\alpha}$, $u^-_\alpha = (u^{+\alpha})^*$, $u^{+\alpha}u^-_\alpha = 1$ parametrize the R-symmetry group SU(2) of the $\mathcal{N}=4$ superalgebra, and on fermionic variables $\theta^\pm = \theta^\alpha u^\pm_\alpha$, $\bar\theta^\pm = \bar\theta^\alpha u^\pm_\alpha$. The most important feature of HSS is the presence of an *analytic subspace* $\left\{t_{\rm A}, \theta^+, \bar\theta^+, u^\pm_\alpha\right\}$ in it involving the "analytic time" $t_{\rm A} = t + i(\theta^+\bar\theta^- + \theta^-\bar\theta^+)$ and containing twice as less fermionic coordinates. Spinor derivatives D^+ and $\bar D^+$ in the *analytic basis* $\left\{t_{\rm A}, \theta^\pm, \bar\theta^\pm, u^\pm_\alpha\right\}$ are [21]

$$D^+ = \frac{\partial}{\partial\theta^-}, \qquad \bar D^+ = -\frac{\partial}{\partial\bar\theta^-}. \tag{C.2.1}$$

Other important objects used in what follows are the harmonic derivatives D^{++}, D^{--} preserving the $\mathcal{N}=4$ analyticity :

$$D^{++} = u^+_\alpha\frac{\partial}{\partial u^-_\alpha} + \theta^+\frac{\partial}{\partial\theta^-} + \bar\theta^+\frac{\partial}{\partial\bar\theta^-} + 2i\theta^+\bar\theta^+\frac{\partial}{\partial t_{\rm A}}, \tag{C.2.2}$$

$$D^{--} = u^-_\alpha\frac{\partial}{\partial u^+_\alpha} + \theta^-\frac{\partial}{\partial\theta^+} + \bar\theta^-\frac{\partial}{\partial\bar\theta^+} + 2i\theta^-\bar\theta^-\frac{\partial}{\partial t_{\rm A}}. \tag{C.2.3}$$

Also, for further use, we give how the coordinates of the analytic subspace transform under $\mathcal{N}=4$ supersymmetry :

$$\delta\theta^+ = \epsilon^\alpha u^+_\alpha\,, \quad \delta\bar\theta^+ = \bar\epsilon^\alpha u^+_\alpha\,, \quad \delta t_A = 2i\left(\epsilon^\alpha u^-_\alpha\bar\theta^+ - \bar\epsilon^\alpha u^-_\alpha\theta^+\right), \quad \delta u^\pm_\alpha = 0\,, \ \ \bar\epsilon^\alpha = (\epsilon_\alpha)^*. \tag{C.2.4}$$

In this paper, we shall deal with the analytic superfields L^{++} and $v^+, \widetilde{v^+}$ which encompass, respectively, the multiplets $(\mathbf{3},\mathbf{4},\mathbf{1})$ and $(\mathbf{4},\mathbf{4},\mathbf{0})$ and are subjected to the constraints

$$\text{(a) } D^+L^{++} = \bar D^+L^{++} = 0\,, \qquad \text{(b) } D^{++}L^{++} = 0\,,\ \ \widetilde{(L^{++})} = -L^{++}, \tag{C.2.5}$$

$$\text{(a) } D^+(v^+,\ \widetilde{v^+}) = \bar D^+(v^+,\ \widetilde{v^+}) = 0\,, \quad \text{(b) } (D^{++}+iV^{++})v^+ = (D^{++}-iV^{++})\widetilde{v^+} = 0\,. \tag{C.2.6}$$

The U(1) gauge superfield V^{++} appearing in Eqs. (C.2.6b) is analytic,

$$D^+V^{++} = \bar D^+V^{++} = 0\,, \tag{C.2.7}$$

and pseudoreal, $V^{++} = \widetilde{(V^{++})}$. It ensures the covariance of (C.2.6b) under the gauge U(1) transformations with the analytic parameter Λ [22]

$$V^{++} \ \rightarrow\ V^{++} + D^{++}\Lambda, \qquad v^+ \ \rightarrow\ e^{-i\Lambda}v^+, \ \ \widetilde{v^+} \ \rightarrow\ e^{i\Lambda}\widetilde{v^+}, \quad D^+\Lambda = \bar D^+\Lambda = 0\,. \tag{C.2.8}$$

In what follows, we shall use the WZ gauge for V^{++},

$$V^{++} \ = \ 2i\,\theta^+\bar\theta^+B\,. \tag{C.2.9}$$

Here $B(t)$ is a real $d=1$ "gauge field", it transforms as $B \;\to\; B+\dot\lambda$, with $\lambda(t)$ being the parameter of the residual gauge U(1) symmetry.

The constraints (C.2.5a), (C.2.6a) and (C.2.7) are the $\mathcal{N}=4$ Grassmann analyticity conditions just implying that the superfields L^{++}, v^+, $\widetilde{v}^+$, V^{++} live on the analytic superspace $\left\{t_{\rm A}, \theta^+, \bar\theta^+, u^\pm_\alpha\right\}$. The basic conditions are those with the harmonic derivatives, i.e. (C.2.5b) and (C.2.6b). They constrain the analytic superfields L^{++} and v^+, $\widetilde{v^+}$ to have the appropriate off-shell component field contents, namely $(\mathbf{3},\mathbf{4},\mathbf{1})$ and $(\mathbf{4},\mathbf{4},\mathbf{0})$:

$$L^{++}=\ell^{\alpha\beta}u^+_\alpha u^+_\beta+i\theta^+\chi^\alpha u^+_\alpha+i\bar\theta^+\bar\chi^\alpha u^+_\alpha+\theta^+\bar\theta^+[F-2i\dot\ell^{\alpha\beta}u^+_\alpha u^-_\beta], \tag{C.2.10}$$

with $(\ell_{\alpha\beta})^*=-\ell^{\alpha\beta}$, $(\chi^\alpha)^*=\bar\chi_\alpha$, and

$$v^+=\phi^\alpha u^+_\alpha+\theta^+\omega_1+\bar\theta^+\bar\omega_2-2i\theta^+\bar\theta^+(\dot\phi^\alpha+iB\phi^\alpha)u^-_\alpha\,, \tag{C.2.11}$$

$$\widetilde{v^+}=\bar\phi^\alpha u^+_\alpha+\theta^+\omega_2-\bar\theta^+\bar\omega_1-2i\theta^+\bar\theta^+(\dot{\bar\phi}^\alpha-iB\bar\phi^\alpha)u^-_\alpha\,, \tag{C.2.12}$$

with $\bar\phi^\alpha=(\phi_\alpha)^*$, $\bar\omega_{1,2}=(\omega_{1,2})^*$. The multiplet L^{++} involves the 3-dimensional target space coordinates $\ell^{\alpha\beta}=\ell^{\beta\alpha}$, their fermionic partners and a real auxiliary field F, while v^+ accommodates the auxiliary degrees of freedom needed to arrange a coupling to the external non-Abelian SU(2) Yang-Mills field [11].

The full Lagrangian $\mathcal{L}$ entering the $\mathcal{N}=4$ invariant off-shell action $S=\int dt\mathcal{L}$ consists of the three pieces

$$\mathcal{L}=\mathcal{L}_{\rm kin}+\mathcal{L}_{\rm int}+\mathcal{L}_{\rm FI}=\int du\,d^4\theta\,R_{\rm kin}(L^{++},L^{+-},L^{--},u)$$
$$-\frac12\int du\,d\bar\theta^+d\theta^+\,K(L^{++},u)v^+\widetilde{v^+}-\frac{ik}{2}\int du\,d\bar\theta^+d\theta^+\,V^{++}\,, \tag{C.2.13}$$

where $L^{+-}=\frac12 D^{--}L^{++}$ and $L^{--}=D^{--}L^{+-}$. The superfield functions $R_{\rm kin}$ and K bear an arbitrary dependence on their arguments. The meaning of three terms in (C.2.13) will be explained in the next Section.

C.3 From harmonic superspace to components

The first, sigma-model-type term in Eq. (C.2.13), after integrating over Grassmann and harmonic variables, yields the generalized kinetic terms for $\ell^{\alpha\beta},\chi^\alpha,\bar\chi_\alpha$:

$$\mathcal{L}_{\rm kin}=\frac18 f^{-2}\left(-2\dot\ell_{\alpha\beta}\dot\ell^{\alpha\beta}+F^2\right)+\frac{i}{8}f^{-2}\left(\bar\chi_\alpha\dot\chi^\alpha-\dot{\bar\chi}_\alpha\chi^\alpha\right)+\frac{1}{64}\left(\partial_{\alpha\beta}\partial^{\alpha\beta}f^{-2}\right)\chi^4$$
$$+\frac{i}{4f^3}\dot\ell^{\alpha\beta}\left\{\partial_{\alpha\gamma}f\chi_\beta\bar\chi^\gamma+\partial_{\beta\gamma}f\chi^\gamma\bar\chi_\alpha\right\}-\frac{1}{4f^3}F\chi^\alpha\bar\chi^\beta\partial_{\alpha\beta}f, \tag{C.3.1}$$

where $\chi^4 = \chi^\alpha \chi_\alpha \bar{\chi}^\beta \bar{\chi}_\beta$, $\partial_{\alpha\beta} \equiv \frac{\partial}{\partial \ell^{\alpha\beta}}$ and $f(\ell)$ is a conformal factor.[5] The fermionic kinetic term can be brought to the canonical form by the change of variables

$$\chi^\alpha = 2f\psi^\alpha, \qquad \bar{\chi}_\alpha = 2f\bar{\psi}_\alpha. \tag{C.3.2}$$

It is worth pointing out that the R-symmetry SU(2) group amounts to the rotational SO(3) group in the $\mathbb{R}^3$ target space parametrized by $\ell^{\alpha\beta}$. The conformal factor $f(\ell)$ can bear an arbitrary dependence on $\ell^{\alpha\beta}$, so this SO(3) can be totally broken in the Lagrangian (C.3.1).

The second piece in Eq. (C.2.13) describes the coupling to an external non-Abelian gauge field. Performing the integration over θ^+, $\bar{\theta}^+$ and $u^\pm_\alpha$, eliminating the auxiliary fermionic fields $\omega_{1,2}$ and, finally, rescaling the bosonic doublet variables as $\varphi_\alpha = \phi_\alpha \sqrt{h(\ell)}$, where

$$h(\ell) = \int du\, K\left(\ell^{\alpha\beta} u^+_\alpha u^+_\beta, u^\pm_\gamma\right), \tag{C.3.3}$$

after some algebra we obtain

$$\mathcal{L}_{\text{int}} = i\bar{\varphi}^\alpha\left(\dot{\varphi}_\alpha + iB\varphi_\alpha\right) + \bar{\varphi}^\gamma\varphi^\delta \frac{1}{2}\left(\mathcal{A}_{\alpha\beta}\right)_{\gamma\delta}\dot{\ell}^{\alpha\beta} - \frac{1}{2}F\,\bar{\varphi}^\gamma\varphi^\delta\,U_{\gamma\delta} + \frac{1}{4}\chi^\alpha\bar{\chi}^\beta\bar{\varphi}^\gamma\varphi^\delta\nabla_{\alpha\beta}U_{\gamma\delta}. \tag{C.3.4}$$

Here the non-Abelian background gauge field and the scalar (matrix) potential are fully specified by the function h defined in (C.3.3) :

$$\left(\mathcal{A}_{\alpha\beta}\right)_{\gamma\delta} = \frac{i}{2h}\Big\{\varepsilon_{\gamma\beta}\partial_{\alpha\delta}h + \varepsilon_{\gamma\alpha}\partial_{\beta\delta}h + \varepsilon_{\delta\beta}\partial_{\alpha\gamma}h + \varepsilon_{\delta\alpha}\partial_{\beta\gamma}h\Big\}, \qquad U_{\gamma\delta} = \frac{1}{h}\partial_{\gamma\delta}h\,. \tag{C.3.5}$$

By definition, the function h obeys the 3-dimensional Laplace equation,

$$\partial^{\alpha\beta}\partial_{\alpha\beta}\,h = 0\,. \tag{C.3.6}$$

Using the explicit expressions (C.3.5), it is straightforward to check the relation

$$\left(\mathcal{F}_{\alpha\beta}\right)_{\gamma\delta} = 2i\nabla_{\alpha\beta}U_{\gamma\delta}, \tag{C.3.7}$$

where

$$\left(\mathcal{F}_{\alpha\beta}\right)_{\gamma\delta} = -2\partial^{\lambda}_{\alpha}\left(\mathcal{A}_{\lambda\beta}\right)_{\gamma\delta} + i\left(\mathcal{A}_\alpha{}^\lambda\right)_{\gamma\sigma}\left(\mathcal{A}_{\lambda\beta}\right)_\delta{}^\sigma + (\alpha\leftrightarrow\beta)\,, \tag{C.3.8}$$

$$\nabla_{\alpha\beta}U_{\gamma\delta} = -2\partial_{\alpha\beta}U_{\gamma\delta} + i\left(\mathcal{A}_{\alpha\beta}\right)_{\gamma\lambda}U_\delta{}^\lambda + i\left(\mathcal{A}_{\alpha\beta}\right)_{\delta\lambda}U_\gamma{}^\lambda, \tag{C.3.9}$$

and $\left(\mathcal{F}_{\alpha\beta}\right)_{\gamma\delta}$ is related to the standard gauge field strength in the vector notation, see below. As we shall see soon, the condition (C.3.7) is none other than the static form of the general self-duality condition for the SU(2) Yang-Mills field on $\mathbb{R}^4$, i.e. the Bogomolny equations for BPS monopoles [18], while (C.3.5) provides a particular solution to these equations, being a static form of the renowned 't Hooft ansatz [12].

[5]The calculations are most easy in the central basis, where $L^{++} = u^+_\alpha u^+_\beta L^{\alpha\beta}\left(t, \theta_\gamma, \bar{\theta}^\delta\right)$. Then

$$f^{-2}(\ell) = -\partial_{\alpha\beta}\partial^{\alpha\beta}\int R_{\text{kin}}\left(\ell^{\alpha\beta}u^+_\alpha u^+_\beta, \ell^{\alpha\beta}u^+_\alpha u^-_\beta, \ell^{\alpha\beta}u^-_\alpha u^-_\beta\right)du.$$

Note that the relation (C.3.7) is covariant and the Lagrangian (C.3.4) is form-invariant under the following "target space" SU(2) gauge transformations :

$$\begin{gathered} \varphi_\alpha \to \left(U^\dagger \varphi\right)_\alpha, \quad \bar{\varphi}^\alpha \to (\bar{\varphi} U)^\alpha \\ \mathcal{A}_{\alpha\beta} \to \Lambda^\dagger \mathcal{A}_{\alpha\beta} \Lambda + i\Lambda^\dagger \partial_{\alpha\beta} \Lambda, \qquad U \to \Lambda^\dagger U \Lambda, \end{gathered} \tag{C.3.10}$$

with $\Lambda(\ell) \in \mathrm{SU}(2)$. This is not a genuine symmetry; rather, it is a reparametrization of the Lagrangian which allows one to cast the background potentials (C.3.5) in some different equivalent forms. It is worth noting that the gauge group indices coincide with those of the R-symmetry group, like in the four-dimensional case [11]. Nevertheless, the "gauge" reparametrizations (C.3.10) do not affect the doublet indices of the target space coordinates $\ell^{\alpha\beta}$ and their superpartners accommodated by the superfield L^{++}. They act only on the semi-dynamical spin variables $\varphi_\alpha, \bar{\varphi}^\alpha$ and gauge and scalar potentials (C.3.5).

Finally, the last piece in Eq. (C.2.13) yields the Fayet-Iliopoulos term,

$$\mathcal{L}_{\mathrm{FI}} = kB\,. \tag{C.3.11}$$

In the quantum case, the coefficient k is quantized, $k \in \mathbb{Z}$, on the same ground as in the 4-dimensional case [11]. As is obvious from Eqs. (C.3.4) and (C.3.11), the auxiliary gauge field B serves as a Lagrange multiplier for the constraint

$$\bar{\varphi}^\alpha \varphi_\alpha = k\,. \tag{C.3.12}$$

In the classical case it implies (together with the residual U(1) gauge freedom) that $\bar{\varphi}^\alpha, \varphi_\alpha$ describe coordinates on a sphere S^2 in the target space, while in the quantum case the constraint (C.3.12) is imposed on the wave function requiring it to span an irreducible SU(2) multiplet with spin $|k|/2$ [11].

It is instructive to rewrite the above relations and expressions, including the full Lagrangian (C.2.13) in a vector notation. To this end, we associate a vector v_i to any traceless bi-spinor v_α^β by the general rule

$$v_\alpha^\beta = v_i \left(\sigma_i\right)_\alpha^{\ \beta}, \qquad v_i = \frac{1}{2} v_\beta^\alpha \left(\sigma_i\right)_\alpha^{\ \beta}, \qquad i = 1,2,3, \tag{C.3.13}$$

where σ_i are Pauli matrices. In particular, the $3D$ spinor coordinates $\ell^{\alpha\beta}$ (restricted by the condition $(\ell^{\alpha\beta})^* = -\ell_{\alpha\beta}$) correspond to real vector coordinates ℓ_i. The only exception from the rule (C.3.13) is the relation between the partial derivatives $\partial_{\alpha\beta} = \partial/\partial\ell^{\alpha\beta}$ and $\partial_i = \partial/\partial\ell_i$,

$$\partial_{\alpha\beta} = -\frac{1}{2}\left(\sigma_i\right)_{\alpha\beta} \partial_i, \qquad \partial_i = -\left(\sigma_i\right)_\alpha^{\ \beta} \partial_\beta^\alpha\,. \tag{C.3.14}$$

We also make a similar conversion of the gauge group indices,

$$M_\gamma^{\ \delta} = \frac{1}{2} M^a \left(\sigma_a\right)_\gamma^{\ \delta}, \qquad M^a = M_\delta^{\ \gamma} \left(\sigma_a\right)_\gamma^{\ \delta}, \qquad a = 1,2,3\,, \tag{C.3.15}$$

for any Hermitian traceless 2×2 matrix M, and define

$$T^a = \frac{1}{2} \bar{\varphi}^\alpha \left(\sigma_a\right)_\alpha^{\ \beta} \varphi_\beta\,. \tag{C.3.16}$$

In the new notations, the total Lagrangian (C.2.13) takes the following form :

$$\mathcal{L} = \frac{1}{2}f^{-2}\dot{\ell}_i^2 + \mathcal{A}_i^a T^a \dot{\ell}_i + i\bar{\varphi}^\alpha\left(\dot{\varphi}_\alpha + iB\varphi_\alpha\right) + kB + i\bar{\psi}_\alpha\dot{\psi}^\alpha + f^2\nabla_i U^a T^a\,\psi\sigma_i\bar{\psi}$$
$$+\frac{1}{4}\left\{f\partial_i^2 f - 3\left(\partial_i f\right)^2\right\}\psi^4 + 2f^{-1}\varepsilon_{ijk}\partial_i f\,\dot{\ell}_j\,\psi\sigma_k\bar{\psi}$$
$$+\frac{1}{8}f^{-2}F^2 + \frac{1}{2}F\left(U^aT^a - f^{-1}\partial_i f\,\psi\sigma_i\bar{\psi}\right). \quad \text{(C.3.17)}$$

Here

$$\nabla_i U^a = \partial_i U^a + \varepsilon^{abc}\mathcal{A}_i^b U^c \quad \text{(C.3.18)}$$

and the Bogomolny equations (C.3.7) relating $\mathcal{A}_i^a$ and U^a are equivalently rewritten in the more familiar form,

$$\mathcal{F}_{ij}^a = \varepsilon_{ijk}\nabla_k U^a\,, \quad \text{(C.3.19)}$$

where $\mathcal{F}_{ij}^a = \partial_i\mathcal{A}_j^a - \partial_j\mathcal{A}_i^a + \varepsilon^{abc}\mathcal{A}_i^b\mathcal{A}_j^c$. Finally, the gauge field and the matrix potential defined in (C.3.5) are rewritten as

$$\mathcal{A}_i^a = -\varepsilon_{ija}\partial_j \ln h, \qquad U^a = -\partial_a \ln h\,, \qquad \Delta\, h = 0\,. \quad \text{(C.3.20)}$$

The component action corresponding to the Lagrangian (C.3.17) is partly on shell since we have already eliminated the fermionic fields of the auxiliary v^+ multiplet by their algebraic equations of motion. The fields of the coordinate multiplet L^{++} are still off shell. The $\mathcal{N}=4$ transformations leaving invariant the action $S = \int dt\,\mathcal{L}$ look most transparent in terms of the component fields $\ell_i, F, \chi^\alpha,\ \bar{\chi}^\alpha,\ \phi^\beta,\ \bar{\phi}^\beta$:

$$\begin{array}{ll}\delta\ell_i = -\frac{i}{2}\left(\epsilon\sigma_i\chi + \bar{\epsilon}\sigma_i\bar{\chi}\right), & \delta F = \epsilon^\alpha\dot{\chi}_\alpha + \bar{\epsilon}^\alpha\dot{\bar{\chi}}_\alpha\,,\\ \delta\chi^\alpha = iF\bar{\epsilon}^\alpha + 2(\bar{\epsilon}\sigma_i)^\alpha\,\dot{\ell}_i\,, & \delta\bar{\chi}^\alpha = -iF\epsilon^\alpha - 2(\epsilon\sigma_i)^\alpha\,\dot{\ell}_i\,,\\ \delta\phi^\alpha = \frac{i}{2}\left(\epsilon^\alpha\chi\sigma_i\phi + \bar{\epsilon}^\alpha\bar{\chi}\sigma_i\phi\right)\partial_i\ln h\,, & \delta\bar{\phi}^\alpha = \frac{i}{2}\left(\epsilon^\alpha\chi\sigma_i\bar{\phi} + \bar{\epsilon}^\alpha\bar{\chi}\sigma_i\bar{\phi}\right)\partial_i\ln h\,.\end{array} \quad \text{(C.3.21)}$$

These transformations can be deduced from the analytic subspace realization of $\mathcal{N}=4$ supersymmetry (C.2.4), with taking into account the compensating U(1) gauge transformations of the superfields $v^+, \widetilde{v}^+$ and V^{++} needed to preserve the WZ gauge (C.2.9). Note that $\delta B = 0$ under $\mathcal{N}=4$ supersymmetry.[6]

After eliminating the auxiliary field F by its equation of motion,

$$F = 2f^2\left(f^{-1}\partial_i f\,\psi\sigma_i\bar{\psi} - U^aT^a\right), \quad \text{(C.3.22)}$$

the Lagrangian (C.3.17) takes the form

$$\begin{aligned}\mathcal{L} \;=\; & \frac{1}{2}f^{-2}\dot{\ell}_i^2 + \mathcal{A}_i^aT^a\dot{\ell}_i + i\bar{\varphi}^\alpha\left(\dot{\varphi}_\alpha + iB\varphi_\alpha\right) + kB + i\bar{\psi}_\alpha\dot{\psi}^\alpha + f^2\psi\sigma_i\bar{\psi}\left(\nabla_i + f^{-1}\partial_i f\right)U^aT^a\\ & +\frac{1}{4}\left\{f\partial_i^2 f - 4\left(\partial_i f\right)^2\right\}\psi^4 + 2f^{-1}\varepsilon_{ijk}\partial_i f\,\dot{\ell}_j\,\psi\sigma_k\bar{\psi} - \frac{1}{2}f^2(U^aT^a)^2\,.\end{aligned} \quad \text{(C.3.23)}$$

[6] This transformation law matches with the $\mathcal{N}=4, d=1$ superalgebra in WZ gauge, taking into account that the $d=1$ translation of B looks as a particular U(1) gauge transformation of the latter.

It is invariant, modulo a total time derivative, under the transformations (C.3.21) in which F is expressed from (C.3.22). We see that this Lagrangian involves three physical bosonic fields ℓ_i and four physical fermionic fields ψ_α. It is fully specified by two independent functions : the metric conformal factor $f(\ell)$ which can bear an arbitrary dependence on ℓ_i and the function $h(\ell)$ which satisfies the $3D$ Laplace equation and determines the background non-Abelian gauge and scalar potentials. The representation (C.3.3) for h in terms of the analytic function $K(\ell^{++}, u)$ yields in fact a general solution of the $3D$ Laplace equation [21]. The Lagrangian (C.3.23) also contains the "semi-dynamical" spin variables $\varphi_\alpha, \bar\varphi^\alpha$, the role of which is the same as in the $4D$ case [11] : after quantization they ensure that T^a defined in (C.3.16) become matrix SU(2) generators corresponding to the spin $|k|/2$ representation.

C.4 Hamiltonian and supercharges

The Lagrangian (C.3.23) is the point of departure for setting up the Hamiltonian formulation of the model under consideration and quantizing the latter. The main peculiarity of the quantization procedure in the present case is related to the spin variables $\varphi_\alpha, \bar\varphi^\alpha$. The corresponding commutation relations are

$$[\varphi_\alpha, \bar\varphi^\beta] = \delta^\alpha_\beta\,, \qquad [\varphi_\alpha, \varphi_\beta] = [\bar\varphi^\alpha, \bar\varphi^\beta] = 0\,, \tag{C.4.1}$$

whence, e.g., $\varphi_\alpha \to \hat\varphi_\alpha \equiv \partial/\partial\bar\varphi^\alpha$ and the constraint (C.3.12) becomes the condition on the wave functions

$$\bar\varphi^\alpha \frac{\partial}{\partial\bar\varphi^\alpha}\Psi = k\Psi \tag{C.4.2}$$

(hereafter, without loss of generality, we assume that $k > 0$). It implies that Ψ is a collection of homogeneous monomials of $\bar\varphi^\alpha$ of an integer degree k and, thus, carries an irreducible SU(2) multiplet with spin $k/2$ (the number of such independent monomials is equal just to $k+1$). The SU(2) vector T^a defined in (C.3.16) satisfy the SU(2) commutation relations

$$[T^a, T^b] = i\varepsilon^{abc}T^c\,, \tag{C.4.3}$$

and, as a consequence of the constraint (C.4.2), is subject to the condition

$$T^aT^a = \frac{k}{2}\left(\frac{k}{2}+1\right). \tag{C.4.4}$$

In this way, T^a can be treated as generators of the irreducible unitary representation of SU(2) with spin $k/2$. [7]

[7] The crucial role of the constraint (C.4.2) is to restrict the space of quantum states of the considered model to the *finite* set of irreducible SU(2) multiplets of fixed spins (e.g., of the spin $k/2$ in the bosonic sector). This is an essential difference of our approach from that employed, e.g., in [14] (and, lately, in [17, 13]) where no any analog of the constraints (C.3.12) and (C.4.2) is imposed, thus allowing the space of states to involve an *infinite* number of SU(2) multiplets of all spins. The quantization scheme which we follow here can be traced back to the work [23]. In the SQM context, it was already used in [16] and [11].

The system (C.3.23) is a generalization, to the non-Abelian case, of the Abelian $\mathcal{N}=4$ $3D$ system found in [4], which, in turn, is a generalization, to the conformal metric, of the system in a flat space invented by de Crombrugghe and Rittenberg [3]. After substitution of SU(2) spin-$k/2$ generators instead of T^a [11], the (quantum) Hamiltonian of this system takes the form

$$H = \frac{1}{2} f\left(\hat{p}_i - \mathcal{A}_i\right)^2 f + \frac{1}{2} f^2 U^2 - f^2 \nabla_i U \psi \sigma_i \bar{\psi}$$
$$+ \left\{ \varepsilon_{ijk} f \partial_i f \left(\hat{p}_j - \mathcal{A}_j\right) - f \partial_k f U \right\} \psi \sigma_k \bar{\psi} + f \partial^2 f \left\{ \psi^\gamma \bar{\psi}_\gamma - \frac{1}{2} \left(\psi^\gamma \bar{\psi}_\gamma\right)^2 \right\}, \quad \text{(C.4.5)}$$

which is just a static $3D$ reduction of the 4-dimensional Hamiltonian given in [8]. In this expression, the gauge field $\mathcal{A}_i = \mathcal{A}_i^a T^a$ and the scalar potential $U = U^a T^a$ are SU(2) matrices subjected to the constraint (C.3.19). It is also easy to find the supercharges $Q_\alpha, \bar{Q}^\beta$,

$$\begin{aligned} Q_\alpha &= f\left(\sigma_i \bar{\psi}\right)_\alpha \left(\hat{p}_i - \mathcal{A}_i\right) - \psi^\gamma \bar{\psi}_\gamma \left(\sigma_i \bar{\psi}\right)_\alpha i \partial_i f - i f U \bar{\psi}_\alpha, \\ \bar{Q}^\alpha &= \left(\psi \sigma_i\right)^\alpha \left(\hat{p}_i - \mathcal{A}_i\right) f + i \partial_i f \left(\psi \sigma_i\right)^\alpha \psi^\gamma \bar{\psi}_\gamma + i f U \psi^\alpha, \end{aligned} \quad \text{(C.4.6)}$$

$$\{Q_\alpha, \bar{Q}^\beta\} = 2\delta_\alpha^\beta H, \quad \{Q_\alpha, Q_\beta\} = \{\bar{Q}^\alpha, \bar{Q}^\beta\} = 0. \quad \text{(C.4.7)}$$

The ordering ambiguity arising in the case of the general conformal factor $f(\ell)$ can be fixed, as in [8], by the arguments of Ref. [24].

We would like to emphasize that the only condition required from the background matrix fields $\mathcal{A}_i$ and U for the generators Q_α and $\bar{Q}^\beta$ to form $\mathcal{N}=4$ superalgebra (C.4.7) is that these fields satisfy the Bogomolny equations (C.3.19). Thus the expressions (C.4.5) and (C.4.6) define the $\mathcal{N}=4$ SQM model in the field of *arbitrary* BPS monopole, not necessarily restricted to the ansatz (C.3.20). Also, one can extend the gauge group SU(2) to SU(N) in (C.4.5) and (C.4.6). The 't Hooft type ansatz (C.3.20) and the choice of SU(2) as the gauge group are required for the existence of *off-shell* Lagrangian formulation of this SQM system. We do not know whether the most general system can be derived from some off-shell superfield formalism, though the corresponding component Lagrangian with the on-shell realization of $\mathcal{N}=4$ supersymmetry can certainly be constructed. It is a straightforward extension of the Lagrangian (C.3.17) or (C.3.23), with the properly enlarged set of semi-dynamical spin variables, and the external potentials $\mathcal{A}_i$, U taking values in the $su(N)$ algebra and obeying Eq. (C.3.19). This situation is quite similar to what was observed in [8, 11] in the case of $4D$ SQM with self-dual gauge fields.

Finally, as a simple example of the monopole background consistent with the off- and on-shell $\mathcal{N}=4$ supersymmetry, let us consider a particular $3D$ spherically symmetric case. It corresponds to the most general SO(3) invariant solution of the Laplace equation for the function h

$$h_{\mathrm{so}(3)}(\ell) = c_0 + c_1 \frac{1}{\sqrt{\ell^2}}. \quad \text{(C.4.8)}$$

The corresponding potentials calculated according to Eqs. (C.3.20) read

$$\mathcal{A}_i^a = \varepsilon_{ija} \frac{\ell_j}{\ell^2} \frac{c_1}{c_1 + c_0\sqrt{\ell^2}}, \quad U^a = \frac{\ell_a}{\ell^2} \frac{c_1}{c_1 + c_0\sqrt{\ell^2}}. \quad \text{(C.4.9)}$$

This configuration becomes the Wu-Yang monopole [20] for the choice $c_0 = 0$. It is easy to find the analytic function $K(\ell^{++}, u)$ which generates the solution (C.4.8) (see [6]) :

$$h_{\mathrm{so}(3)}(\ell) = \int du\, K_{\mathrm{so}(3)}(\ell^{++}, u)\,, \quad K_{so(3)}(\ell^{++}, u) = c_0 + c_1 \left(1 + a^{--}\hat{\ell}^{++}\right)^{-\frac{3}{2}}, \tag{C.4.10}$$
$$\ell^{++} \equiv \hat{\ell}^{++} + a^{++}\,, \quad a^{\pm\pm} = a^{\alpha\beta} u^{\pm}_{\alpha} u^{\pm}_{\beta}\,, \quad a^{\alpha}_{\beta} a^{\beta}_{\alpha} = 2\,.$$

One could equally choose as $h(\ell)$, e.g., the well-known multi-center solution to the Laplace equation, with the broken SO(3). Note that the $\mathcal{N} = 4$ mechanics with coupling to Wu-Yang monopole was recently constructed in [17], proceeding from a different approach, with the built-in SO(3) invariance and the treatment of spin variables in the spirit of Ref. [14]. Our general consideration shows, in particular, that the demand of SO(3) symmetry is not necessary for the existence of $\mathcal{N} = 4$ SQM models with non-Abelian monopole backgrounds.

C.5 Relation to four-dimensional $\mathcal{N} = 4$ SQM model

It is instructive to show that (C.3.20) can indeed be viewed as a $3D$ reduction of 't Hooft ansatz for the solution of general self-duality equation in $\mathbb{R}^4$ for the gauge group SU(2), with the identification $U^a = \mathcal{A}^a_0$, while the condition (C.3.19) as $3D$ reduction of this equation.

To establish this relation, we use the following dictionary between the SO(4) $\sim$ SU(2) $\times$ SU(2) spinor formalism of Refs. [8, 11] and its SU(2) reduction :

$$\begin{array}{c} (\sigma_\mu)_{\alpha\dot{\beta}} \to \left\{ i\delta^{\beta}_{\alpha},\, (\sigma_i)_{\alpha}{}^{\beta} \right\}, \quad \varepsilon^{\dot{\alpha}\dot{\beta}} \to -\varepsilon_{\alpha\beta}\,, \varepsilon_{\dot{\alpha}\dot{\beta}} \to -\varepsilon^{\alpha\beta} \\ x_{\alpha\dot{\beta}} \to \ell^{\beta}_{\alpha}, \qquad x^{\alpha\dot{\beta}} \to -\ell^{\alpha}_{\beta} \qquad \psi_{\dot{\alpha}} \to \psi^{\alpha}. \end{array} \tag{C.5.1}$$

This reflects the fact that the R-symmetry SU(2) in the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ models can be treated as a diagonal subgroup in the symmetry group SO(4) $\sim$ SU(2) $\times$ SU(2) of the $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ models, with the SU(2) factors acting, respectively, on the undotted and dotted indices.

The self-dual $\mathbb{R}^4$ SU(2) gauge field in the 't Hooft ansatz used in [11] can be written in the spinor notation as

$$(\mathcal{A}_{\alpha\dot{\rho}})_{\beta}{}^{\gamma} = -\frac{2i}{h}\left(\varepsilon_{\alpha\beta}\, \partial^{\gamma}_{\dot{\rho}} h - \frac{1}{2}\,\delta^{\gamma}_{\beta}\, \partial_{\alpha\dot{\rho}} h\right), \qquad \partial_{\alpha\dot{\rho}} \equiv \frac{\partial}{\partial x^{\alpha\dot{\rho}}}\,,$$
$$h = h(x^{\alpha\dot{\beta}})\,, \quad \partial^{\alpha\dot{\beta}} \partial_{\alpha\dot{\beta}}\, h = 0\,. \tag{C.5.2}$$

Then, using the rules (C.5.1), one performs the reduction $\mathbb{R}^4 \to \mathbb{R}^3$ as

$$\begin{array}{l} (\mathcal{A}_{\alpha\dot{\beta}})_{\gamma}{}^{\delta} \;\to\; iU_{\gamma}{}^{\delta}\delta^{\beta}_{\alpha} + (\mathcal{A}^{\beta}_{\alpha})_{\gamma}{}^{\delta}\,, \qquad (\mathcal{A}^{\alpha}_{\alpha})_{\gamma}{}^{\delta} = 0\,, \\ h(x) \;\to\; h(\ell), \quad \partial^{\alpha}_{\beta}\partial^{\beta}_{\alpha}\, h = 0\,. \end{array} \tag{C.5.3}$$

Upon this reduction, the four-dimensional ansatz (C.5.2) yields precisely (C.3.5), while the general self-duality condition

$$2\partial_{\alpha\dot{\rho}}(\mathcal{A}^{\dot{\rho}}_{\beta})_{\gamma}{}^{\delta} + i(\mathcal{A}_{\alpha\dot{\rho}})_{\gamma}{}^{\lambda}(\mathcal{A}^{\dot{\rho}}_{\beta})_{\lambda}{}^{\delta} + (\alpha \leftrightarrow \beta) = 0 \tag{C.5.4}$$

goes over into the Bogomolny equations (C.3.7). Of course, the same reduction can be performed in the vector notation, with $\mathcal{F}_{\mu\nu} \to \{\mathcal{F}_{ij}, \mathcal{F}_{0k} = \nabla_k U\}$, and Eqs. (C.3.19), (C.3.20) as an output.

Thus, the general gauge field background prescribed by the off-shell $\mathcal{N}=4$ supersymmetry in our $(\mathbf{3},\mathbf{4},\mathbf{1})$ model is a static form of the 't Hooft ansatz for the self-dual SU(2) gauge field in $\mathbb{R}^4$. As was shown in [11], this particular form of the self-dual field is prescribed by the same off-shell $\mathcal{N}=4$ supersymmetry in the 4D SQM model based on the supermultiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$. This suggests that the above bosonic target space reduction has its superfield counterpart relating the model of [11] to the one considered in the present paper.

Indeed, the superfield $(\mathbf{3},\mathbf{4},\mathbf{1})$ action (C.2.13) can be obtained from the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet action of Ref. [11] via the "automorphic duality" [25] by considering a restricted class of the $(\mathbf{4},\mathbf{4},\mathbf{0})$ actions with U(1) isometry and performing a superfield gauging of this isometry by an extra gauge superfield $V^{++\prime}$ along the general line of Ref. [22]. Actually, the bosonic target space reduction we have just described corresponds to the shift isometry of the analytic superfield $q^{+\dot\alpha}$ accommodating the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet, namely, to $q^{+\dot\alpha} \to q^{+\dot\alpha} + \omega u^{+\dot\alpha}$. It is the invariant projection $q^{+\dot\alpha} u^{+}_{\dot\alpha}$ which is going to become the $(\mathbf{3},\mathbf{4},\mathbf{1})$ superfield L^{++} upon gauging this isometry and choosing the appropriate manifestly $\mathcal{N}=4$ supersymmetric gauge. Another type of possible isometry of the $q^{+\dot\alpha}$ actions of Ref. [11] is the phase one, with $q^{+\dot1} q^{+\dot2}$ as the appropriate invariant. It can also be gauged, with the same L^{++} action as a result.

An important impact of this superfield reduction on the structure of the component action is the appearance of the new induced potential bilinear in the gauge group generators $\sim U^2 = U^a U^b T^a T^b$. It comes out as a result of eliminating the auxiliary field F in the off-shell $(\mathbf{3},\mathbf{4},\mathbf{1})$ multiplet, and so is necessarily prescribed by $\mathcal{N}=4$ supersymmetry. It is interesting that analogous potential terms were introduced in [26] at the bosonic level for ensuring the existence of some hidden symmetries in the models of $3D$ particle in a non-Abelian monopole background.

The same reduction $\mathbb{R}^4 \to \mathbb{R}^3$ can be performed at the level of Hamiltonian and supercharges. In particular, the reduction of the Hamiltonian of the $4D$ system of [8] yields the $3D$ Hamiltonian (C.4.5).

C.6 Conclusions

In this paper, we constructed some rather general off-shell $\mathcal{N}=4$ supersymmetric coupling of the $d=1$ multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ to an external SU(2) gauge field. The off-shell $\mathcal{N}=4$ supersymmetry restricts the latter to be a $3D$ reduction of the 't Hooft ansatz for self-dual SU(2) gauge field in $\mathbb{R}^4$, that is a particular solution of the Bogomolny monopole equations. At the component level, the coupling to a gauge field is necessarily accompanied by an induced potential which is bilinear in the SU(2) generators and arises as a result of eliminating an auxiliary field. Our main devices, as in [11], were the HSS approach and the use of an analytic "semi-dynamical" multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ with the WZ type action. This multiplet incorporates SU(2) doublet bosonic spin variables which are crucial for arranging couplings to non-Abelian gauge fields. We also presented the explicit form of the corresponding Hamiltonian and $\mathcal{N}=4$ supercharges which can be equally used for an arbitrary monopole BPS background, though with the on-shell realization of $\mathcal{N}=4$ supersymmetry.

Like in the case of $4D$, $\mathcal{N}=4$ mechanics coupled to a self-dual non-Abelian gauge field [11], in the $3D$ case considered here there remains a problem of extending the model to a generic SU(N) gauge group, as well as to general monopole backgrounds obtained as a $3D$ reduction of ADHM construction [27]. It would be also interesting to study SQM models with nonlinear counterparts of the target space multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ [6, 28] and/or of the semi-dynamical multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ [22]. Such models exhibit more general target geometries as compared to the conformally-flat ones associated with the linear $(\mathbf{3},\mathbf{4},\mathbf{1})$ multiplet and are capable to yield also more general background gauge fields.

Finally, it is worthwhile to note that similar constraints (Bogomolny equations) on the external non-Abelian $3D$ gauge field were found in [29], while considering an $\mathcal{N}=4$ extension of Berry phase in quantum mechanics. However, no invariant actions and/or the explicit expressions for the Hamiltonian and $\mathcal{N}=4$ supercharges were presented there.

Acknowledgments

We cordially thank Andrei Smilga for numerous useful discussions and comments. E.I. thanks the Directorate of SUBATECH, Nantes, where this work was completed, for warm hospitality. His work was supported in part by the RFBR grants 09-02-01209, 09-02-91349 and 09-01-93107.

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Supersymétrie étendue et ses applications dans des modèles de mécanique quantique associés aux champs de jauge auto-duaux

Nous étudions certains nouveaux modèles de la mécanique quantique supersymétrique. La forme explicite des lagrangiens dans la langage des superchamps et dans la langage des composantes, ainsi que les hamiltoniens quantiques et les supercharges sont donné.
Il est montré que l'hamiltonien $H = \not{\mathcal{D}}^2$, où $\not{\mathcal{D}}$ est l'opérateur plat de Dirac en quatre dimensions dans un champ de jauge externe auto-dual, abélien ou non abélien, est supersymétrique avec la supersymétrie $\mathcal{N} = 4$. Une généralisation de cet hamiltonien pour un espace courbé et conformément plat en quatre dimensions existe. Pour un champ de jauge abélien auto-dual, le lagrangien correspondant peut être déduit dans le cadre d'approche de superespace harmonique. Si le hamiltonien implique un champ de jauge non abélien auto-dual, on peut construire la formulation lagrangienne par l'introduction des variables bosoniques auxiliaires avec une action de type Wess-Zumino. Pour une classe spéciale des lagrangiens quand le groupe de jauge est SU(2) et le champ de jauge est exprimé sous la forme de l'ansatz de 't Hooft, il est possible de donner une description dans la langage des superchamps en utilisant le formalisme de superespace harmonique. Comme un exemple, la mécanique $\mathcal{N} = 4$ de monopôle de Yang dans $\mathbb{R}^5$ (qui coïncide avec un instanton sur S^4) est considérée.
Indépendamment, un système similaire avec la supersymétrie $\mathcal{N} = 4$ en trois dimensions admet la description dans le langage des superchamps. La supersymétrie off-shell $\mathcal{N} = 4$ exige que le champ de jauge a une forme statique de ansatz de 't Hooft, qui est une solution particulière des équations de Bogomolny pour les monopôles BPS.

Mots-clés : mécanique quantique supersymétrique, supersymétrie étendue, superchamps, superespace harmonique, auto-dualité, champs de jauge, ADHM, supercharges.

Extended supersymmetry and its applications in quantum mechanical models associated with self-dual gauge fields

We study certain new models of supersymmetric quantum mechanics. The explicit form of the corresponding superfield and component actions, as well as of the quantum Hamiltonians and supercharges is given.
It is shown that the Hamiltonian $H = \not{\mathcal{D}}^2$, where $\not{\mathcal{D}}$ is flat four-dimensional Dirac operator in an external self-dual gauge background, Abelian or non-Abelian, is supersymmetric with $\mathcal{N} = 4$ supersymmetry. A generalization of this Hamiltonian to the motion on a curved conformally flat four-dimensional manifold exists. For an Abelian self-dual background, the corresponding Lagrangian can be derived from certain harmonic superspace expressions.
If the Hamiltonian involves a non-Abelian self-dual gauge field, one can construct the Lagrangian formulation of it by introducing auxiliary bosonic variables with Wess-Zumino type action. For a special class of such Lagrangians when the gauge group is SU(2) and the gauge field is expressed in the 't Hooft ansatz form, it is possible to give a superfield description using the harmonic superspace formalism. As a new explicit example, the $\mathcal{N} = 4$ mechanics with Yang monopole in $\mathbb{R}^5$ (= instanton on S^4) is considered.
Independently, a similar system with $\mathcal{N} = 4$ supersymmetry in three dimensions also admits the superfield description. Although the three-dimensional system involves different superfields, its component Lagrangian and Hamiltonian appear to be the three-dimensional reduction of the mentioned four-dimensional system. The off-shell $\mathcal{N} = 4$ supersymmetry requires the gauge field to be a static form of the 't Hooft ansatz for the four-dimensional self-dual SU(2) gauge fields, that is a particular solution of Bogomolny equations for BPS monopoles.

Keywords : supersymmetric quantum mechanics, extended supersymmetry, superfields, harmonic superspace, self-duality, gauge fields, ADHM, supercharges.